Introduction to treewidth

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Rencontres virtuelles en théorie des graphes
JCRAALMA – 29 mars 2021

Outline of the talk

1. Definition and simple properties
2. Dynamic programming on tree decompositions
   - Two simple algorithms
   - Courcelle's theorem
   - Introduction to parameterized complexity
3. Brambles and duality
4. Computing treewidth

Next section is...

The multiples origins of treewidth

- 1972: Bertelè and Brioschi (dimension).
- 1976: Halin (S-functions of graphs).
- 1984: Arnborg and Proskurowski (partial k-trees).
- 1984: Robertson and Seymour (treewidth).

A measure of the similarity with a tree

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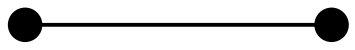

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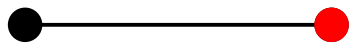

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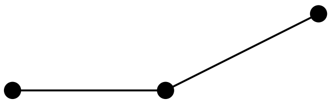

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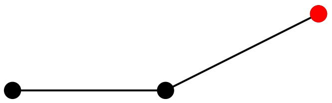

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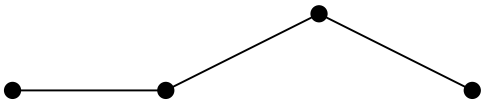

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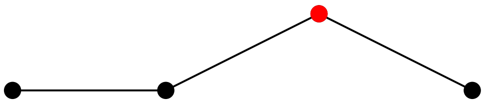

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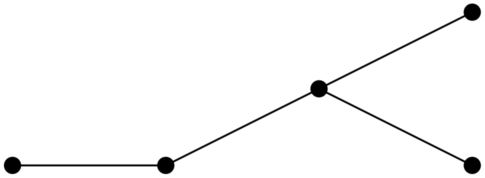

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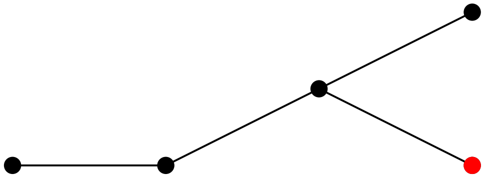

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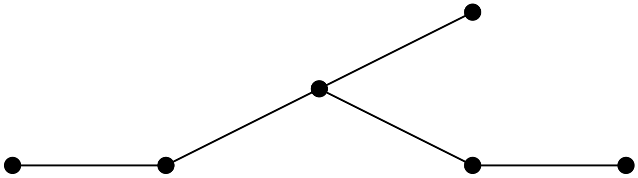

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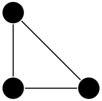

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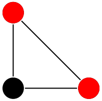

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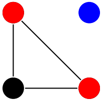

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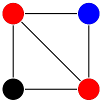

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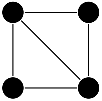

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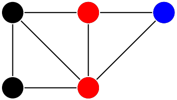

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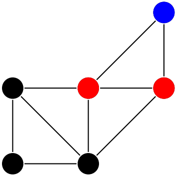

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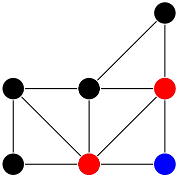

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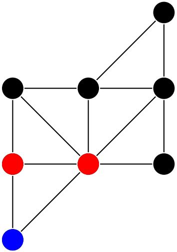

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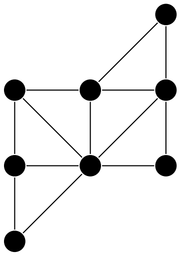

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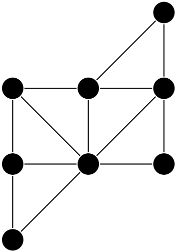

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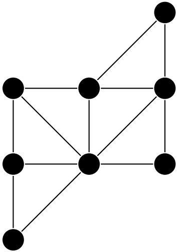

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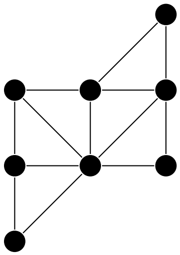

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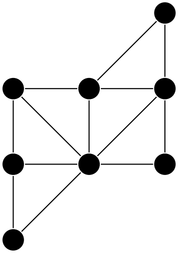

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Construction suggests the notion of tree decomposition: small separators.

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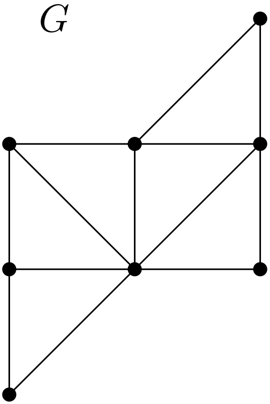

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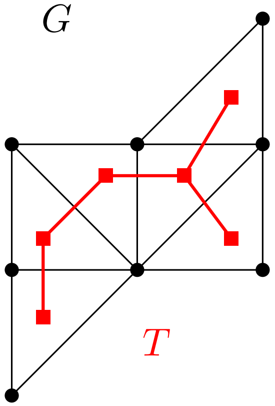

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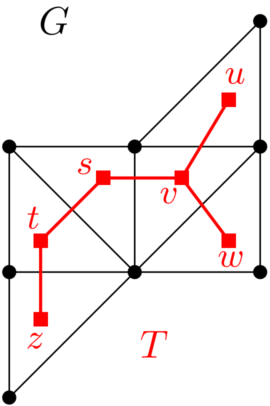

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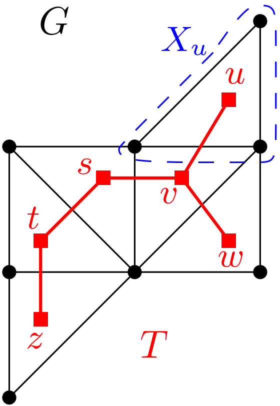

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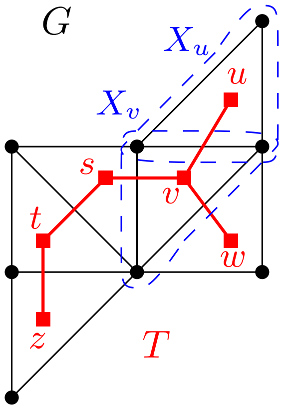

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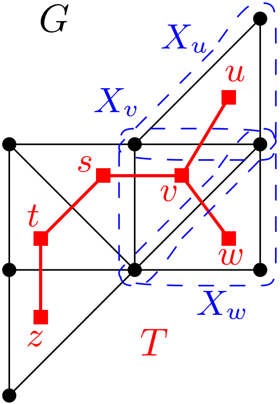

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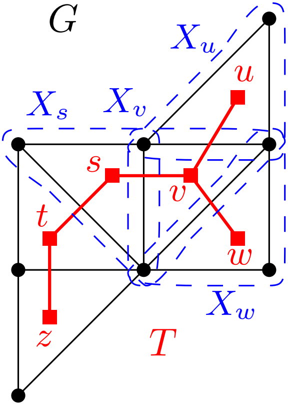

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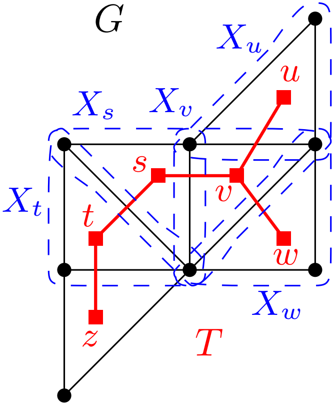

An equivalent (and more common) definition of treewidth

- Tree decomposition of a graph G:

  pair $(T, \{X_t \mid t \in V(T)\})$, where
  T is a tree, and
  $X_t \subseteq V(G) \ \ \forall t \in V(T)$ (bags),

  satisfying the following:

  - $\bigcup_{t \in V(T)} X_t = V(G)$,
  - $\forall \{u, v\} \in E(G)$, $\exists t \in V(T)$ with $\{u, v\} \subseteq X_t$.
  - $\forall v \in V(G)$, bags containing v define a connected subtree of T.

- Width of a tree decomposition:
  $\max_{t \in V(T)} |X_t| - 1$.

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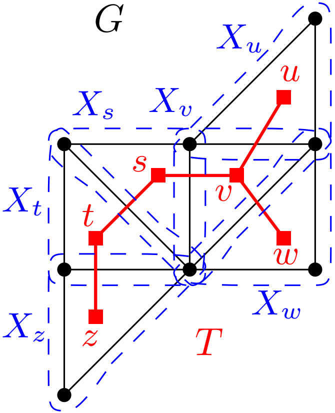

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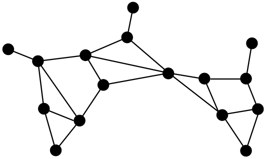

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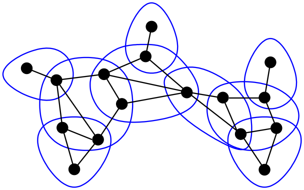

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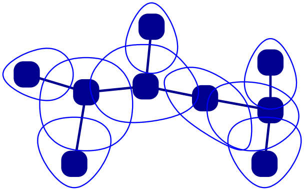

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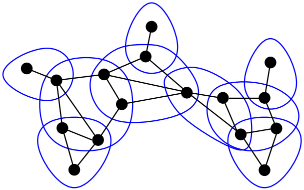

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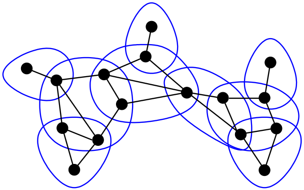

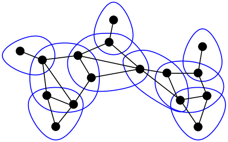

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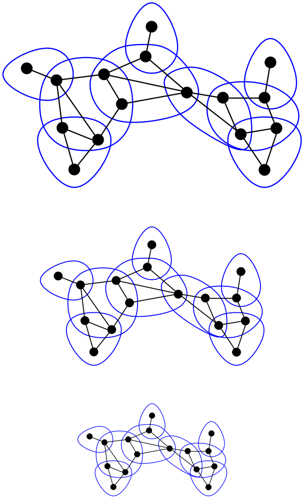

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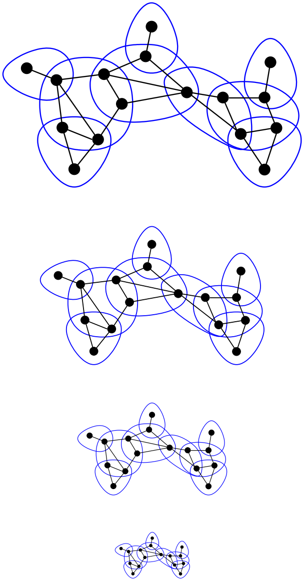

Every bag of a tree decomposition is a separator

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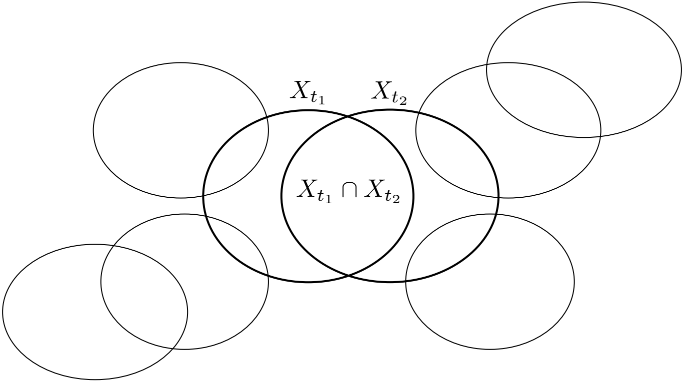

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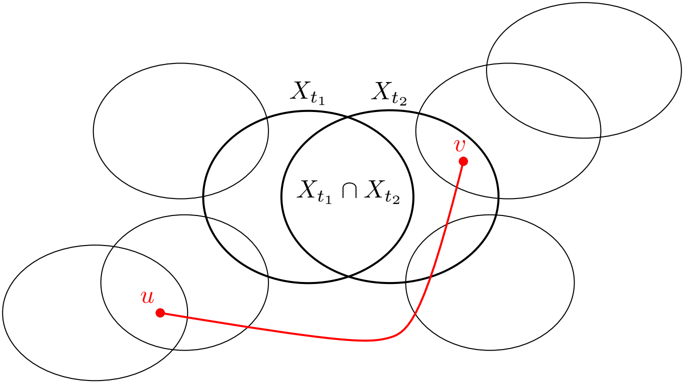

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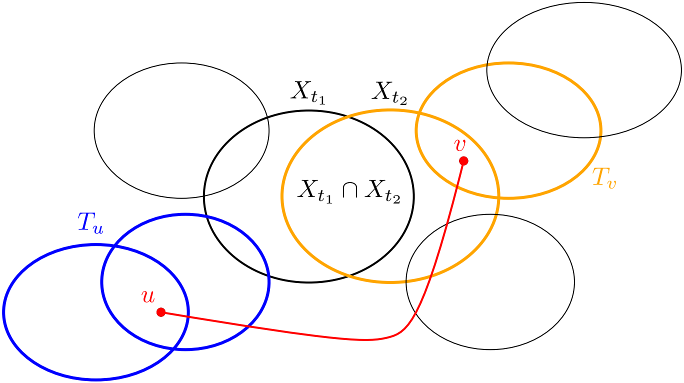

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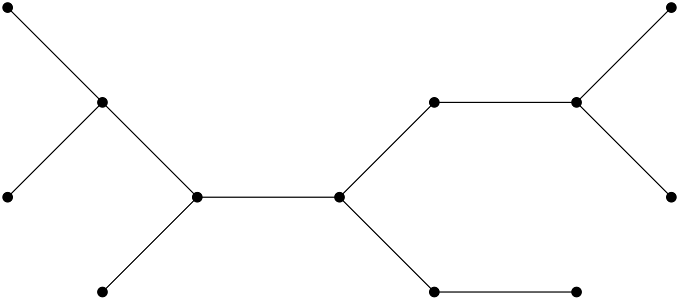

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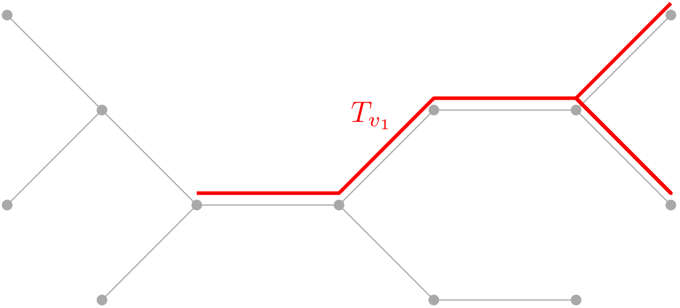

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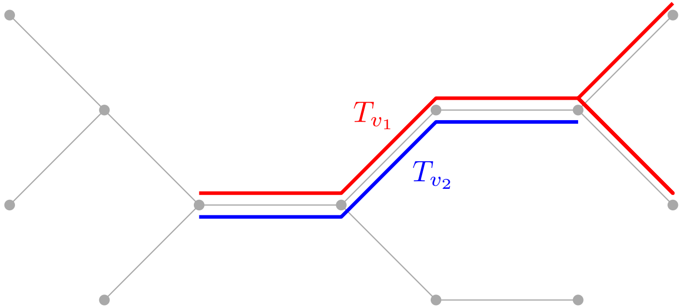

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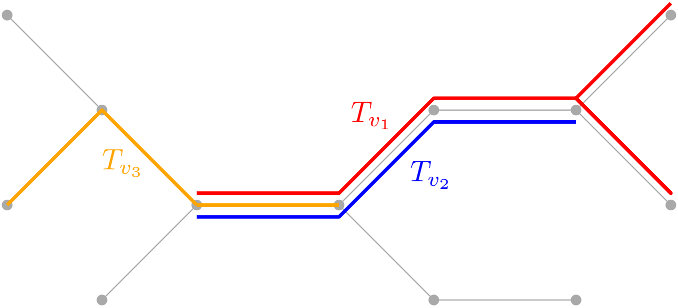

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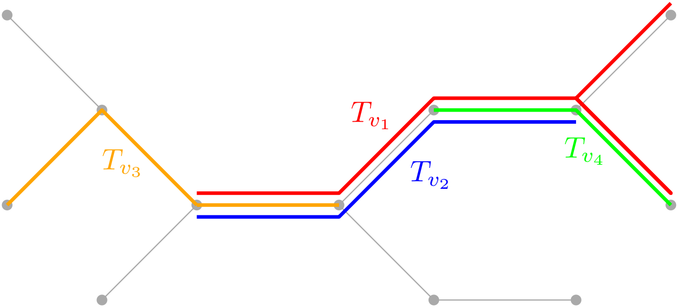

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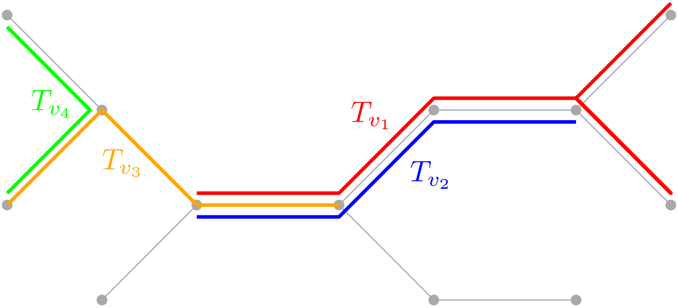

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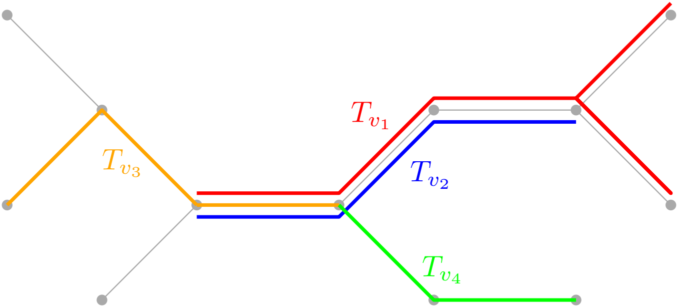

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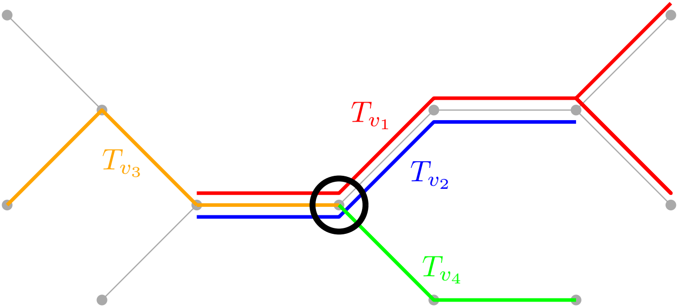

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3. In many practical scenarios, it turns out that the treewidth of the associated graph is small (programming languages, road networks, ...).

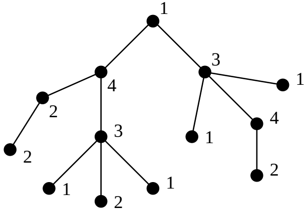
Next section is...

1. Definition and simple properties
2. Dynamic programming on tree decompositions
   - Two simple algorithms
   - Courcelle's theorem
   - Introduction to parameterized complexity
3. Brambles and duality
4. Computing treewidth

Next subsection is...

Weighted Independent Set on trees

[slides borrowed from Christophe Paul]



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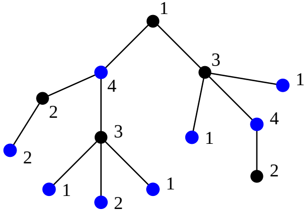

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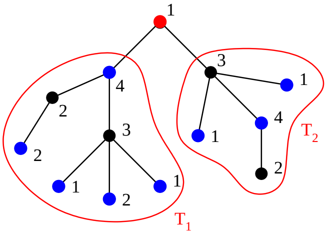

Observations:

1. Every vertex of a tree is a separator.
2. The union of independent sets of distinct connected components is an independent set.

WEIGHTED INDEPENDENT SET on trees

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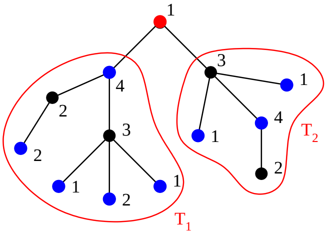

Let x be the root of T, $x_1 \dots x_\ell$ its children, $T_1, \dots T_\ell$ subtrees of $T - x$:

- $wIS(T, x)$: maximum weighted independent set containing x.
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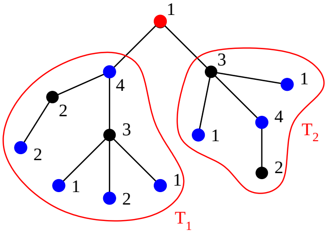

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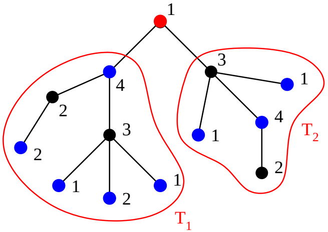

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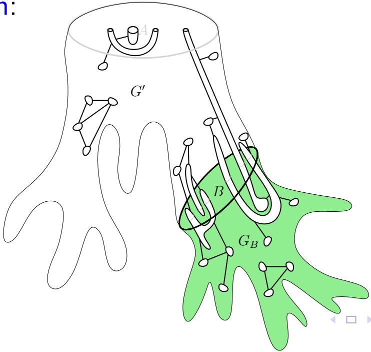
- The way that these partial solutions are defined depends on each particular problem:



[Figure by Valentin Garnero]

Back to tree decompositions

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- V_t: all vertices of G appearing in bags that are descendants of t.
- $G_t = G[V_t]$.

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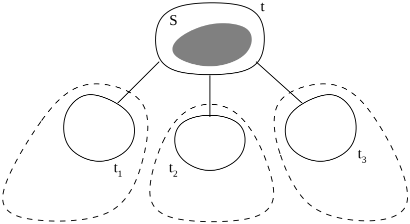

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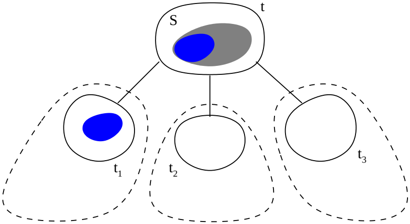

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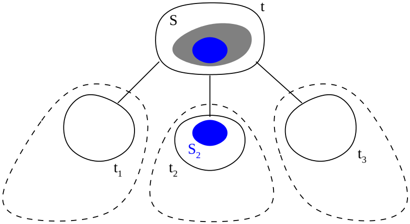

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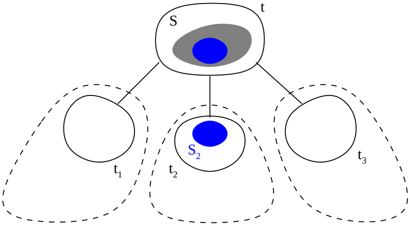

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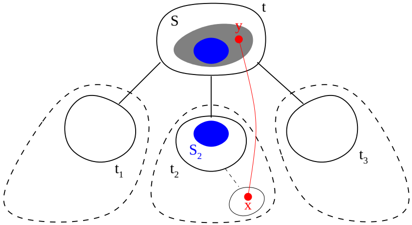
For contradiction: suppose $IS(S,t) \cap V_{t_j}$ is not maximum in G_{t_j}.

$\boxed{\forall S \subseteq X_t,\ IS(S,t)\ = \text{maximum independent set } I \text{ of } G_t \text{ s.t. } I \cap X_t = S}$



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INDEPENDENT SET on tree decompositions

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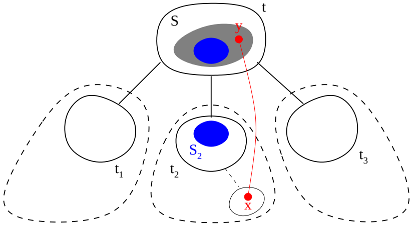

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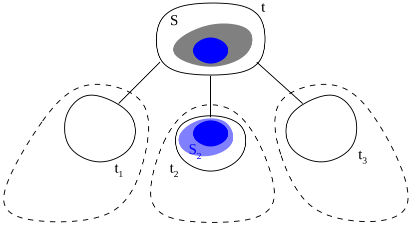
Contradiction! X_{t_j} is not a separator.

Independent Set on tree decompositions

Idea of the dynamic programming algorithm:



How to compute $|IS(S,t)|$ from $|IS(S_j^i, t_j)|$, $\forall j \in [\ell]$, $\forall S_j^i \subseteq X_{t_j}$:

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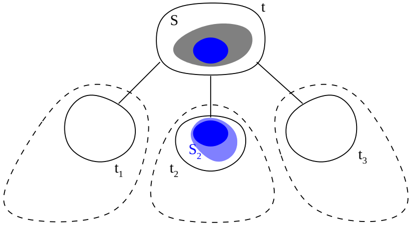

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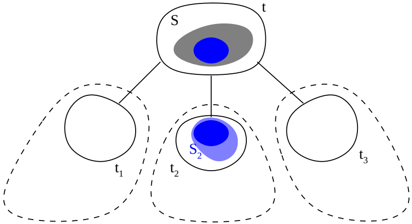

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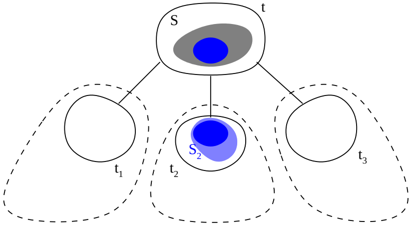

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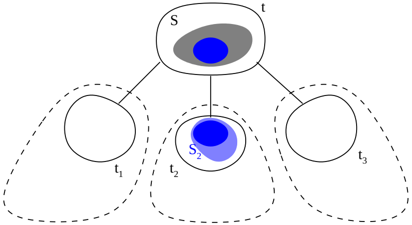

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$$|IS(S,t)| = \left\{ \sum_{i\in[\ell]} \max \begin{array}{l} |S| + \\ \{|IS(S_j^i, t_j)| - |S_j| : \\ S_j^i \cap X_t = S_j \ \wedge \ S_j \subseteq S_j^i \text{ independent}\} \end{array} \right.$$

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INDEPENDENT SET on tree decompositions

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Independent Set on tree decompositions

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★ We have to add the time in order to compute a "good" tree decomposition of the input graph (we discuss this later).

Helpful tool: nice tree decompositions

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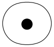

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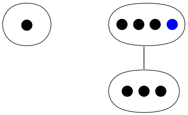

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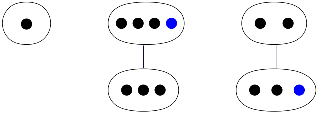

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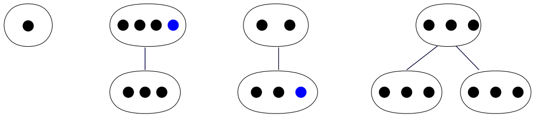

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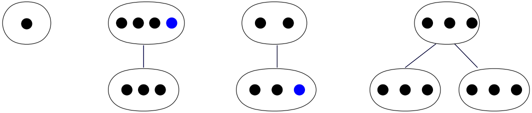

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Lemma

A tree decomposition $(T, \{X_t : t \in T\})$ of width k and x nodes of an n-vertex graph G can be transformed in time $\mathcal{O}(k^2 \cdot n)$ into a nice tree decomposition of G of width k and $k \cdot x$ nodes.

Simpler algorithm for INDEPENDENT SET

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$$|IS(S,t)| = \begin{cases} |IS(S,t')| & \text{if } v \notin S \\ |IS(S \setminus \{v\}, t')| + 1 & \text{if } v \in S \text{ and } S \text{ independent} \\ -\infty & \text{otherwise} \end{cases}$$

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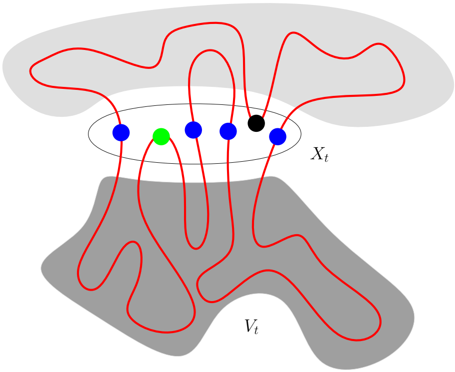
$\boxed{\text{Complexity} : \mathcal{O}(2^k \cdot k^2 \cdot n)}$

[slides borrowed from Christophe Paul]

Let $\mathcal{C}$ be a Hamiltonian cycle.

- Note that $\mathcal{C} \cap G[V_t]$ is a collection of paths.



Hamiltonian Cycle on tree decompositions

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Let $\mathcal{C}$ be a Hamiltonian cycle.

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- Partition of the bag X_t:
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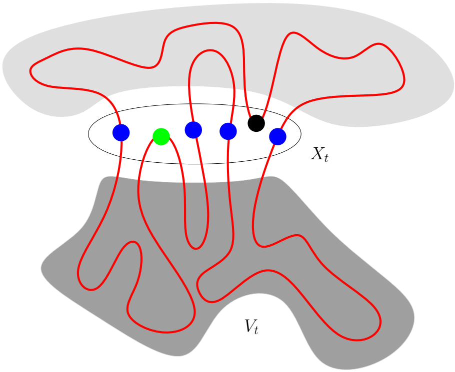

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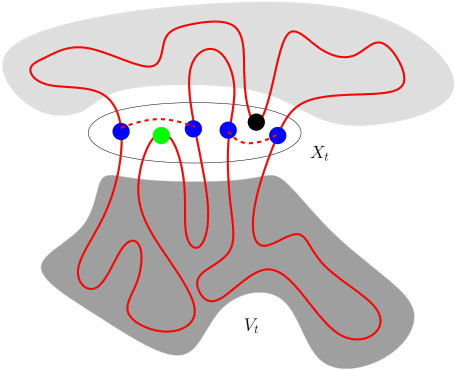


For every node t of the tree decomposition, we need to know if

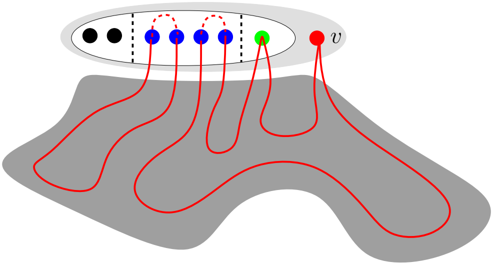
$$(X_t^0, X_t^1, X_t^2, M)$$

where M is a matching on X_t^1, corresponds to a partial solution.

skip

Forget node

Let t be a forget node and t' its child such that $X_t = X_{t'} \setminus \{v\}$.



$\boxed{\text{Claim}}$ X_t is a separator $\Rightarrow$
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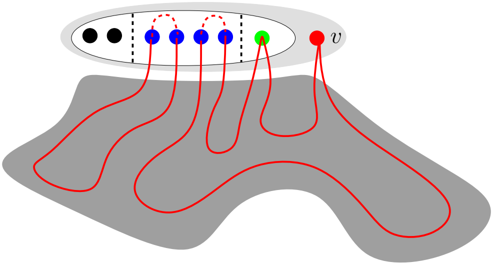

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$$(X^0_{t'}, X^1_{t'}, X^2_{t'} \setminus \{v\}, M) \text{ is a partial solution for } t$$
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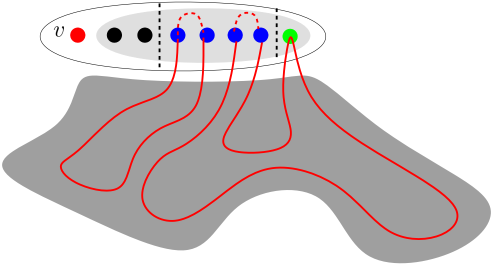

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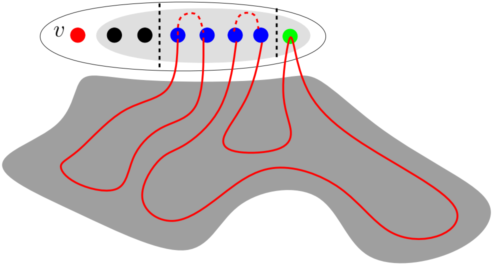

$$(X_{t'}^0 \cup \{v\}, X_{t'}^1, X_{t'}^2, M) \text{ is a partial solution for } t$$
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Let t be an introduce node and t' its child such that $X_t = X_{t'} \cup \{v\}$.

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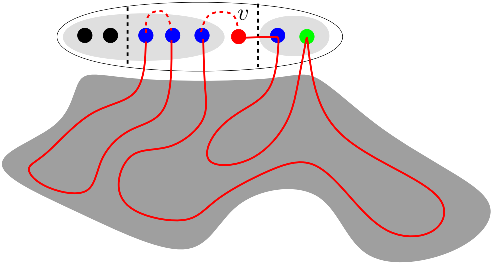

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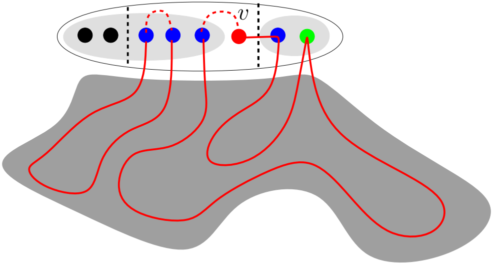

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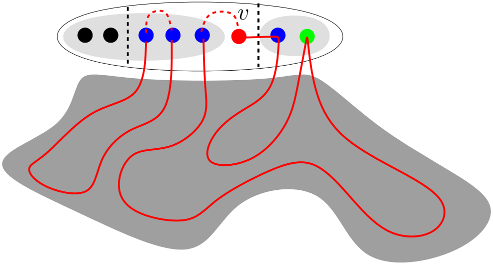

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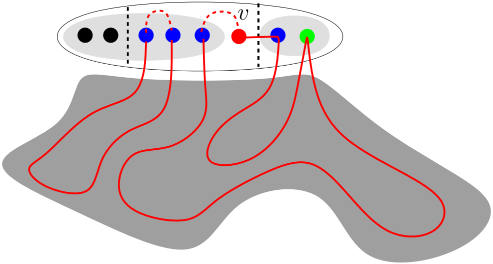

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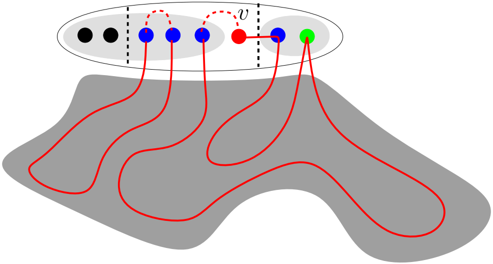

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- a vertex $u \in X_{t'}^1$ becomes internal $\Rightarrow$ $u \in X_t^2$.
- or a vertex $w \in X_{t'}^0$ becomes extremity of a path $\Rightarrow$ $w \in X_t^1$ (similar).

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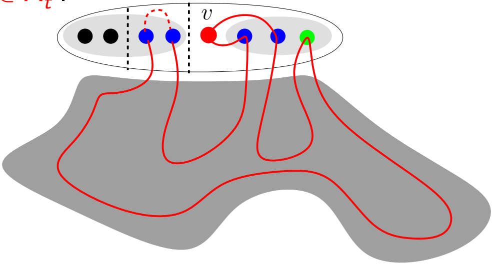

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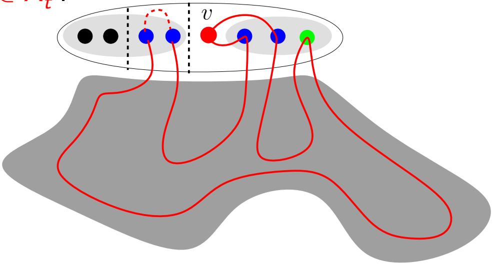

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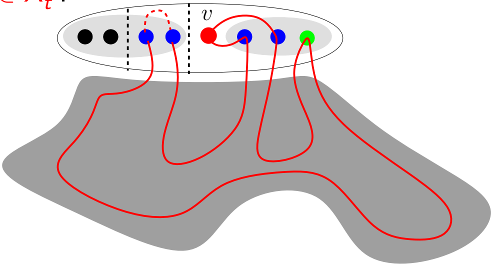

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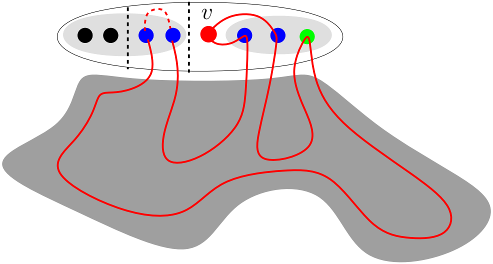

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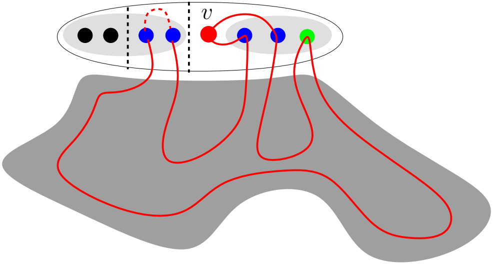

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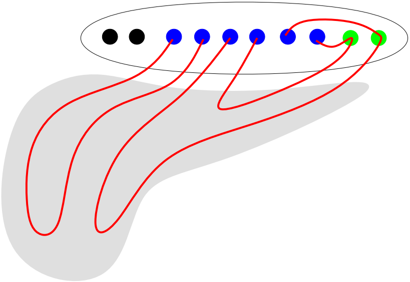

$\boxed{\text{Fact}}$ For being compatible, partial solutions should verify:

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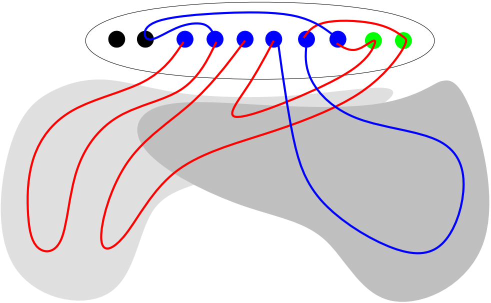

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Can this approach be generalized to more problems?

Next subsection is...

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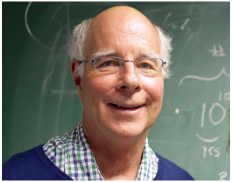
In parameterized complexity: FPT parameterized by treewidth.

Next subsection is...

Parameterized complexity in a nutshell

Idea Measure the complexity of an algorithm in terms of the input size and an additional parameter.

This theory started in the late 80's, by Downey and Fellows:



Today, it is a well-established and very active area.

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- k-Vertex Cover: Does a graph G contain a set $S \subseteq V(G)$, with $|S| \leq k$, containing at least an endpoint of every edge?
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These three problems are NP-hard, but are they $\boxed{\text{equally}}$ hard?

They behave quite differently...

- k-Vertex Cover: Solvable in time $\mathcal{O}(2^k \cdot (m+n))$

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skip

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Working hypothesis of parameterized complexity: $\boxed{k\text{-Clique is not FPT}}$

(in classical complexity: 3-SAT cannot be solved in poly-time)

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1. (x, k) is a YES-instance of A $\Leftrightarrow$ (x', k') is a YES-instance of B.
2. $k' \leq g(k)$ for some computable function $g : \mathbb{N} \to \mathbb{N}$.

W[1]-hard problem: $\exists$ parameterized reduction from k-CLIQUE to it.

W[2]-hard problem: $\exists$ param. reduction from k-DOMINATING SET to it.

W[i]-hard: strong evidence of not being FPT. Hypothesis: $\boxed{\text{FPT} \neq \text{W[1]}}$

Back to treewidth: only good news?

Theorem (Courcelle. 1990)

Every problem expressible in MSO_2 *can be solved in time* $f(\mathrm{tw}) \cdot n$ *on graphs on* n *vertices and treewidth at most* tw.

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2. Most natural problems (Vertex Cover, Dominating Set, ...) do not admit polynomial kernels parameterized by treewidth.

Next section is...

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Theorem (Robertson and Seymour. 1993)

For every $k \geq 0$ and graph G, the treewidth of G is at least k if and only if G contains a bramble of order at least $k+1$.

Another dual notion to treewidth: linkedness

[slides borrowed from Christophe Paul]

- Two sets $Y, Z \subseteq V(G)$, with $|Y| = |Z|$, are separable if there is a set $S \subseteq V(G)$ with $|S| < |Y|$ and such that $G - S$ contains no path between $Y \setminus S$ and $Z \setminus S$.

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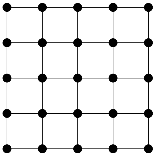

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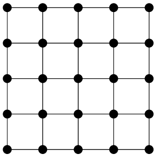

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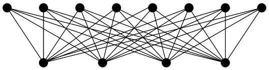

$K_{2k,k}$ is also k-linked

Highly linked graphs have large treewidth

Lemma

If G contains a $(k+1)$-linked set X with $|X| \geqslant 3k$, then $\mathrm{tw}(G) \geq k$.

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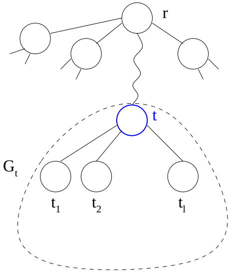

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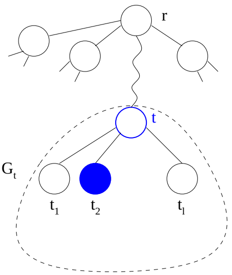

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If $\exists i \in [\ell]$ such that $|V_{t_i} \cap X| \geqslant k$, then we can choose $Y \subseteq V_{t_i} \cap X$, $|Y| = k$ and $Z \subseteq (V \setminus V_{t_i}) \cap X$, $|Z| = k$.

But $S = X_{t_i} \cap X_t$ separates Y and Z and $|S| \leq k-1$.

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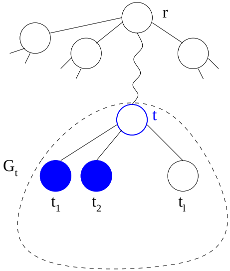

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Otherwise, let $W = V_{t_1} \cup \cdots \cup V_{t_i}$ with $|W \cap X| > k$ and $|(W \setminus V_{t_j}) \cap X| < k$ for $1 \leq j \leq i$.

$Y \subseteq W \cap X$, $|Y| = k+1$ and $Z \subseteq (V \setminus W) \cap X$, $|Z| = k+1$.

But $S = X_t$ separates Y from Z and $|S| \leqslant k$.

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Remark If X is not k-linked we can find, within the same running time, two separable subsets $Y, Z \subseteq X$.

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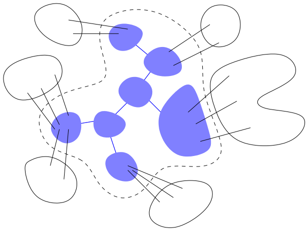

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4-approximation of Robertson and Seymour

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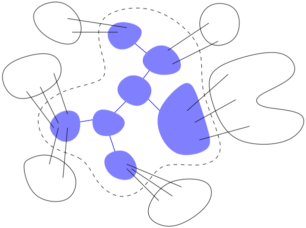

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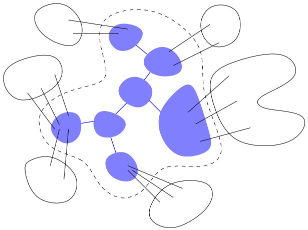

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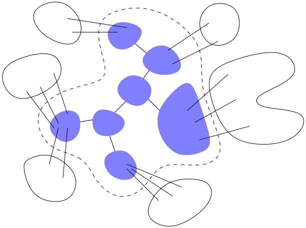

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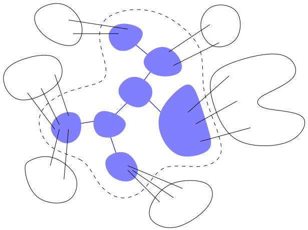

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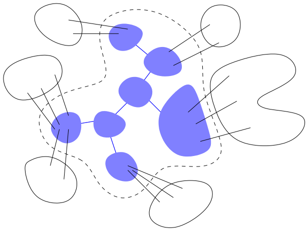

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Initially, we start with U being any set of $3k$ vertices.

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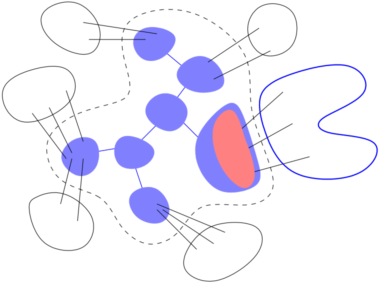

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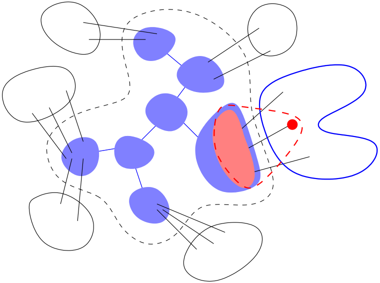

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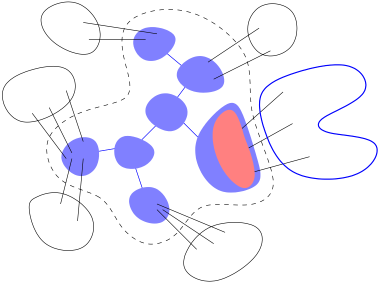

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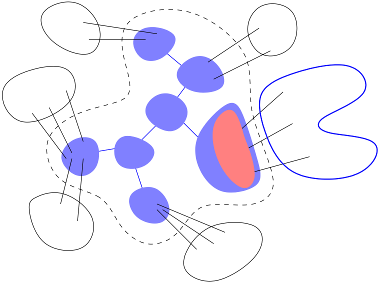

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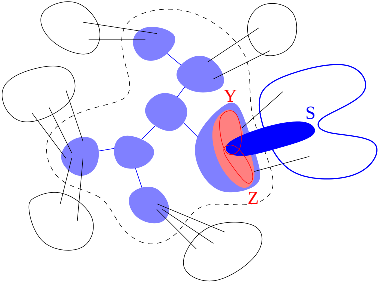

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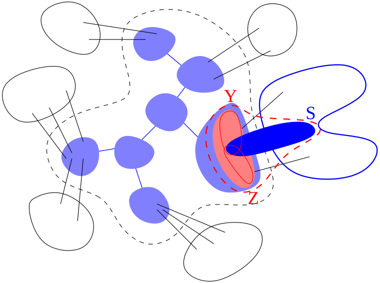

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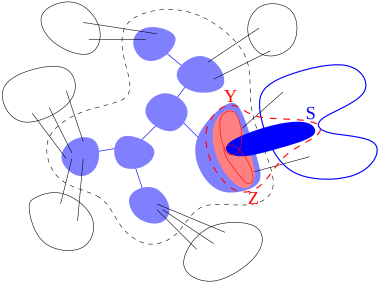

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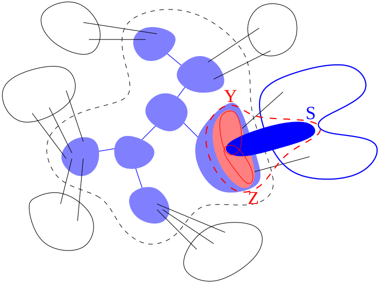
- If $|X| = 3k$: we test whether X is $(k+1)$-linked in time $f(k) \cdot n^{\mathcal{O}(1)}$:
  1. If X is $(k+1)$-linked, then $tw(G) \geq k$, and we stop.
  2. Otherwise, we find sets Y, Z, S with $|S| < |Y| = |Z| \leq k+1$ and such that S separates Y and Z.
     We create a node t' neighbor of t s.t. $X_{t'} = (S \cap C) \cup X$.

     Obs: the neighbors of every new component $C' \subseteq C$ are in $(X \setminus Z) \cup (S \cap C)$ or in $(X \setminus Y) \cup (S \cap C)$

4-approximation of Robertson and Seymour (2)



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     Obs: the neighbors of every new component $C' \subseteq C$ are in $(X \setminus Z) \cup (S \cap C)$ or in $(X \setminus Y) \cup (S \cap C) \Rightarrow \leq 3k$ neighbors.

Gràcies!