Colourings of oriented graphs

Éric SOPENA
LaBRI, Université de Bordeaux

Eric.Sopena@labri.fr

Outline

Preliminary (basic) notions

Oriented vertex-colourings

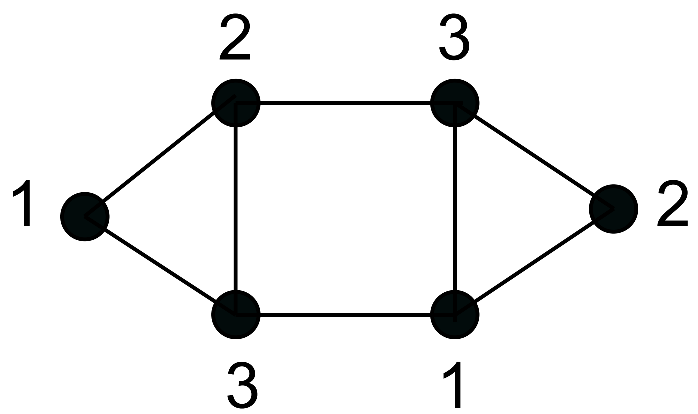
Oriented arc-colourings

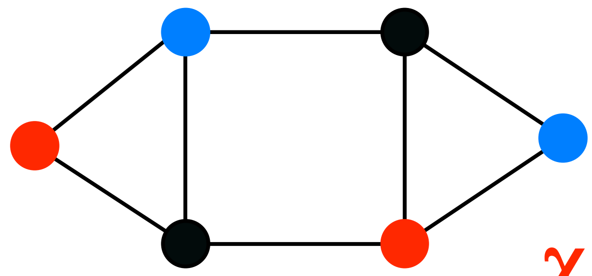
Definitions
Results
Open problems

Preliminary (basic) notions (1)

Vertex-Colourings of undirected graphs

A (proper) **k-vertex-colouring** of a graph *G* is a mapping c : V(G) → {1,2,...,k} such that every two *adjacent vertices* are assigned *distinct colours*.





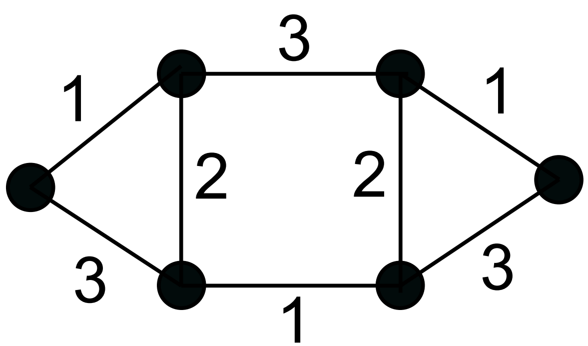
$\chi = 3$

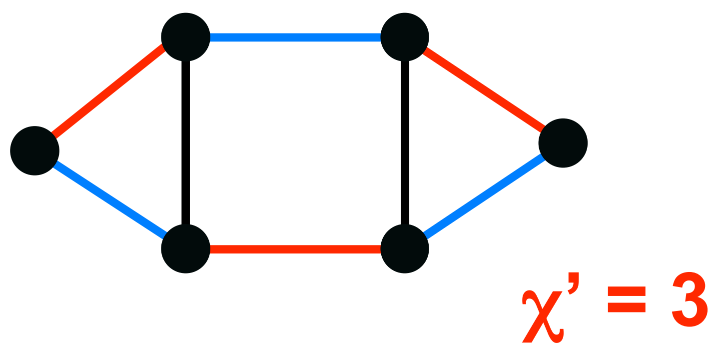
The **chromatic number** $\chi(G)$ of *G* is the smallest k for which *G* has a k-vertex-colouring.

Preliminary (basic) notions (2)

Edge-Colourings of undirected graphs

A (proper) **k-edge-colouring** of a graph *G* is a mapping $c : E(G) \rightarrow \{1,2,\ldots,k\}$ such that every two *incident edges* are assigned *distinct colours*.





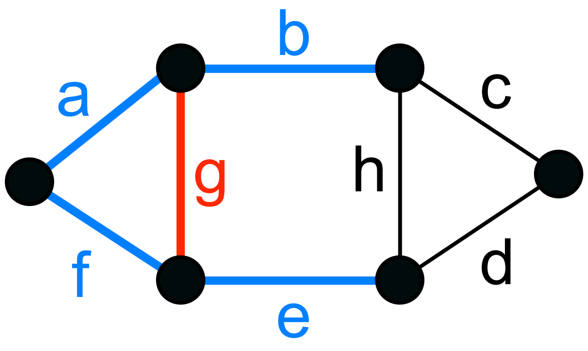
The **chromatic index $\chi'(G)$** of *G* is the smallest k for which *G* has a k-edge-colouring.

Preliminary (basic) notions (3)

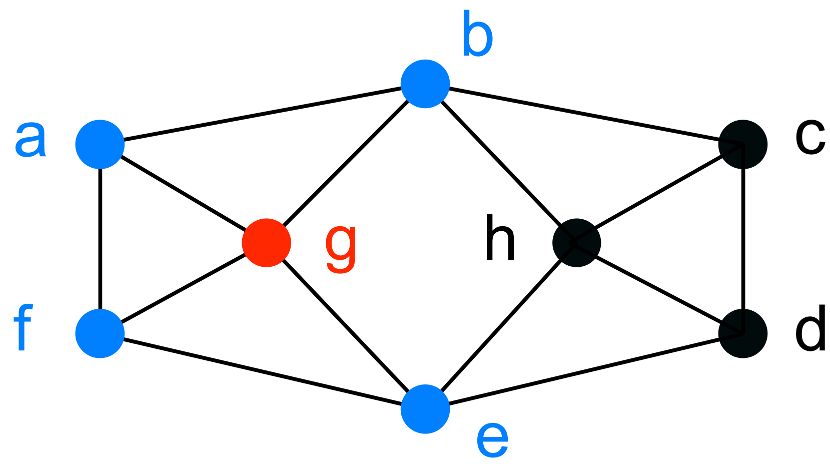
Line-graphs

The **line-graph L(G)** of G is given by :

- V(L(G)) = E(G),
- (e,f) ∈ E(L(G)) iff e and f are *incident* in G.



The graph G

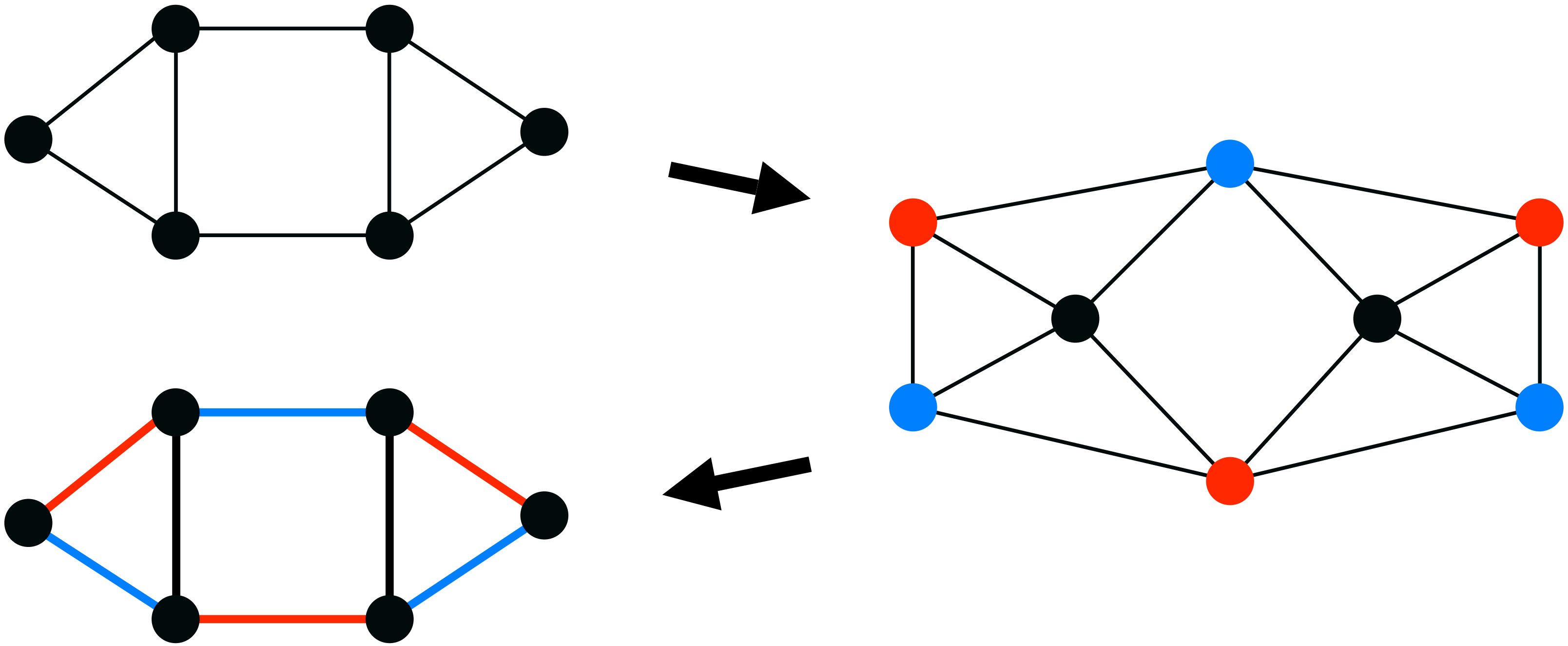


The line-graph L(G) of G

Preliminary (basic) notions (4)

Vertex-colourings *vs* edge-colourings

An edge-colouring of G is nothing but a vertex-colouring of $L(G)$.

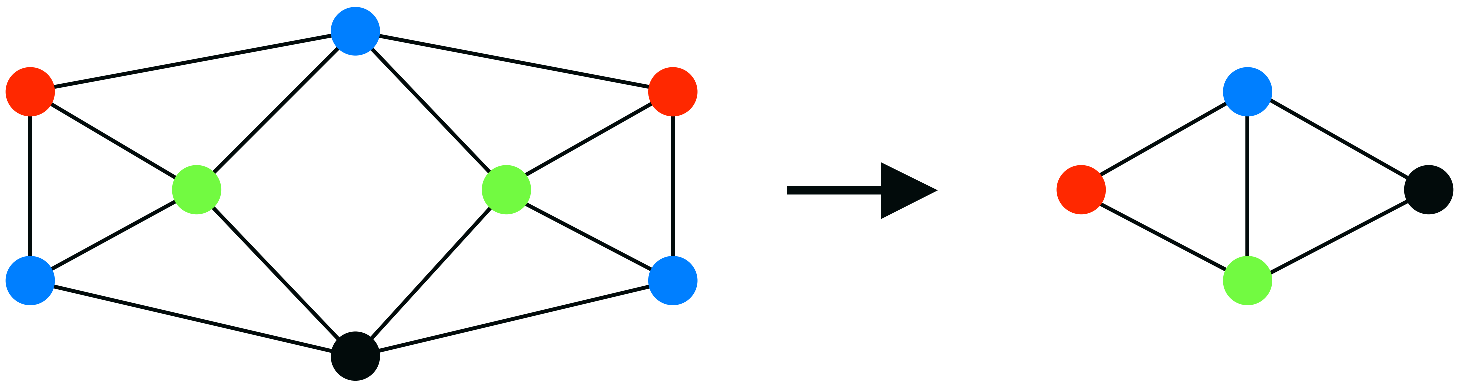


Preliminary (basic) notions (5)

Homomorphisms of undirected graphs

A **homomorphism** from G to H is a mapping $h : V(G) \to V(H)$ such that :

$$xy \in E(G) \Rightarrow h(x)h(y) \in E(H)$$



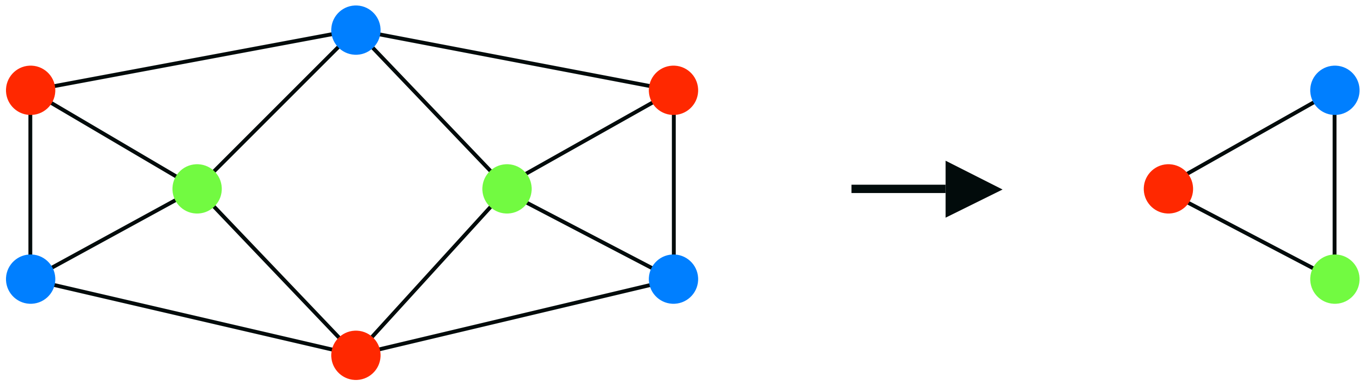
Notation.

$G \to H$: there exists a homomorphism from G to H

Preliminary (basic) notions (6)

Vertex-colourings vs homomorphisms

A k-vertex-colouring of G is nothing but a *homomorphism* from G to K_k, the *complete graph* on k vertices.



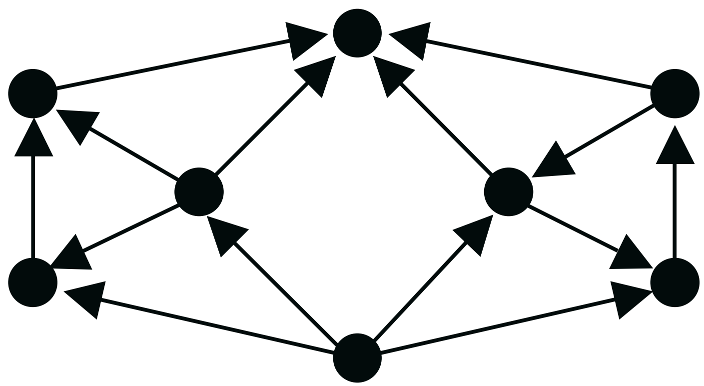
Remark.

$\chi(G) = k$ if and only if $G \to K_k$ and $G \nrightarrow K_{k-1}$

Preliminary (basic) notions (7)

Oriented graphs

An **oriented graph** is an *antisymmetric* digraph (no directed cycle of length 1 or 2).



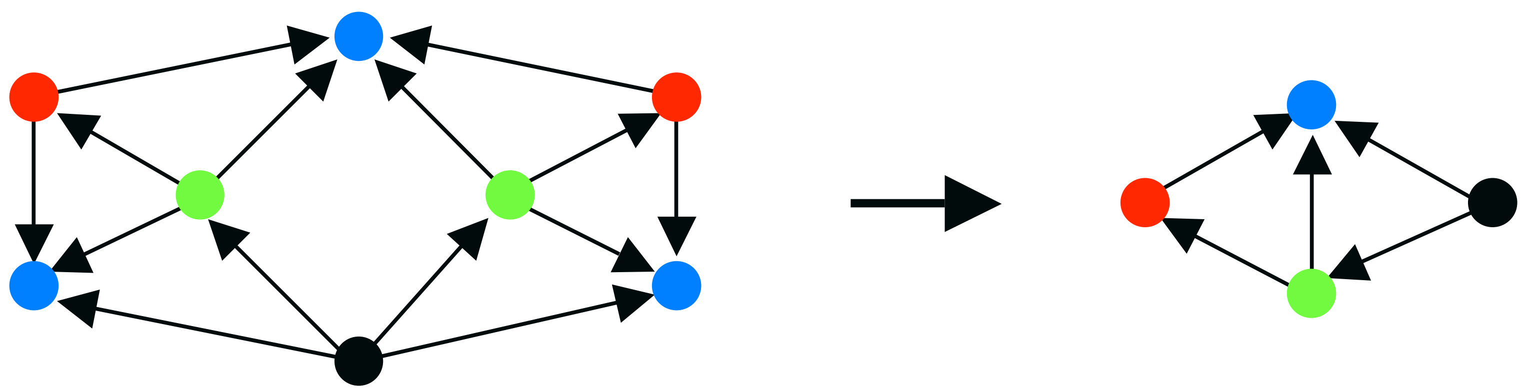
An oriented graph is an *orientation* of some undirected graph, obtained by giving to each edge one of its two possible orientations.

Preliminary (basic) notions (8)

Homomorphisms of oriented graphs

A **homomorphism** from G to H is a mapping $h : V(G) \rightarrow V(H)$ such that :

$$xy \in E(G) \quad \Rightarrow \quad h(x)h(y) \in E(H)$$



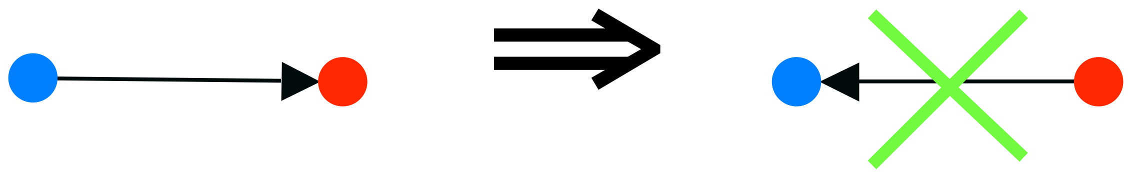
Oriented vertex colourings (1)

Oriented Vertex-Colourings of oriented graphs

An **oriented k-vertex-colouring** of an oriented graph G is a mapping $c : V(G) \rightarrow \{1,2,\ldots,k\}$ such that:

(1) $uv \in E(G) \quad \Rightarrow \quad c(u) \neq c(v)$

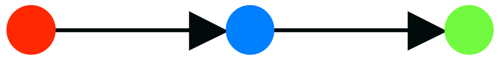
(2) $uv,\ wx \in E(G),\ c(u) = c(x) \quad \Rightarrow \quad c(v) \neq c(w)$

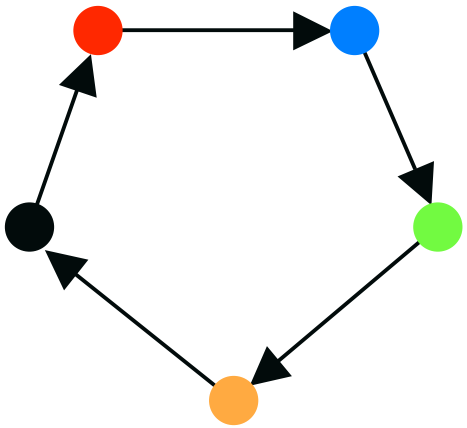


Hence, all the arcs linking two colour classes (independent sets) have the same direction.

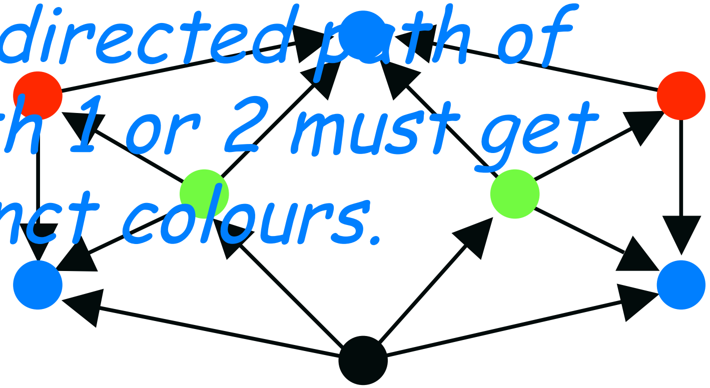
Oriented vertex colourings (2)

Examples.





Any two vertices linked by a directed path of length 1 or 2 must get distinct colours.



Remark.

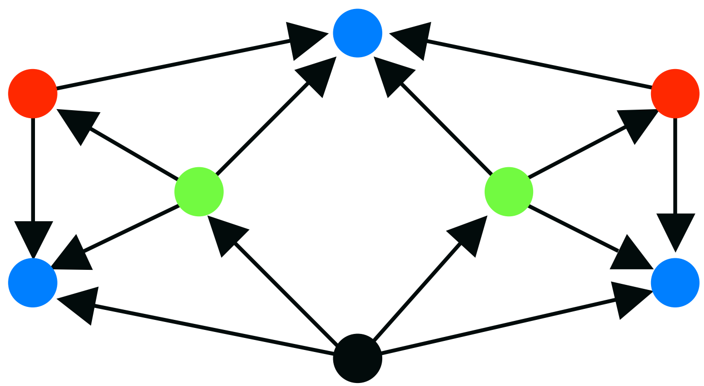
An oriented k-colouring of an oriented graph is nothing but a *homomorphism* to a given oriented graph (or *tournament*) of order k.

Oriented chromatic number (1)

Oriented chromatic number of oriented graphs

The **oriented chromatic number** $\chi_o(G)$ of an oriented graph G is the smallest k such that G admits an oriented k-vertex-colouring.

(Or, equivalently, the minimal order of an oriented graph H such that $G \to H$)



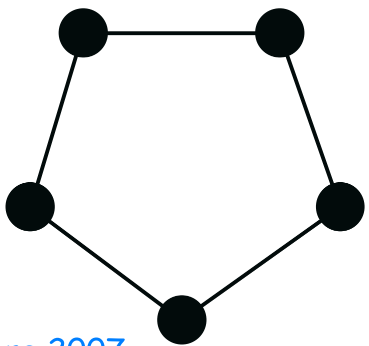
$\chi_o = 4$

Oriented chromatic number (2)

Oriented chromatic number of undirected graphs

The **oriented chromatic number** $\chi_o(U)$ of an *undirected* graph U is the smallest k such that *every orientation* of U admits an oriented k-vertex-colouring:

$$\chi_o(U) = \max \{ \chi_o(G) \, ; \, G \text{ orientation of } U \}$$



$\chi_o = 5$

Observation.
$\chi(U) = \min \{ \ldots \}$

Oriented chromatic number (3)

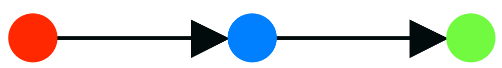
Oriented chromatic number of graph families

The **oriented chromatic number** $\chi_o(\mathbf{F})$ of a family **F** of (undirected or oriented) graphs is given by:

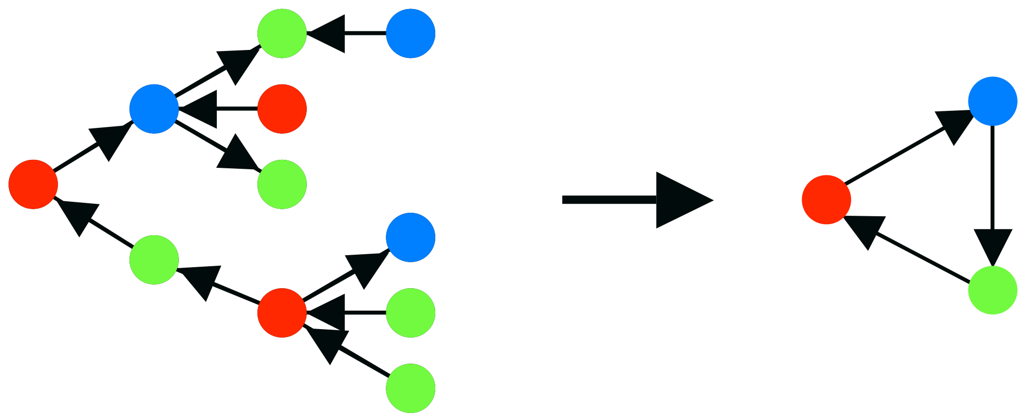
$$\chi_o(\mathbf{F}) = \max \{ \chi_o(G) \,;\, G \in \mathbf{F} \}$$

Example.

χ_o(**trees**) = 3



at least 3...



at most 3...

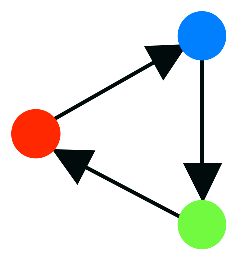
Universal graphs (1)

An oriented graph U is said to be **universal** for a family **F** of oriented (resp. undirected) graphs if every member (resp. every orientation of every member) of **F** admits a homomorphism to U.

$\mathbf{F} \to U$ means $\forall G \in \mathbf{F}, G \to U$

Example.

The directed cycle on 3 vertices is universal for the family of trees.



Remark.

If U is universal for **F** then $\chi_o(\mathbf{F}) \leq |V(U)|$

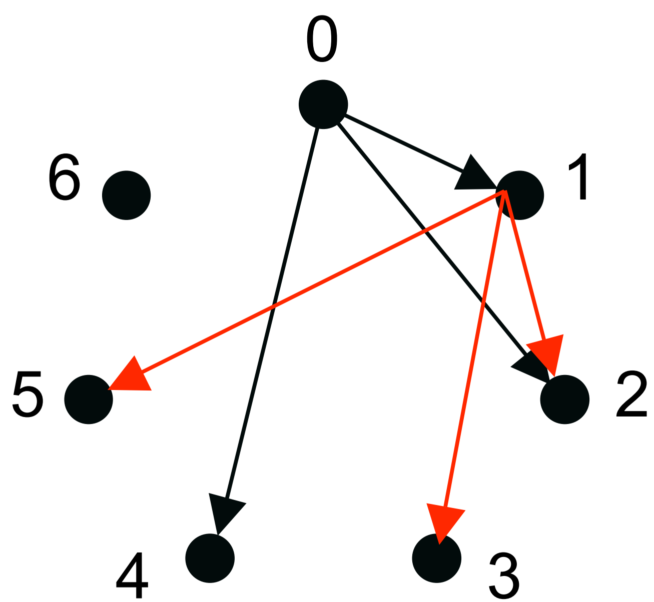
Universal graphs (2)

Outerplanar graphs

Let QR_7 be the tournament given by:

- $V(QR_7) = \{0, 1, \ldots, 6\}$
- $uv \in E(QR_7)$ iff $v - u \pmod 7 = 1, 2$ or 4

(non-zero quadratic residues of 7)



Claim.

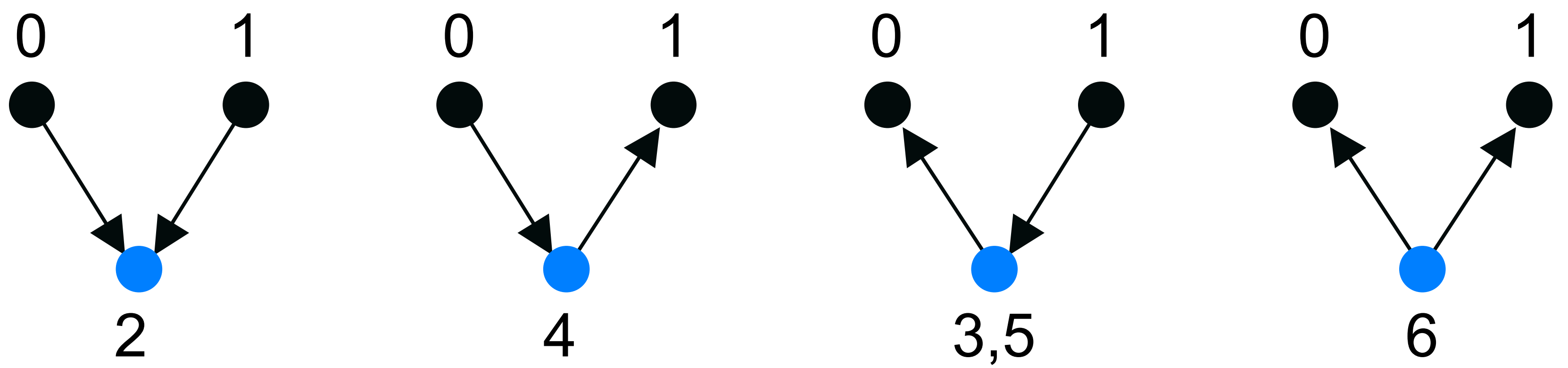
QR_7 is universal for the family **O** of outerplanar graphs.

Universal graphs (3)

Proof of our claim

The graph QR_7 has the following property:

(P_2) For every arc $uv \in E(QR_7)$, there exists a vertex w for every possible orientation of the edges uw and vw:

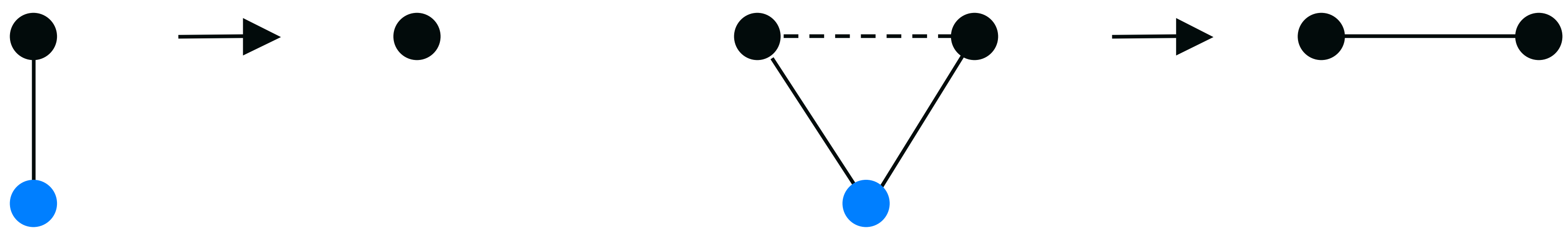


Universal graphs (4)

Proof of our claim (cont.)

Every outerplanar graph has a vertex of degree at most 2.

Simple induction...



1-vertex 2-vertex

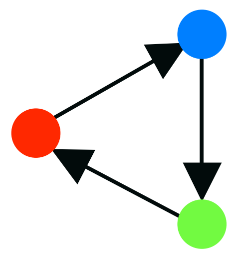
Corollary. $\chi_o(\mathcal{O}) \leq 7$

Oriented cliques (o-cliques) (1)

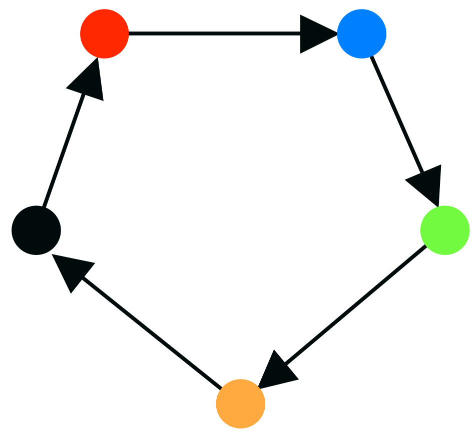
Oriented cliques

An **oriented clique** C is (a subraph of) a graph satisfying $\chi_o(C) = |V(C)|$.

Examples.



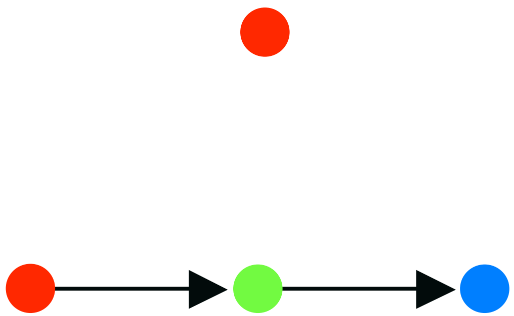
(tournaments...)

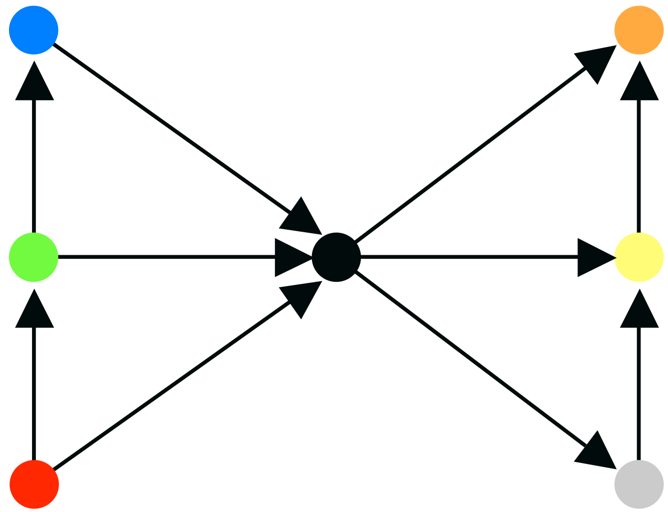


Remark. An o-clique is nothing but an oriented graph in which any two vertices are linked by a directed path (in any direction) of length ≤ 2...

Oriented cliques (o-cliques) (2)

Oriented o-cliques of order 2^k-1

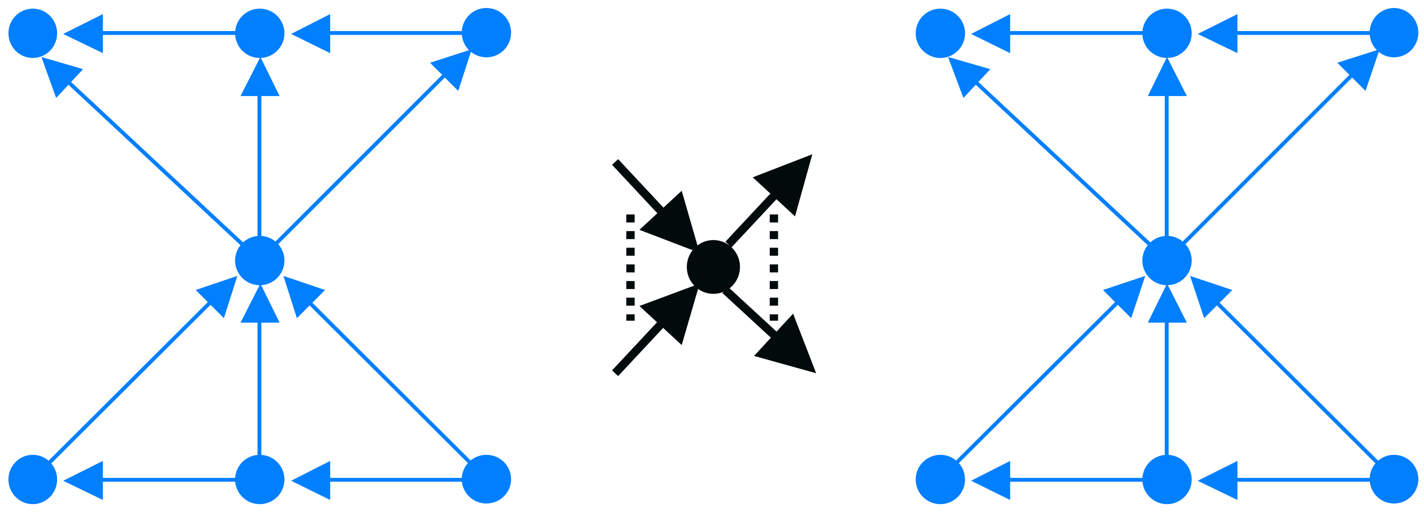




outerplanar...

<u>**Corollary**</u>. $\chi_o(\mathcal{O}) = 7$

Oriented cliques (o-cliques) (3)



planar...

Corollary. $\chi_o(\mathbf{P}) \geq 15$

($\mathbf{P}$ is the family of planar graphs)

Oriented cliques (o-cliques) (4)

Question 1.

What is the maximal order (number of vertices) of a planar o-clique ?

Klostermeyer and MacGillivray (2002)

The order of a planar o-clique is at most 36.

Conjecture. No planar o-clique has more than 15 vertices.

Complete families of graphs (1)

A family of graphs **F** is said to be **complete** if for any two graphs A,B in **F** there exists a graph C in **F** having both A and B as *subgraphs*.

Examples.

Outerplanar graphs, planar graphs, graphs with bounded degree, k-trees, etc.

The family of *connected cubic graphs* is not complete...

Property. Every complete family **F** with bounded χ_o admits a universal graph U with $|U| = \chi_o(\mathbf{F})$.

Complete families of graphs (2)

Oriented chromatic number of complete families

To find an upper bound for this oriented chromatic number, it suffices to find a *universal graph*.

Example.

The tournament QR_{11} is universal for the family of graphs with *degree at most 3*.

Hence,

$$\chi_o(\mathbf{D_3}) \leq 11$$

Complete families of graphs (3)

Open Problem A.

Determine the oriented chromatic number of the family of connected graphs with degree at most 3 (known to be at least 7).

This family is not complete...

Conjecture. The answer is 7.

Best known upper bound : 11 [Vignal, S., 1996]

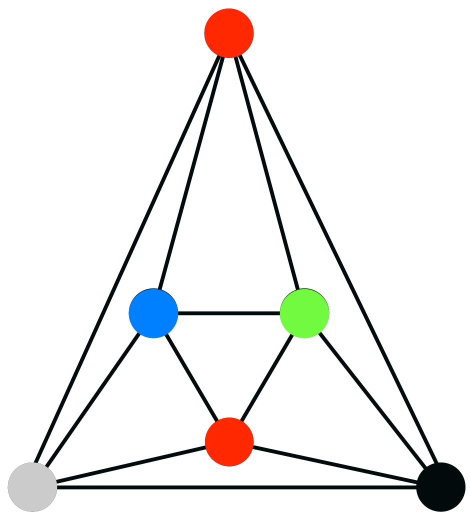
Planar graphs (1)

Acyclic colourings of undirected graphs

A k-vertex-colouring of an undirected graph G is **acyclic** if every cycle in G uses *at least 3 colours*.
(in other words, any two colours induce a forest)

Theorem [Borodin, 1979].
Every planar graph admits an acyclic 5-colouring.

And this bound is tight…



Planar graphs (2)

Oriented colourings of planar graphs

Theorem [Raspaud, S., 1994]

If G admits an acyclic k-vertex-colouring, then

$$\chi_o(G) \leq k \cdot 2^{k-1}$$

And this bound is tight [Ochem, 2005].

Corollary.

Every oriented planar graph admits an oriented 80-colouring.

$$\chi_o(P) \leq 80$$

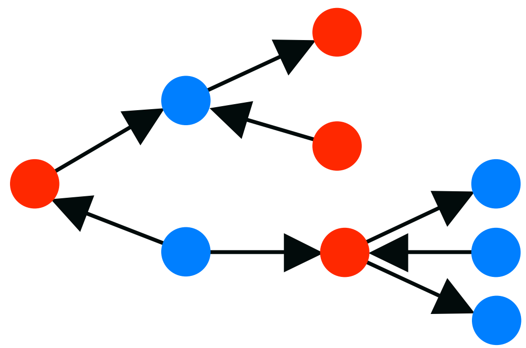
Planar graphs (3)

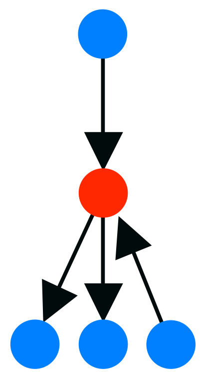
Sketch of proof

Let G be a planar graph and c a k-acyclic colouring of G.

Let a,b be any two colours with $a < b$. Let H be any orientation of G and consider the subgraph $H_{a,b}$ induced by vertices *with colour a or b* (which is a forest).

$a < b$





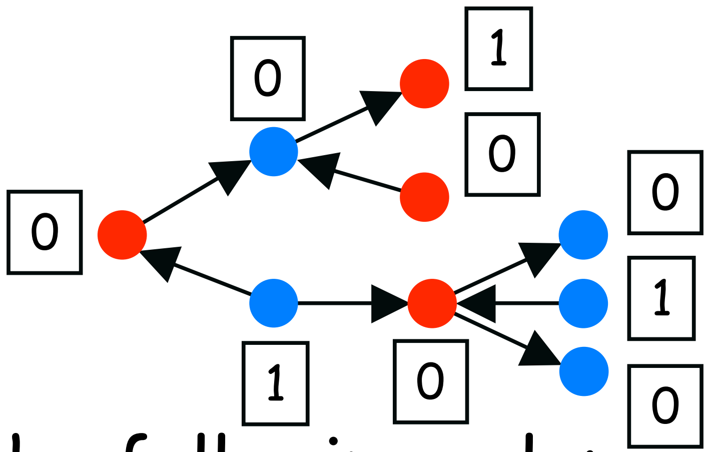
Planar graphs (4)

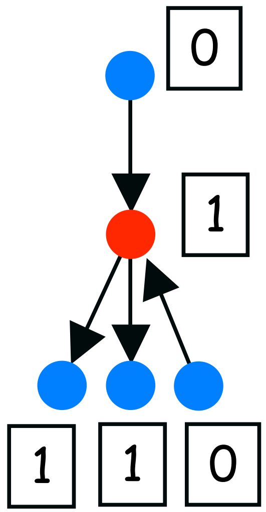
Sketch of proof (cont.)

Choose a vertex in each component of $H_{a,b}$.

Associate a bit with value 0 with each of them.

a < b





Use the following rule:

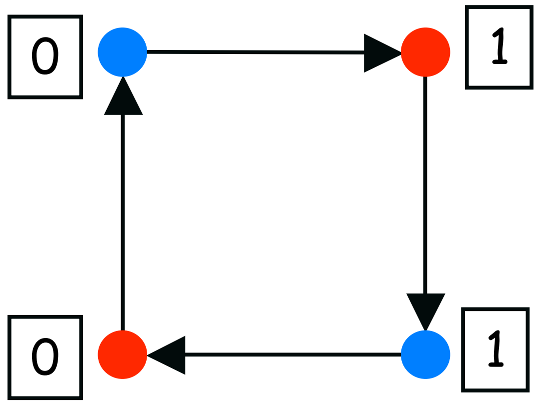
 (a < b) : keep the same bit

 (b > a) : change the bit

Planar graphs (5)

Sketch of proof (cont.)

Doing that, we have constructed a homomorphism from $H_{a,b}$ to the following graph:



With any colour a, we associate k-1 such bits (one for each other colour).

We thus obtain an oriented colouring of H using at most $k \cdot 2^{k-1}$ colours.

Planar graphs (6)

Open Problem B.

Determine the oriented chromatic number of the family of planar graphs.

Conjecture. no conjecture !...

Best known lower bound : 17 [Marshall, 2001]

Planar graphs (7)

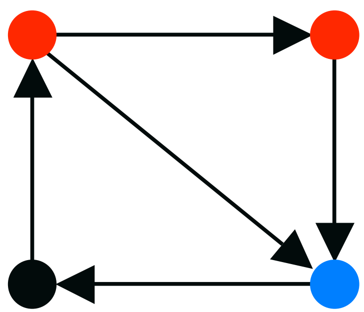
Simple colourings [Smolíková, 2000]

A **simple k-vertex-colouring** of an oriented graph G is a mapping $c : V(G) \to \{1,2,\ldots,k\}$ such that:

(1) $E(G) \neq \emptyset \Rightarrow \exists\, u,v$ s.t. $c(u) \neq c(v)$

(2) $uv, wx \in E(G)$, $c(u) \neq c(v)$, $c(u) = c(x)$ $\Rightarrow c(v) \neq c(w)$

Example.



$\chi_s(G)$ = simple chromatic number of G.

Theorem. $\chi_o(P) = \chi_s(P)$

Planar graphs (8)

Open Problem C.

Determine the oriented chromatic number of the family of triangle-free planar graphs.

Conjecture. no conjecture !...

Best known lower bound : 11 [Ochem, 2004]

Best known upper bound : 47

[Borodin, Ivanova, 2005]

Some other results on χ_o (1)

Planar graphs with large girth

The **girth** g(G) of a graph G is the size of a *shortest* cycle in G.

girth	lower bound	upper bound	
3	17	80	Marshall 2001 – Raspaud, S., 1994
4	11	47	Ochem, 2004 – Borodin, Ivanova 2005
5	6	16	Borodin, Kostochka, Nešetřil, Raspaud, S., 1999 – Pinlou, 2007
6	6	11	Borodin, Kostochka, Nešetřil, Raspaud, S., 1999
$7 \leq g \leq 10$	5	7	Nešetřil, Raspaud, S., 1997 – Borodin, Ivanova 2005
11	5	6	id. – Pinlou, S., 2006
$g \geq 12$	5	5	id. – Borodin, Ivanova, Kostochka, 2007

Some other results on χ_o (2)

Outerplanar graphs with large girth

girth	lower bound	upper bound	
3	7	7	S., 1997
4	6	6	Hosseini Dolama, 2006 - Pinlou, S., 2006
$g \geq 5$	5	5	Nešetřil, Raspaud, S., 1997 - Pinlou, S., 2006

Question 2.

What is the oriented chromatic number of the family of k-outerplanar graphs ?

$\chi_o(2\text{-}O) \leq 67$ [Esperet, Ochem, 2007]

Some other results on χ_o (3)

Graphs with bounded degree

We have seen before :

$$7 \leq \chi_o(\mathbf{D_3}) \leq 11$$

Theorem (Kostochka, S., Zhu, 1997).

If G has maximum degree Δ, then

$$\chi_o(G) \leq 2.\Delta^2.2^\Delta$$

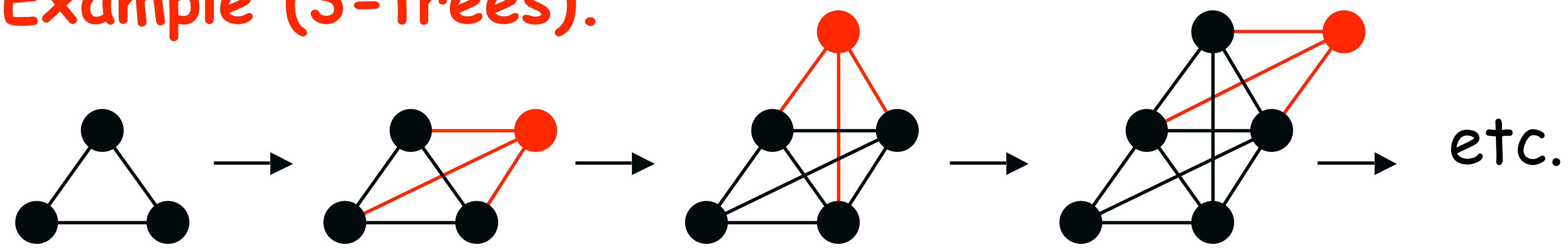
For every Δ, there exists a graph with maximum degree at most Δ and oriented chromatic number at least $2^{\Delta/2}$.

Some other results on χ_o (4)

(partial) k-trees

A **k-tree** is a graph obtained from the *complete graph K_k*, by repeatedly adding new vertices linked to a *clique of size k* (a 1-tree is a tree...).

Example (3-trees).



A **partial k-tree** is a *subgraph* of a k-tree.

Some other results on χ_o (5)

(partial) k-trees (cont.)

Theorem (S., 1997).

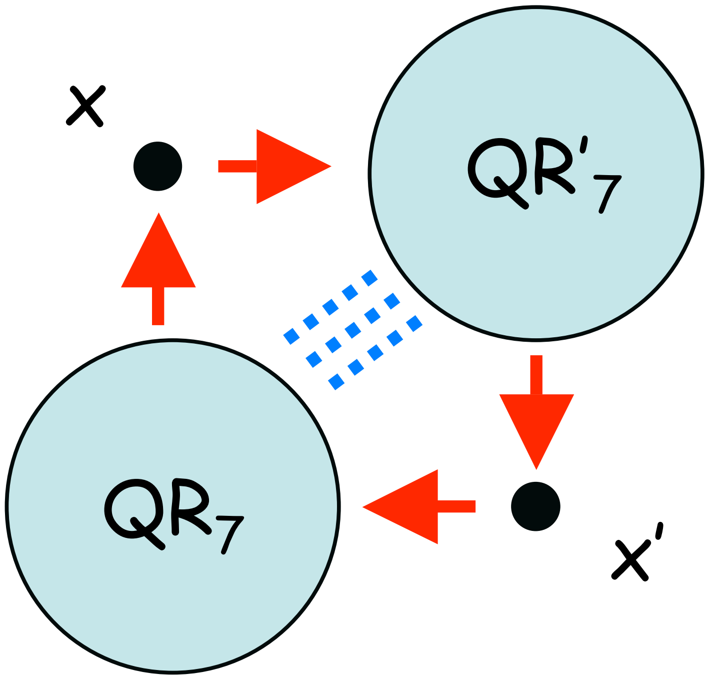
- $\chi_o(T_2) = 7$
- $\chi_o(T_3) = 16$
- $2^{k+1} - 1 \leq \chi_o(T_k) \leq (k + 1) \cdot 2^k$

O-cliques of order $2^{k+1} - 1$ are k-trees...

Universal graph for T_3 : construction due to **J. Tromp** [1990's]

Some other results on χ_o (6)

Tromp's construction (the graph T_{16})



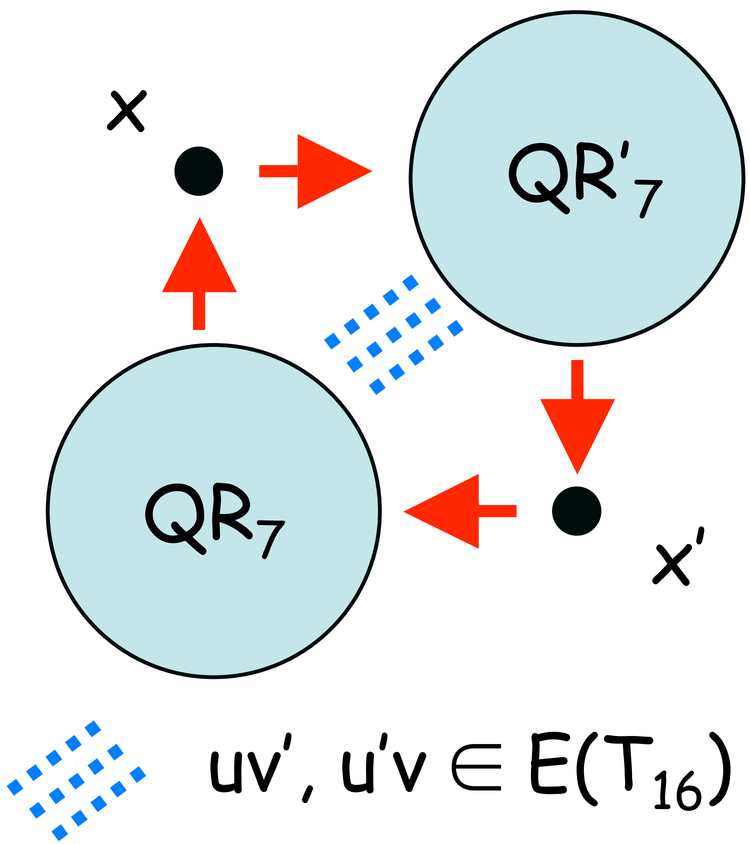
Property.
For every vertex u,
$\Gamma^{+}(u) \sim \Gamma^{-}(u) \sim QR_7$

This property is true for any QR_p, p prime, $p = 4k + 3$.


$uv', u'v \in E(T_{16}) \Leftrightarrow vu \in E(QR_7)$
$u - v \equiv 1, 2 \text{ or } 4 \pmod 7$

Some other results on χ_o (7)

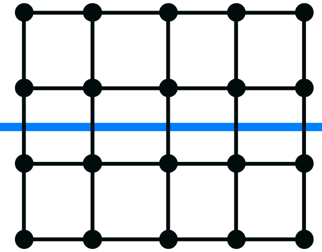
Tromp's construction (the graph T_{16})



Property P_3.
For every *triangle* u,v,w, there exists a vertex z for every possible orientation of the edges uz, vz and wz.
(no graph of order 15 satisfies this property...)

$uv', u'v \in E(T_{16}) \Leftrightarrow vu \in E(QR_7)$
$u - v \equiv 1, 2 \text{ or } 4 \pmod 7$

Some other results on χ_o (8)



Open Problem D.

Determine the oriented chromatic number of the family of 2-dimensional grids.

Conjecture. 7 (proven ≤ 11) [Fertin, Raspaud, Roychowdhury, 2003]

QR_7 is not universal [Szepietowski, Targan, 2004]

Open Problem E.

Determine the oriented chromatic number of the family of hypercubes.

Oriented arc colourings (1)

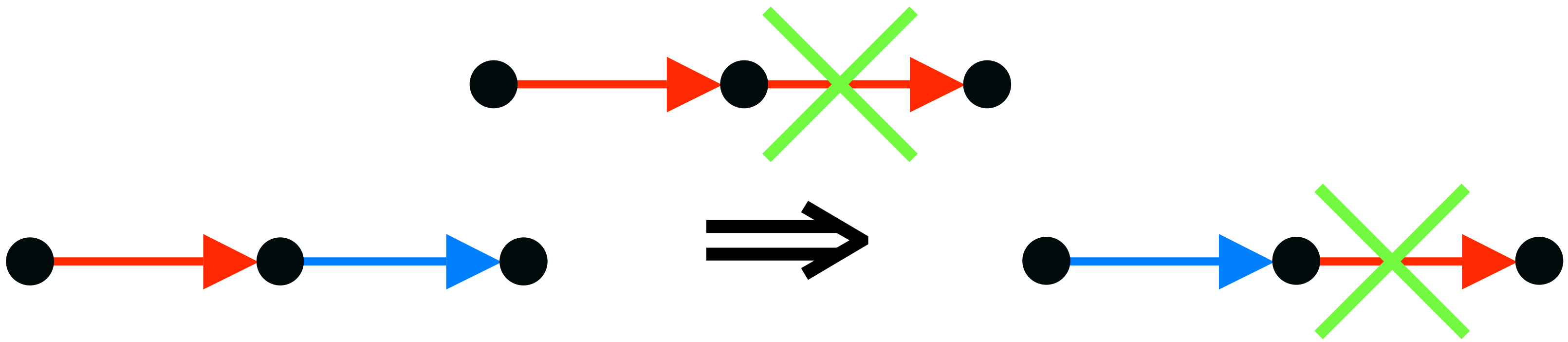
Oriented Arc-Colourings of oriented graphs

An **oriented k-arc-colouring** of an oriented graph G is a mapping $c : E(G) \rightarrow \{1,2,\ldots,k\}$ such that:

(1) $uv, vw \in E(G) \Rightarrow c(uv) \neq c(vw)$

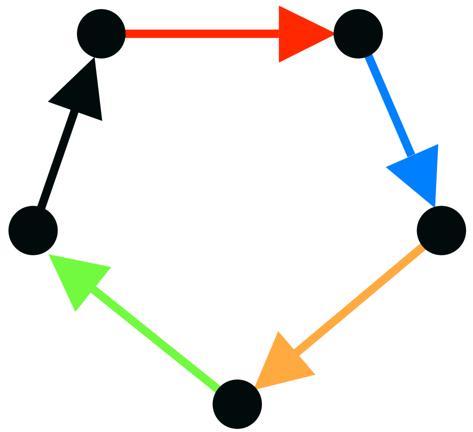
(2) $uv, vw, xy, yz \in E(G), c(uv) = c(yz)$

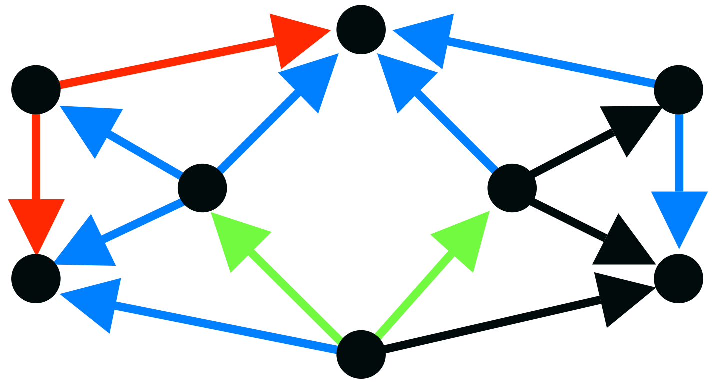
$\Rightarrow c(vw) \neq c(xy)$



Oriented arc colourings (2)

Examples.





Oriented chromatic index of oriented graphs

The **oriented chromatic index** $\chi'_o(G)$ of an oriented graph G is the smallest k such that G admits an oriented k-arc-colouring.

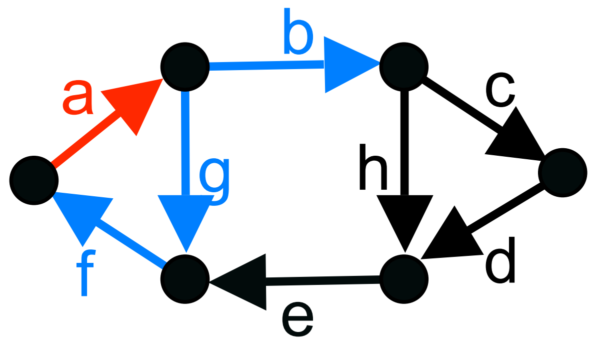
Oriented arc colourings (3)

Line digraph

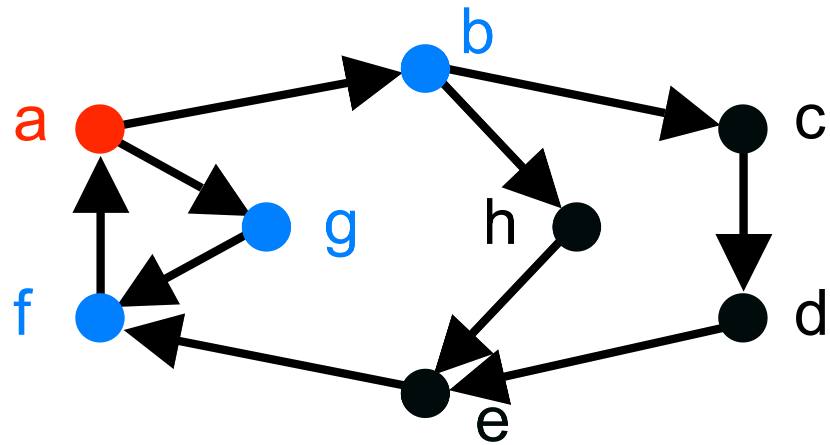
The **line-digraph LD(G)** of a digraph G is given by:

- V(LD(G)) = E(G),
- (e,f) ∈ E(LD(G)) iff f "*follows*" e

(that is e = uv and f = vw)



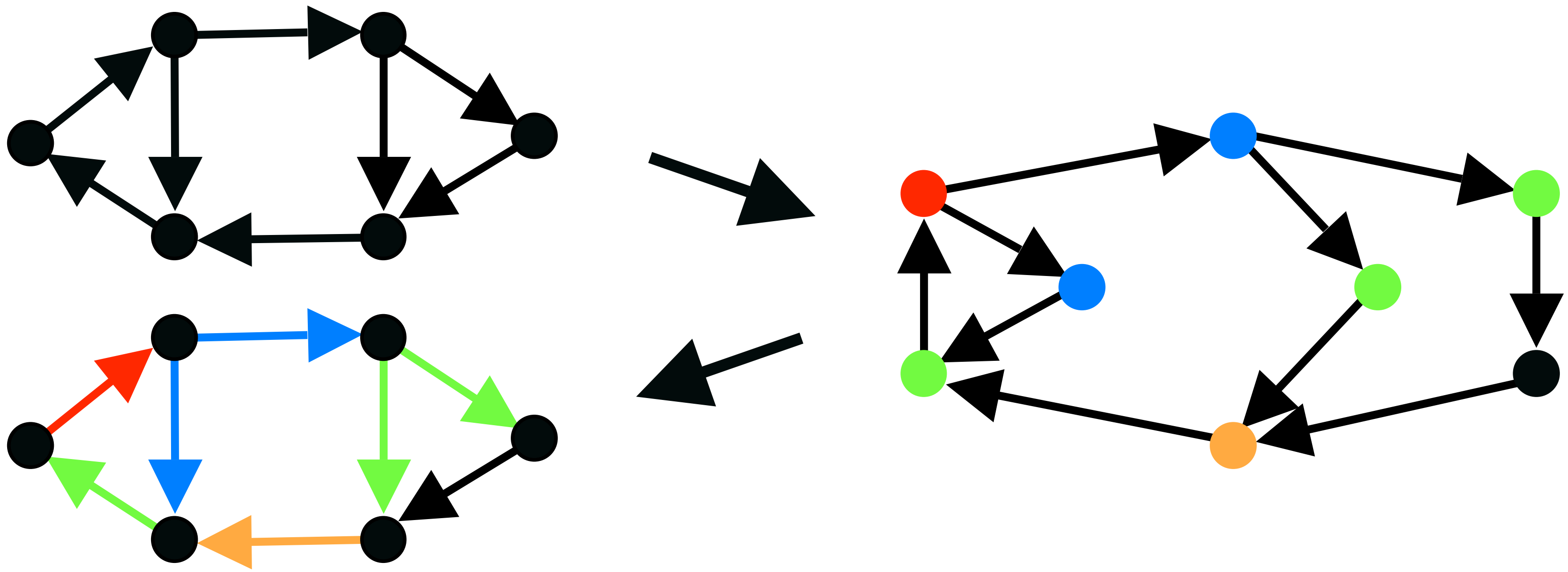
The graph G



The line-digraph LD(G) of G

Oriented arc colourings (4)

An oriented arc-colouring of G is nothing but an oriented *vertex-colouring* of the *line-digraph* of G.



Hence, $\chi'_o(G)$ is the minimum order of an oriented graph H such that $LD(G) \rightarrow H$.

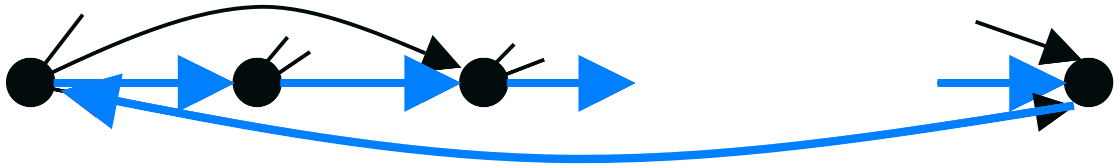
χ'_o versus χ_o (1)

Observation.

For every graph G, $\chi'_o(G) \leq \chi_o(G)$

vertex-colouring arc-colouring

And this bound is tight:



$\chi_o(T_n) = n$ $\chi'_o(T_n) = n$

χ'_o *versus* χ_o (2)

Question.

Is there a function Φ such that $\chi_o(G) \leq \Phi(\chi'_o(G))$?

The answer is yes (optimal Φ).

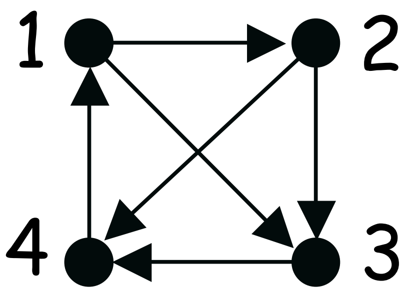
[Ochem, Pinlou, S., 2007]

χ'_o versus χ_o (3)

Let G be an oriented graph. We define the mapping μ_G as follows:

$$\mu_G : 2^{|V(G)|} \rightarrow 2^{|V(G)|}$$
$$s \rightarrow s \cup \{ \Gamma^+(u) \,;\, u \in s \}$$

Example.



$\emptyset \rightarrow \emptyset$

$\{1,2\}, \{1\} \rightarrow \{1,2,3\}$

$\{2,3\}, \{2\} \rightarrow \{2,3,4\}$

$\{3\} \rightarrow \{3,4\}$

$\{4\} \rightarrow \{1,4\}$

$\{3,4\} \rightarrow \{1,3,4\}$

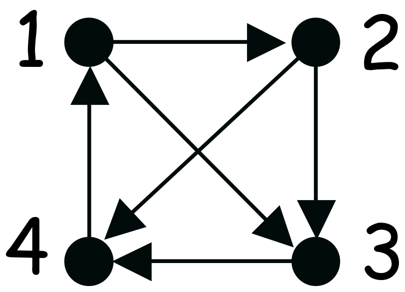
$\{1,3\}, \{1,4\}, \{2,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\} \rightarrow \{1,2,3,4\}$

χ'_o versus χ_o (4)

Now, let **Q(G)** be defined as:

$$Q(G) = \{ \mu_G(s) \; ; \; s \in 2^{|V(G)|} \}$$

Example.



$\varnothing \to \varnothing$

$\{1,2\}, \{1\} \to \{1,2,3\}$

$\{2,3\}, \{2\} \to \{2,3,4\}$

$\{3\} \to \{3,4\}$

$\{4\} \to \{1,4\}$

$\{3,4\} \to \{1,3,4\}$

$\{1,3\}, \{1,4\}, \{2,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\} \to \{1,2,3,4\}$

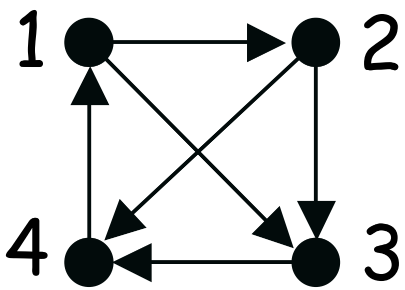
$$Q(T) = \{ \varnothing, \{1,4\}, \{3,4\}, \{1,2,3\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\} \}$$

χ'_o versus χ_o (5)

For each subset s in **Q(G)** define:

$$s^+ = \{ u \in s \,;\, \Gamma^+(u) \subseteq s \}$$

Example.



$\{1,2,3\}^+ = \{1\}$

$\{2,3,4\}^+ = \{2,3\}$

$\varnothing^+ = \varnothing$, $\{1,2,3,4\}^+ = \{1,2,3,4\}$

Q(T) = $\{ \varnothing, \{1,4\}, \{3,4\}, \{1,2,3\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\} \}$

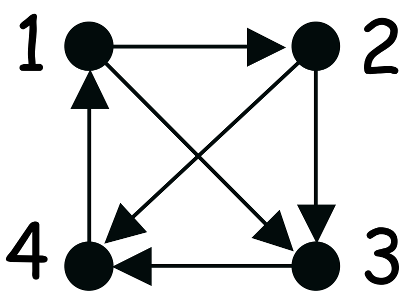
χ'_o versus χ_o (6)

Lemma.

Let G be an oriented graph with $\chi'_o(G) = k$ and T_k be any oriented graph s.t. $LD(G) \to T_k$.

Then, $\chi_o(G) \leq |Q(T_k)|$.

Example.



$c : V(G) \to Q(T_k)$

$u \to \mu_{Tk}(c'(\Gamma^+(u)))$

Hence, $LD(G) \to T \Rightarrow \chi_o(G) \leq 7$

Q(T) = { ∅, {1,4}, {3,4}, {1,2,3}, {1,3,4}, {2,3,4}, {1,2,3,4} }

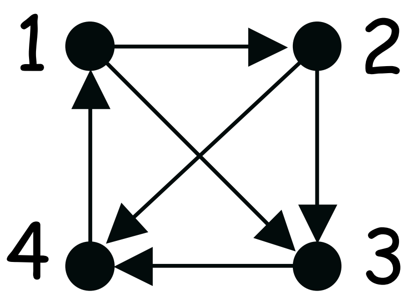
χ'_o versus χ_o (7)

Lemma.

For every tournament T_k of order k, there exists an oriented graph G with:

$$LD(G) \rightarrow T_k \quad \text{and} \quad \chi_o(G) = |Q(T_k)|$$

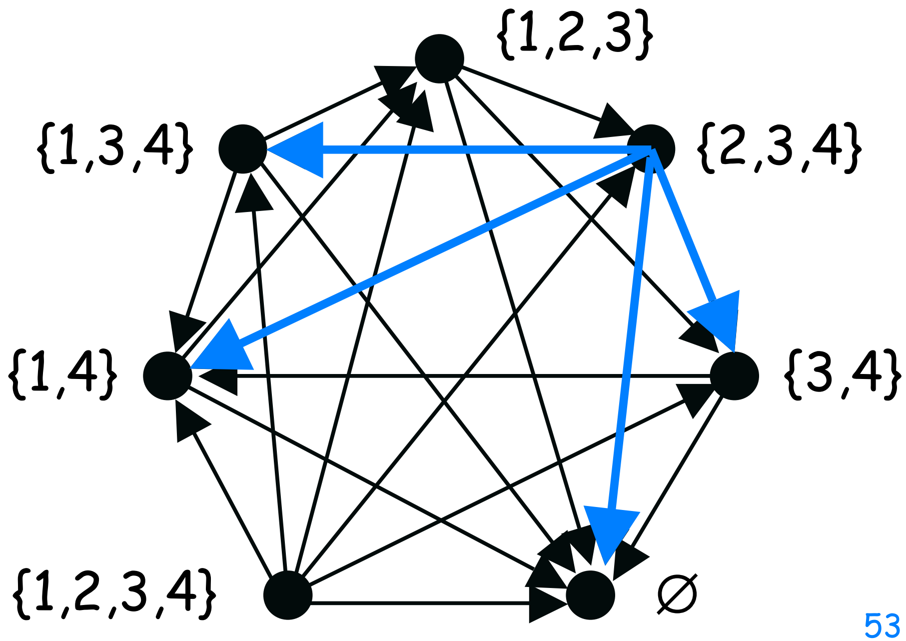
Example.



$V(G) = Q(T_k)$

$s \rightarrow s'$ iff $\exists u,\ u \in s^+,\ u \notin s'$

$\{2,3,4\}^+ = \{2,3\}$



χ'_o versus χ_o (8)

Finally, let

$\Phi(k) = \max \{ |Q(T_k)| ; T_k$ tournament of order $k \}$

Theorem. If G is an oriented graph with $\chi'_o(G) = k$ then $\chi_o(G) \leq \Phi(k)$.

Moreover, this bound is tight.

Values of Φ.

k	0	1	2	3	4	5	6	7	8	9	$k \geq 10$
$\Phi(k)$	1	2	3	5	7	12	15	25	31	51	$\alpha 2^{k/2} - 1 \leq \Phi(k) \leq (\lfloor k/2 \rfloor + 2).2^{\lfloor (k-1)/2 \rfloor}$

$\alpha = 2$ if k is even, $\alpha = 13/(4.2^{1/2})$ if k is odd

χ'_o versus acyclic colourings

Oriented arc-colouring and acyclic colouring

Theorem [Ochem, Pinlou, S., 2007]

If G admits an acyclic k-vertex-colouring, then

$$\chi'_o(G) \leq 2k(k-1) - \lfloor k/2 \rfloor$$

($k.2^{k-1}$ for χ_o)

(Idea of) proof. We have **k(k-1)/2** bicoloured forests, each of one being 4-arc-colourable. Among them, we select **⌊k/2⌋** "independent" forests which can be 3-arc-coloured.

χ'_o *versus* χ_o: outerplanar graphs

Outerplanar graphs

girth	χ_o	χ'_o
3	7	7
4	6	6
$5 \leq g \leq 9$	5	5
$g \geq 10$	5	4

χ'_o versus χ_o : planar graphs (1)

girth	χ_o	χ'_o
3	17 - 80	10 - 38
4	11 - 47	6 - 38
5	6 - 16	5 - 16
6	6 - 11	5 - 11
$7 \leq g \leq 10$	5 - 7	5 - 7
11	5 - 6	5 - 6
$12 \leq g \leq 15$	5	5
$16 \leq g \leq 45$	5	4 - 5
$g \geq 46$	5	4

χ'_o versus χ_o : planar graphs (2)

Open Problem F.

Determine the oriented chromatic index of the family of planar graphs.

Known to lie between 10 and 38…

Open Problem G.

Determine the smallest k for which every planar graph with girth at least k has oriented chromatic index 4.

χ'_o *versus* χ_o: graphs with bounded degree

Graphs with bounded degree

Theorem [Ochem, Pinlou, S., 2006]

If G has maximum degree Δ, then

$$\chi'_o(G) \leq 2 \, . \, (\lfloor \Delta^2/2 \rfloor + \Delta)$$

($2.\,\Delta^2.2^\Delta$ for χ_o)

For every Δ, there exists oriented graphs with oriented chromatic index 2Δ – 1.

Thank you for your attention!

Open Problem A. *Determine the oriented chromatic number of the family of connected graphs with degree at most 3.*

Open Problem B. *Determine the oriented chromatic number of the family of planar graphs.*

Open Problem C. *Determine the oriented chromatic number of the family of triangle-free planar graphs.*

Open Problem D. *Determine the oriented chromatic number of the family of 2-dimensional grids.*

Open Problem E. *Determine the oriented chromatic number of the family of hypercubes.*

Open Problem F. *Determine the oriented chromatic index of the family of planar graphs.*

Open Problem G. *Determine the smallest k for which every planar graph with girth at least k has oriented chromatic index 4.*

Complexity

k-O-COL problem

Instance: an oriented graph G

Question: is $\chi_o(G) \leq k$?

This problem is polynomial for $k \leq 3$, NP-complete for $k \geq 4$.

[Klostermeyer, MacGillivray, 2004]

It is still NP-complete for bipartite planar circuit-free graphs.

[Culus, Demange, 2007]