On the complexity of finding large odd induced subgraphs and odd colorings

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Outline of the talk

1. Introduction
2. Our results
3. Some proofs
4. Further research

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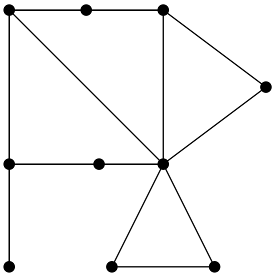

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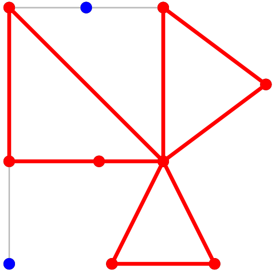

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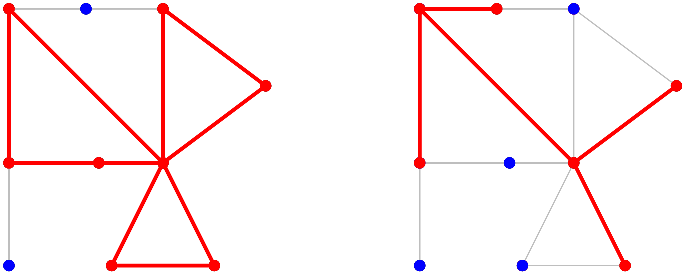

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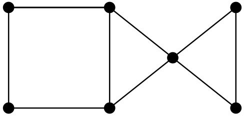

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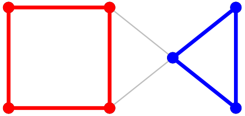

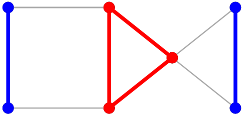

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Corollary

Every graph G contains an even induced subgraph with at least $|V(G)|/2$ vertices.

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What about $\text{mos}(G)$ and $\chi_{\text{odd}}(G)$?

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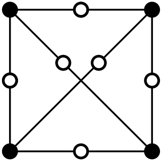

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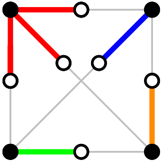

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Our goal Computational aspects of the parameters mos and χ_{odd}.

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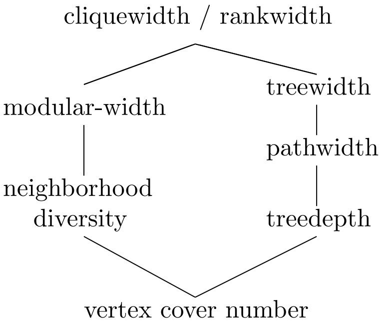

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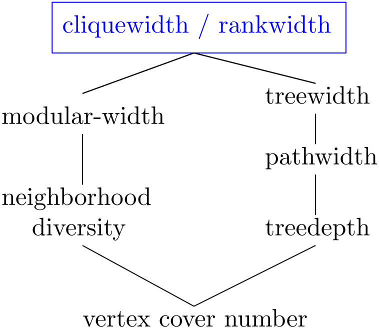

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- Is the function $2^{\mathcal{O}(\mathsf{rw})}$ optimal under the ETH? ($\nexists 2^{o(n)}$ algo for 3-SAT)
  - For deciding whether $\chi_{\mathsf{odd}}(G) \leq q$, our NP-hardness reduction implies that $\nexists\ 2^{o(n)}$ algo under the ETH $\Rightarrow\ \nexists\ 2^{o(\mathsf{rw})} \cdot n^{\mathcal{O}(1)}$ ✓
  - For computing mes(G) and mos(G), existing NP-hardness reduction implies only that $\nexists\ 2^{o(\sqrt{n})}$ algo under the ETH. [Cai, Yang. 2011]

    We provide a linear NP-hardness reduction for mes(G) and mos(G), hence $\nexists\ 2^{o(n)}$ algo under ETH $\Rightarrow\ \nexists\ 2^{o(\mathsf{rw})} \cdot n^{\mathcal{O}(1)}$ ✓

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- Note that cographs are exactly P_4-free graphs.
  We show that χ_{odd} is unbounded for P_5-free graphs.

skip

Next section is...

1. Introduction
2. Our results
3. Some proofs
4. Further research

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For $q = 1$ the problem is trivial: G needs to be an odd graph itself.

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- $\chi_{\text{odd}}(G) \leq 2 \iff$ the above system is feasible.

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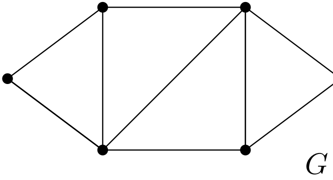

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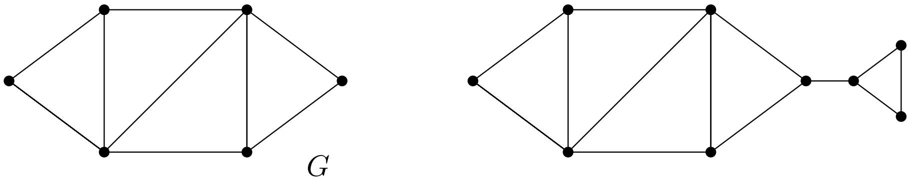

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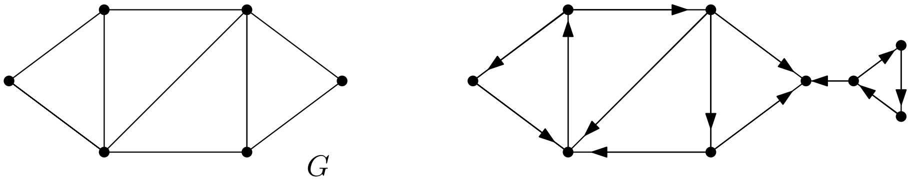

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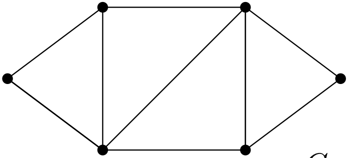

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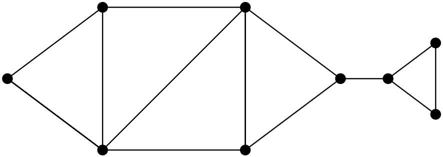

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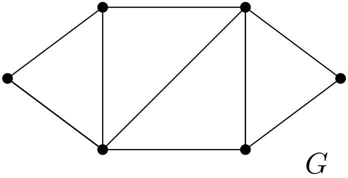

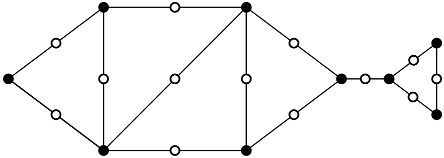

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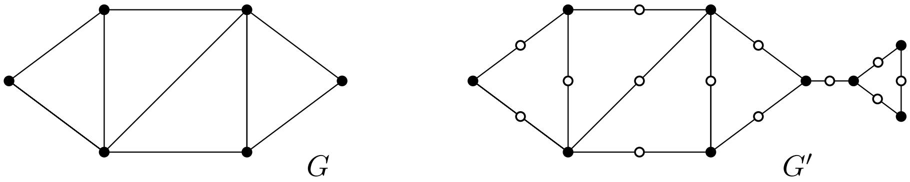

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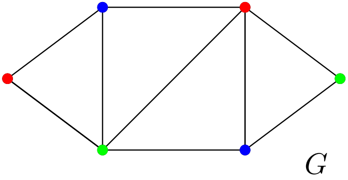

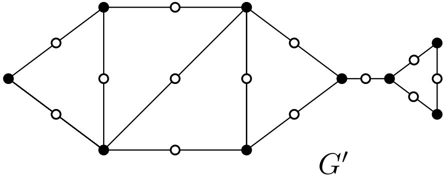

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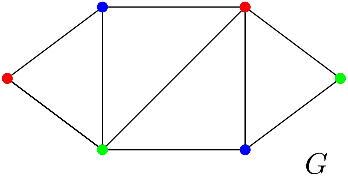

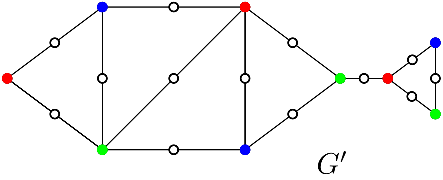

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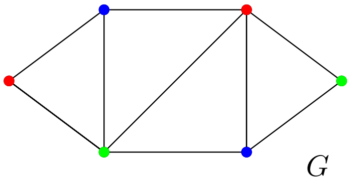

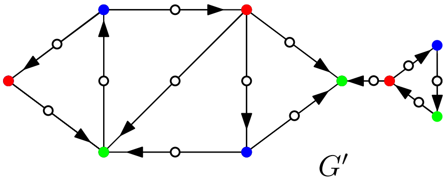

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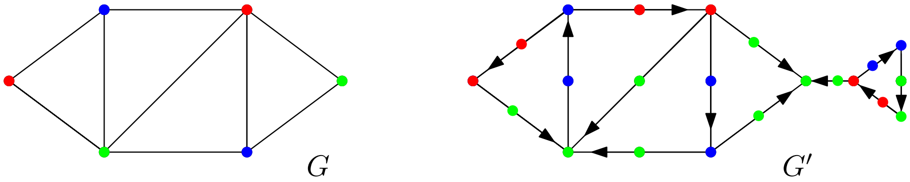

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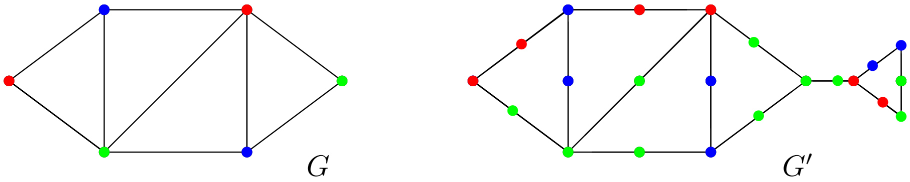

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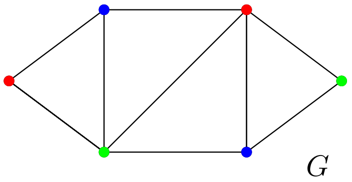

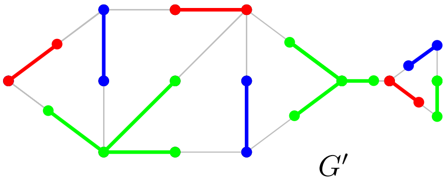

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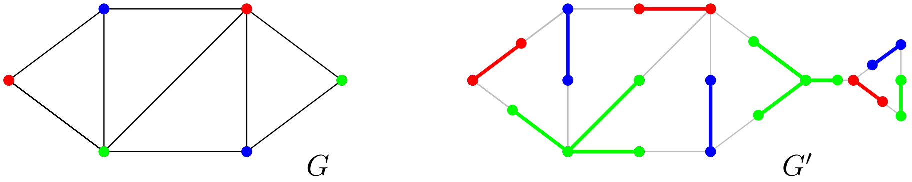

Thus, G is 3-colorable $\iff$ $\chi_{\text{odd}}(G') \leq 3$.

skip

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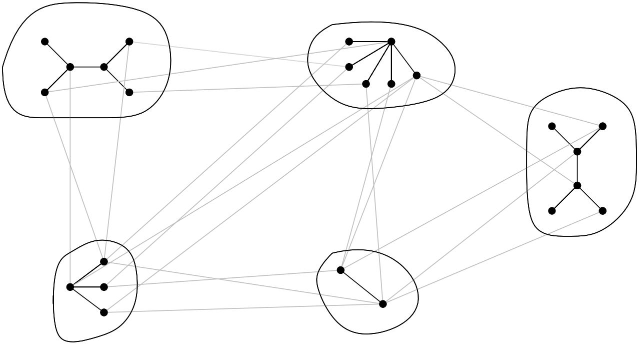

Given G, consider a partition of $V(G)$ into induced odd trees.

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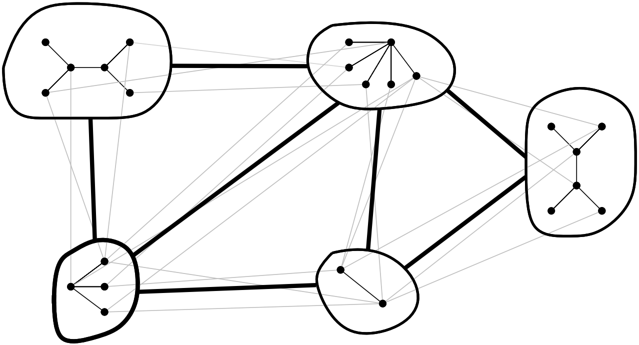

Let G' be obtained from G by contracting each tree to a single vertex.

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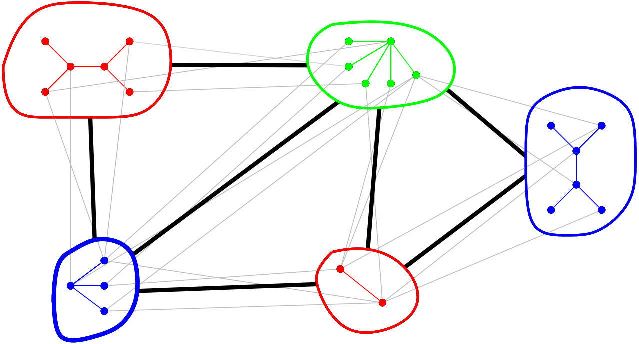

Consider a proper vertex coloring of G' using $\chi(G')$ colors.

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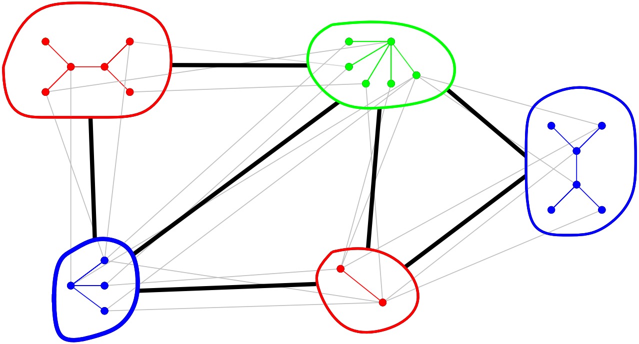

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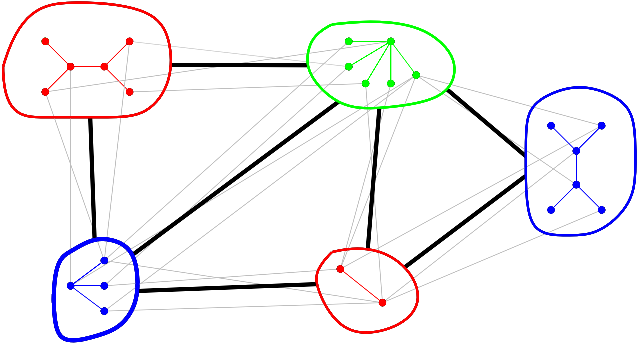

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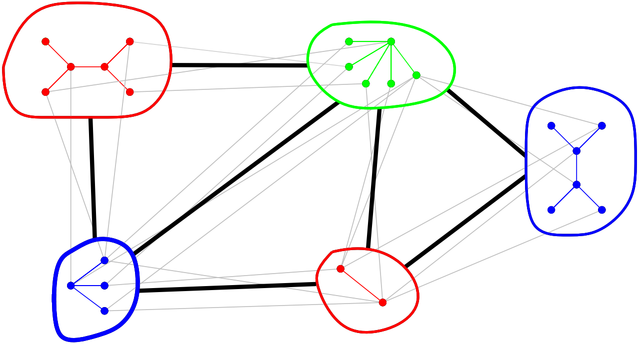

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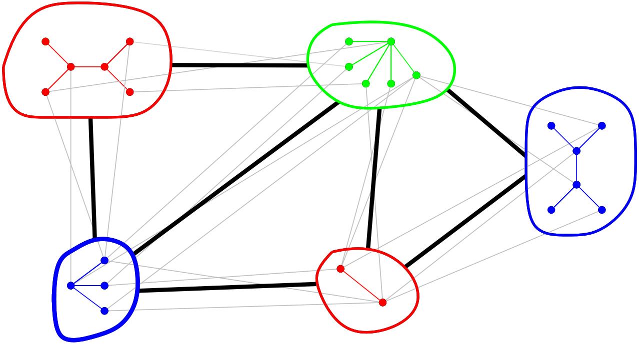

Bound is tight: let G be subdivided n-clique with $n \equiv 0, 3 \pmod 4$. ⏩ skip

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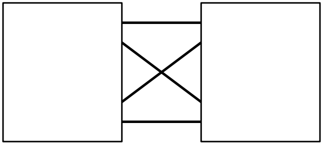

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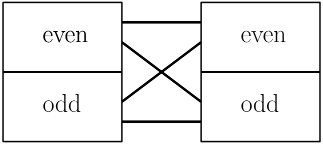

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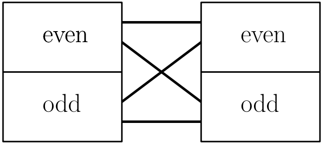

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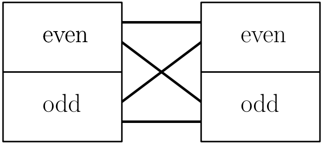

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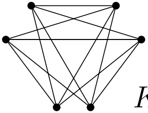
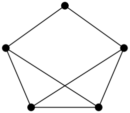


$K_{2,2,2}$



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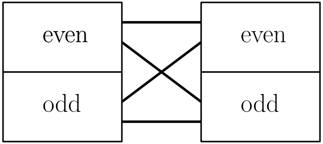

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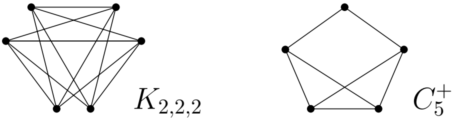

Odd graphs on four vertices: K_4, $K_{1,3}$, and $2K_2$.

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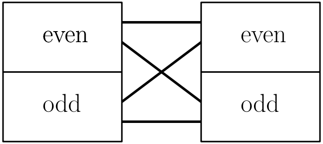

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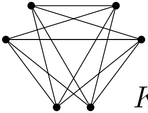

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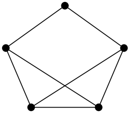

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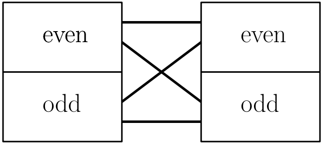

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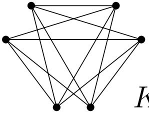

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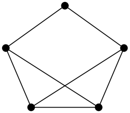

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Next section is...

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4. Deciding whether $\chi_{\mathrm{odd}}(G) \leq q$ parameterized by tw:
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Gràcies!