Valid Inequalities
for
Mixed Integer Linear Programs

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Brief history

First Algorithms | Polynomial Algorithms

Solving systems of linear equations

- Babylonians 1700BC
- Gauss 1801
- Edmonds 1967

Solving systems of linear inequalities

- Fourier 1822
- Dantzig 1951
- Khachyan 1979
- Karmarkar 1984

Solving systems of linear inequalities in integers

- Gomory 1958
- Lenstra 1983

Improvements in MILP software in the last 15 years

Based on Bixby, Gu, Rothberg, Wunderling 2004
and Laundy, Perregaard, Tavares, Tipi, Vazacopoulos 2007

Instances that would have required years of computing time 15 years ago can be solved in seconds today.

- LP Algorithms — 1000 times faster
- MILP Algorithms — 1000 times faster
- Computers — 1000 times faster

- Overall speedup — 1 000 000 000

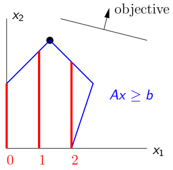
Sources of improvement for MILP :

- Preprocessors Factor 2
- Heuristics Factor 1,5
- Cutting Planes Factor 300

Mixed Integer Linear Programming

$$\min \quad cx$$
$$x \in S$$

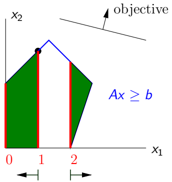
where $S := \{x \in \mathbb{Z}_+^p \times \mathbb{R}_+^{n-p} : \ Ax \geq b\}$



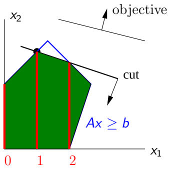
Linear Relaxation

$$\min \quad cx$$
$$x \in P$$

where $P := \{x \in \mathbb{R}_+^n : \ Ax \geq b\}$



Branch-and-bound



Cutting Planes

Polyhedral Theory

$P := \{x \in \mathbb{R}^n_+ : \; Ax \geq b\}$ Polyhedron

$S := P \cap (\mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+)$ Mixed Integer Linear Set

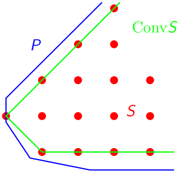
$$\text{Conv } S := \{x \in \mathbb{R}^n : \; \exists x^1, \ldots, x^k \in S, \; \lambda \geq 0, \; \sum \lambda_i = 1 \; \text{ such that } x = \lambda_1 x^1 + \ldots + \lambda_k x^k\}$$

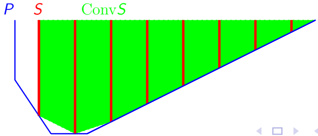
THEOREM Meyer 1974

If A, b have rational entries, then Conv S is a polyhedron.

Proof Using a theorem of Minkowski, Weyl :

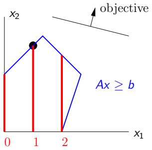
P is a polyhedron if and only if $P = Q + C$ where Q is a polytope and C is a polyhedral cone.





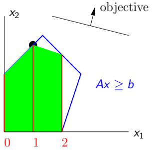
Thus

$$\min \quad cx$$
$$x \in S$$



can be rewritten as the LP

$$\min \quad cx$$
$$x \in \text{Conv } S$$



We are interested in the constructive aspects of Conv S.

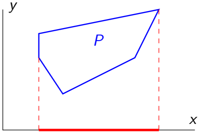
REMARK The number of constraints of Conv S can be exponential in the size of $Ax \geq b$, BUT
1) sometimes a partial representation of Conv S suffices (Example : Dantzig, Fulkerson, Johnson 1954) ;
2) Conv S can sometimes be obtained as the projection of a polyhedron with a polynomial number of variables and constraints.

Projections

Let $P := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^k : \; Ax + Gy \geq b\}$

DEFINITION

$\text{Proj}_x(P) := \{x \in \mathbb{R}^n : \; \exists y \in \mathbb{R}^k \text{ such that } Ax + Gy \geq b\}$



THEOREM

$\text{Proj}_x(P) = \{x \in \mathbb{R}^n : \; vAx \geq vb \text{ for all } v \in Q\}$
where $Q := \{v \in \mathbb{R}^m : \; vG = 0, v \geq 0\}$.

PROOF

Let $x \in \mathbb{R}^n$. Farkas's lemma implies that
$Gy \geq b - Ax$ has a solution y if and only if
$v(b - Ax) \leq 0$ for all $v \geq 0$ such that $vG = 0$. ■

Lift-and-Project

Sherali-Adams 1990
Lovász-Schrijver 1991
Balas-Ceria-Cornuéjols 1993

Let $S := \{x \in \{0,1\}^p \times \mathbb{R}_+^{n-p} : \; Ax \geq b\}$
$P := \{x \in \mathbb{R}_+^n : \; Ax \geq b\}$

LIFT-AND-PROJECT PROCEDURE

STEP 0 Choose an index $j \in \{1, \dots, p\}$.

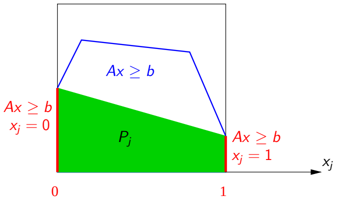
STEP 1 Generate the nonlinear system
$$x_j(Ax - b) \geq 0$$
$$(1 - x_j)(Ax - b) \geq 0$$

STEP 2 Linearize the system by substituting $x_i x_j$ by y_i for $i \neq j$, and x_j^2 by x_j. Denote this polyhedron by M_j.

STEP 3 Project M_j on the x-space. Denote this polyhedron by P_j.

PROPOSITION $\text{Conv}(S) \subseteq P_j \subseteq P$.

THEOREM $P_j = \text{Conv}\{\begin{pmatrix} Ax \geq b \\ x_j = 0 \end{pmatrix} \cup \begin{pmatrix} Ax \geq b \\ x_j = 1 \end{pmatrix}\}$.



THEOREM Balas 1979 $\text{Conv}(P) = P_p(\ldots P_2(P_1)\ldots)$.

LIFT-AND-PROJECT CUT

Given a fractional solution $\bar{x}$ of the linear relaxation $Ax \geq b$, find a cutting plane $\alpha x \geq \beta$ (namely $\alpha\bar{x} < \beta$) that is valid for P_j (and therefore for S).

DEEPEST CUT
$$\begin{array}{c} \max\ \beta - \alpha\bar{x} \\ \alpha x \geq \beta \text{ valid for } P_j \end{array}$$

CUT GENERATION LINEAR PROGRAM

$$M_j := \{x \in \mathbb{R}^n_+, y \in \mathbb{R}^n_+ : \\ Ay - bx_j \geq 0, \\ Ax + bx_j - Ay \geq b, \\ y_j = x_j\}$$

The first two constraints come from the linearization in STEP 1.

In fact, one does not use the variable y_j

$$M_j := \{x \in \mathbb{R}^n_+, y \in \mathbb{R}^{n-1}_+ : \\ B_j x + A_j y \geq 0, \\ D_j x - A_j y \geq b\}$$

To project onto the x-space, we use the cone

$$Q := \{u, v \geq 0 : \; uA_j - vA_j = 0\}$$

$$P_j = \{x \in \mathbb{R}^n_+ : \; (uB_j + vD_j)x \geq vb \text{ for all } (u, v) \in Q\}$$

DEEPEST CUT

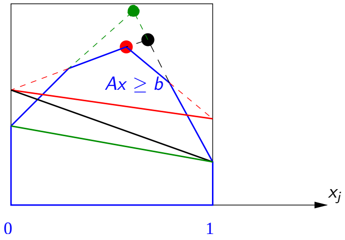
$$\max \; vb - (uB_j + vD_j)\bar{x} \\ uA_j - vA_j = 0 \\ u \geq 0, v \geq 0 \\ \sum u_i + \sum v_i = 1$$

SIZE OF THE CUT GENERATION LP

$$\begin{aligned} \max\ vb - (uB_j + vD_j)\bar{x} \\ uA_j - vA_j = 0 \\ \sum u_i + \sum v_i = 1 \\ u \geq 0, v \geq 0 \end{aligned}$$

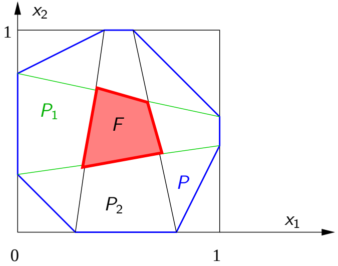
Number of variables : $2m$
Number of constraints : n + nonnegativity

Balas and Perregaard 2003 give a precise correspondance between the basic feasible solutions of the cut generation LP and the basic solutions of the LP $\begin{aligned} \min\ cx \\ Ax \geq b \end{aligned}$



LIFT-AND-PROJECT CLOSURE OF P

$$F := \bigcap_{j=1}^{p} P_j$$



REMARK Balas and Jeroslow 1980 show how to strengthen cutting planes by using the integrality of the other integer variables (lift-and-project only considers the integrality of one variable x_j at a time).

Experiments of Bonami and Minoux 2005 on MIPLIB 3 instances give the amount of duality gap $= \min_{x \in S} cx - \min_{x \in P} cx$ closed by strengthening P :

Lift-and-project closure	Lift-and-project + strengthening
37 %	45 %

Mixed Integer Cuts   Gomory 1963

Consider a single constraint : $S := \{x \in \mathbb{Z}_+^p \times \mathbb{R}_+^{n-p} : \sum_{j=1}^n a_j x_j = b\}$.

Let $b = \lfloor b \rfloor + f_0$ where $0 < f_0 < 1$,
and $a_j = \lfloor a_j \rfloor + f_j$ where $0 \le f_j < 1$.

THEOREM

$$\sum_{\substack{j \le p:\\ f_j \le f_0}} \frac{f_j}{f_0} x_j + \sum_{\substack{j \le p:\\ f_j > f_0}} \frac{1-f_j}{1-f_0} x_j + \sum_{\substack{j \ge p+1:\\ a_j > 0}} \frac{a_j}{f_0} x_j - \sum_{\substack{j \ge p+1:\\ a_j < 0}} \frac{a_j}{1-f_0} x_j \ge 1$$

is a valid inequality for S.

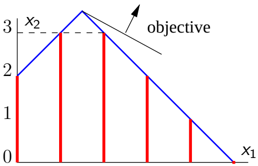
APPLICATION

$$\begin{aligned} \max z &= x_1 + 2x_2 \\ -x_1 + x_2 &\le 2 \\ x_1 + x_2 &\le 5 \\ x_1 &\in \mathbb{Z}_+ \\ x_2 &\in \mathbb{R}_+ \end{aligned}$$

$$\begin{aligned} z + 0.5s_1 + 1.5s_2 &= 8.5 \\ x_1 - 0.5s_1 + 0.5s_2 &= 1.5 \\ x_2 + 0.5s_1 + 0.5s_2 &= 3.5 \end{aligned}$$

Cut $s_1 + s_2 \ge 1$.

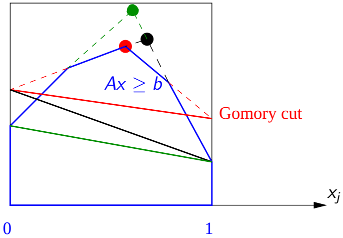
Or $x_2 \le 3$.



HOW GOOD ARE GOMORY CUTS
GENERATED FROM THE OPTIMAL BASIS ?

Bonami and Minoux 2005 MIPLIB 3

Gomory cuts (optimal basis)	Lift-and-project closure	Lift-and-project + strengthening
24 %	37 %	45 %



CAN ONE IMPROVE GOMORY CUTS BY GENERATING THEM FROM OTHER BASES
(Balas, Perregaard 2003)
(Balas, Bonami 2007)
OR OTHER EQUATIONS ?
(Andersen, Cornuéjols, Li 2005)

Reduce-and-split cuts
Andersen, Cornuéjols, Li 2005

Perform linear combinaisons of the constraints $\sum_{j=1}^{n} a_j x_j = b$ in order to reduce the coefficients of the continuous variables, and generate the corresponding Gomory cuts.

Why ?

Remember the Gomory cut formula :

$$\sum_{\substack{j \le p: \\ f_j \le f_0}} \frac{f_j}{f_0} x_j + \sum_{\substack{j \le p: \\ f_j > f_0}} \frac{1 - f_j}{1 - f_0} x_j + \sum_{\substack{j \ge p+1: \\ a_j > 0}} \frac{a_j}{f_0} x_j - \sum_{\substack{j \ge p+1: \\ a_j < 0}} \frac{a_j}{1 - f_0} x_j \ge 1$$

ALGORITHM Consider the lines L of the optimal simplex tableau for the basic variables x_i such that $i \le p$.
For every line $\ell \in L$, reduce the norm $\|(a_{p+1}^{\ell}, \ldots, a_n^{\ell})\|$ by performing integer combinaisons of the other lines of L.

COMPUTATIONAL RESULTS ON MIPLIB INSTANCES

- Reduce-and-split cuts are often very different from Gomory cuts generated directly from the optimal basis.
- Their quality is typically at least as good.
- In some cases, their quality is much better :

Name	20 Times Gomory	20 Times R&S	Nodes Gomory	Nodes R&S
flugpl	14 %	100 %	184	0
gesa2	46 %	97 %	743	116
gesa2o	92 %	98 %	9145	75
mod008	47 %	88 %	1409	82
pp08a	83 %	92 %	7467	745
rgn	15 %	100 %	874	0
vpm1	44 %	98 %	7132	1
vpm2	41 %	61 %	38946	4254

GOMORY CLOSURE

$Ax \geq b \quad x \in \mathbb{Z}_+^p \times \mathbb{R}_+^{n-p}$

- Every valid inequality for $P := \{x \geq 0 : \ Ax \geq b\}$ $(\neq \emptyset)$ is of the form $uAx + vx \geq ub - t$, where $u, v, t \geq 0$.
- Subtract a nonnegative surplus variable $\alpha x - s = \beta$.
- Generate a Gomory inequality.
- Eliminate $s = \alpha x - \beta$ to get the inequality in the x-space.
- The convex set obtained by intersecting all these inequalities with P is called the Gomory closure.

THEOREM Cook, Kannan, Schrijver 1990
The Gomory closure is a polyhedron.

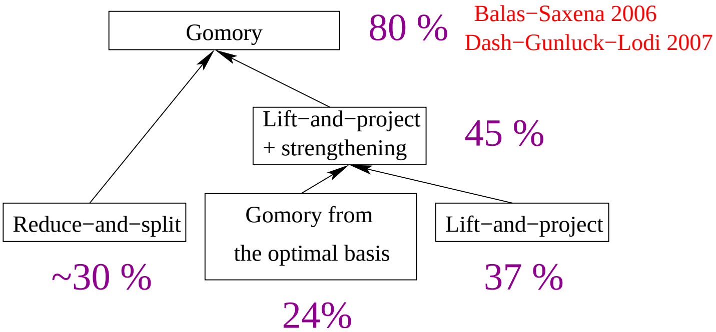
THEOREM Caprara, Letchford 2002 et Cornuéjols, Li 2002
It is NP-hard to optimize a linear function over the Gomory closure.

Nevertheless,
Balas and Saxena 2006 and Dash, Günlück and Lodi 2007
were able to optimize over the Gomory closure by solving a sequence of parametric MILPs.

DUALITY GAP CLOSED BY DIFFERENT CUTS
MIPLIB 3

Gomory cuts (optimal basis)	Reduce-and-split (optimal basis)	Gomory closure
24 %	30 %	80 %

Duality gap closed by different types of cutting planes
MIPLIB 3 instances



Mixed integer rounding

Nemhauser-Wolsey 1990
Wolsey 1999

$S := \{x \in \mathbb{Z}_+^p \times \mathbb{R}_+^{n-p} : \quad Ax \leq b\}$

AGGREGATION $\quad uAx \leq ub, \quad u \geq 0,$ or $\quad \bar{a}x \leq \bar{b}.$

THEOREM

$$\sum_{i \leq p} \left(\lfloor \bar{a}_i \rfloor + \frac{(f_i - f_0)^+}{1 - f_0} \right) x_i + \sum_{\substack{i \geq p+1 : \\ \bar{a}_i < 0}} \frac{\bar{a}_i}{1 - f_0} x_i \leq \lfloor \bar{b} \rfloor$$

is a valid inequality for S. It is called MIR.

REMARK It is the same cut as the Gomory inequality obtained from $\bar{a}x + s = \bar{b}$ where $s \geq 0$.

REMARK Stronger inequalities can be obtained by first adding slack variables to $Ax \leq b$, and then aggregating the equalities $Ax + s = b$ using multipliers u_i (positive or negative).
Bonami and Cornuéjols 2007, and Dash, Günlük and Lodi 2007

Mixed integer rounding continued

$$S := \{x \in \mathbb{Z}_+^p \times \mathbb{R}_+^{n-p} : \quad Ax \leq b\}$$

AGGREGATION $\qquad uAx + us = ub.$

Define the MIR closure as the intersection of the MIR inequalities generated from all the possible multipliers u.

THEOREM The MIR closure is identical to the Gomory closure.

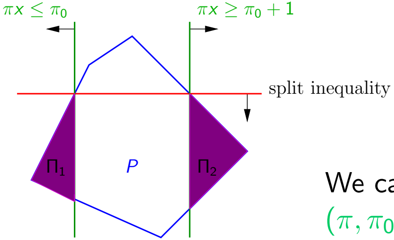
Marchand-Wolsey 2001

- Several classical inequalities for structured problems are MIR inequalities, like "flow cover" inequalities.
- An aggregation heuristic to generate MIR cuts closes around 23 % of the duality gap.

Split Inequalities Cook-Kannan-Schrijver 1990

$P := \{x \in \mathbb{R}^n : \ Ax \geq b\}$
$S := P \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$.

For $\pi \in \mathbb{Z}^n$ such that $\pi_{p+1} = \ldots = \pi_n = 0$ and $\pi_0 \in \mathbb{Z}$, define



$$\Pi_1 := P \cap \{x : \ \pi x \leq \pi_0\}$$

$$\Pi_2 := P \cap \{x : \ \pi x \geq \pi_0 + 1\}$$

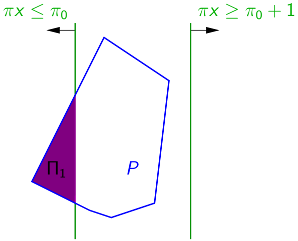
We call $cx \leq c_0$ a split inequality if there exists $(\pi, \pi_0) \in \mathbb{Z}^p \times \mathbb{Z}$ such that $cx \leq c_0$ is valid for $\Pi_1 \cup \Pi_2$.

The split closure is the intersection of all split inequalities.

THEOREM Nemhauser-Wolsey 1990, Cornuéjols-Li 2002
The split closure is identical to the Gomory closure.

Chvátal Inequalities Chvátal 1973

A Chvátal inequality is a split inequality where $\Pi_2 = \emptyset$.

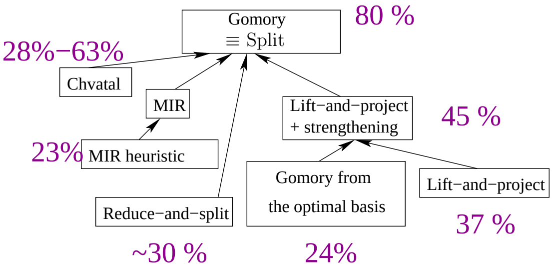


The Chvátal closure is the intersection of all these inequalities.

REMARK Chvátal defined this concept in 1973 in the context of pure integer programs.

The Chvátal closure reduces the duality gap by around 63 % on the pure integer MIPLIB 03 instances (Fischetti-Lodi 2006) and around 28 % on the mixed instances (Bonami-Cornuéjols-Dash-Fischetti-Lodi 2007).

Duality gap closed by different types of cutting planes
MIPLIB 3 instances



Paper available on http ://integer.tepper.cmu.edu/