Polyhedral approach: the p-median polytope

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I. Polyhedral approach

Given a finite set $E = \{e_1, e_2, \ldots, e_n\}$. Let $\mathcal{F}$ be a family of some particular subsets of E (feasible solutions), and let c be a cost function, a mapping

$$c : E \longrightarrow \mathbb{R}.$$

The problem

$$(P): \text{ minimize } \{c(F) \overset{\text{def}}{=} \sum_{e \in F} c(e) : F \in \mathcal{F}\},$$

is called a *combinatorial optimization problem*.

Examples: Traveling Salesman problem, Maximum-Weight Matching problem, Minimum Spanning Tree problem, ...

Integer Polyhedra

For every element $F \in \mathcal{F}$, associate a 0,1 vector $x^F \in \{0,1\}^E$, defined as follows:

$$x^F(e) = \begin{cases} 1 & \text{if } e \in F \\ 0 & \text{otherwise,} \end{cases}$$

x^F is called the *incidence vector of F*.
(P) may be rewritten as

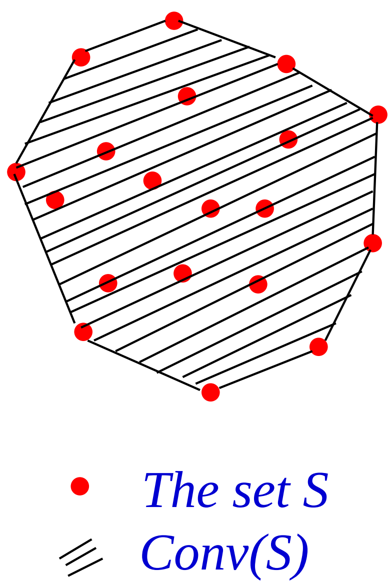
$$(P): \text{ minimize } \{c^T x^F : x^F \in S\},$$

where S is the set of incidence vectors of the elements of $\mathcal{F}$.

Convex hulls: The *convex hull* of a finite set S, denoted by conv(S), is the set of all points that are a convex combination of points in S.

Proposition 1. *Let $S \subseteq \mathbb{R}^n$ be a finite set and let $c \in \mathbb{R}^n$. Then*

$$\min\{c^T x : x \in S\} = \min\{c^T x : x \in \text{conv}(S)\}.$$



Polytopes:

- $P = \{x \in I\!\!R^n : Ax \leq b\}$ where A is an $m \times n$ matrix and $b \in I\!\!R^m$, is called a *polyhedron*.

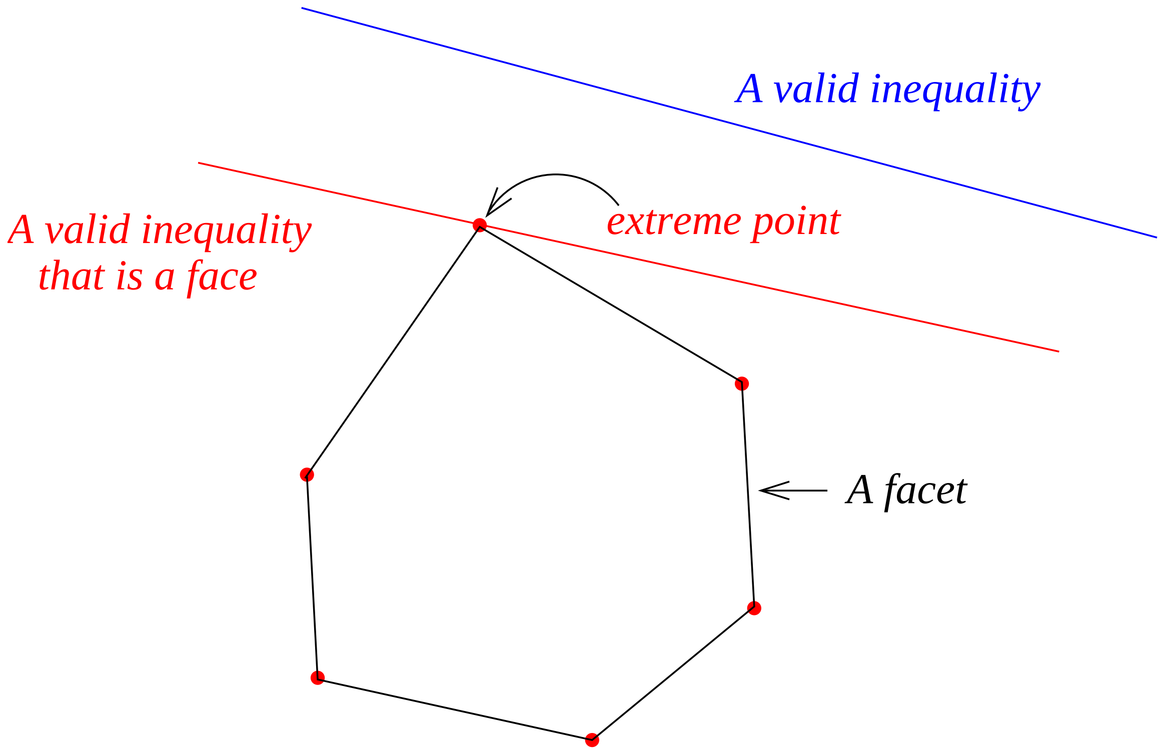
- A *polytope* is a bounded polyhedron.

- An inequality $\alpha^T x \leq \alpha_0$ is *valid* for a polytope P if $P \subseteq \{x : \alpha^T x \leq \alpha_0\}$.

- The *dimension* of a polytope P, $\dim(P)$, is equal to the maximum number of affinely independent points in P minus one.

- Let $\alpha^T x \leq \alpha_0$ be a valid inequality for the polytope P. $F = \{x \in P : \alpha^T x = \alpha_0\}$ is called a *face* of P. F is a *facet* of P if $\dim(F) = \dim(P) - 1$.

- $x \in P$ is an *extreme* point of P if x is a face of P of dimension 0.



Two useful characterizations of extreme points are:

Characterization 1. *$x \in P$ is an extreme point of P if and only if there do not exist $x^1, x^2 \in P$, $x^1 \neq x^2$, such that $x = \frac{1}{2}x^1 + \frac{1}{2}x^2$.*

Characterization 2. *$x \in P$ is an extreme point of P if x is the unique solution of a subsystem of inequalities defining P when replaced by equalities.*

Theorem 1. *A set P is a polytope if and only if there exists a finite set S such that P is the convex hull of S.*

By this theorem, the optimization over conv(S) is equivalent to the optimization over a polytope.

An example

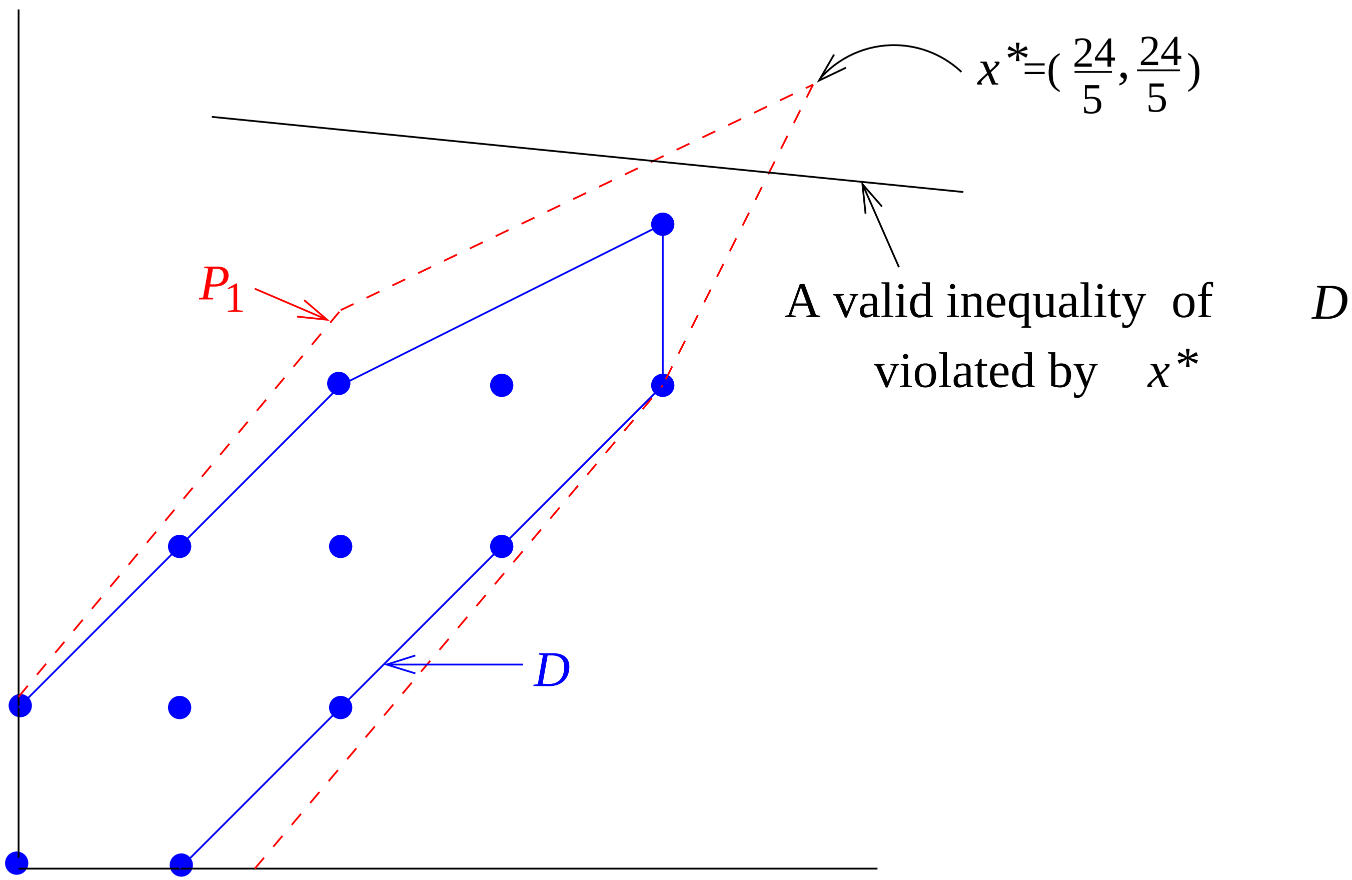
Consider the following combinatorial optimization problem (P) formulated as an integer linear program:

max $x_1 + x_2$
subject to

$$P_1 = \left\{ \begin{array}{rcl} -13x_1 + 28x_2 & \leq & 72 \\ -5x_1 + 4x_2 & \leq & 4 \\ 9x_1 - 4x_2 & \leq & 24 \\ 6x_1 - 5x_2 & \leq & 9 \\ x_1 & \geq & 0 \\ x_2 & \geq & 0 \end{array} \right. \qquad D = \left\{ \begin{array}{rcl} -x_1 + x_2 & \leq & 1 \\ -x_1 + 2x_2 & \leq & 4 \\ x_1 - x_2 & \leq & 1 \\ x_1 & \leq & 4 \\ x_1 & \geq & 0 \\ x_2 & \geq & 0 \end{array} \right.$$

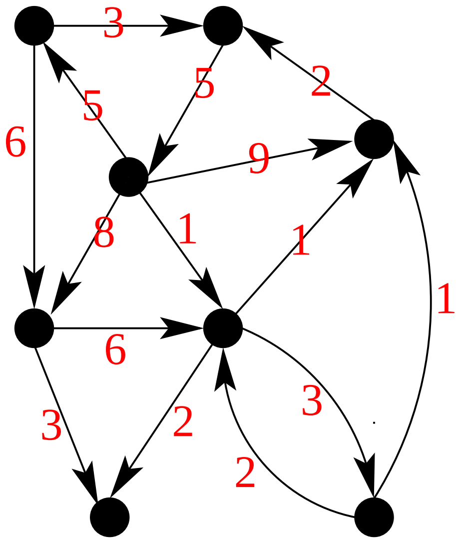
x_1 and x_2 integers

Theorem 1 says that the convex hull of the solutions of (P) is also a polytope; in our example this polytope is denoted D.

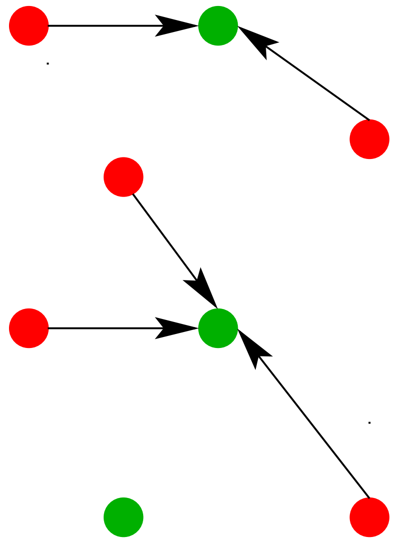

$x^* = (\frac{24}{5}, \frac{24}{5})$
P_1
A valid inequality of D violated by x^*
D

The p-median problem

Given a direct graph $G = (V, A)$ where each arc (u, v) is associated with a cost $c(u, v)$. The problem is to select p nodes, and assign to them the non-selected one such that the assignment cost is minimized.



p=3



Selected nodes

Non−Selected nodes

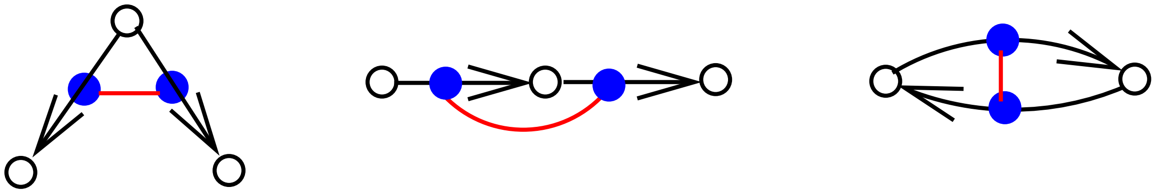
The costof this solution is :

2+3+1+6+2

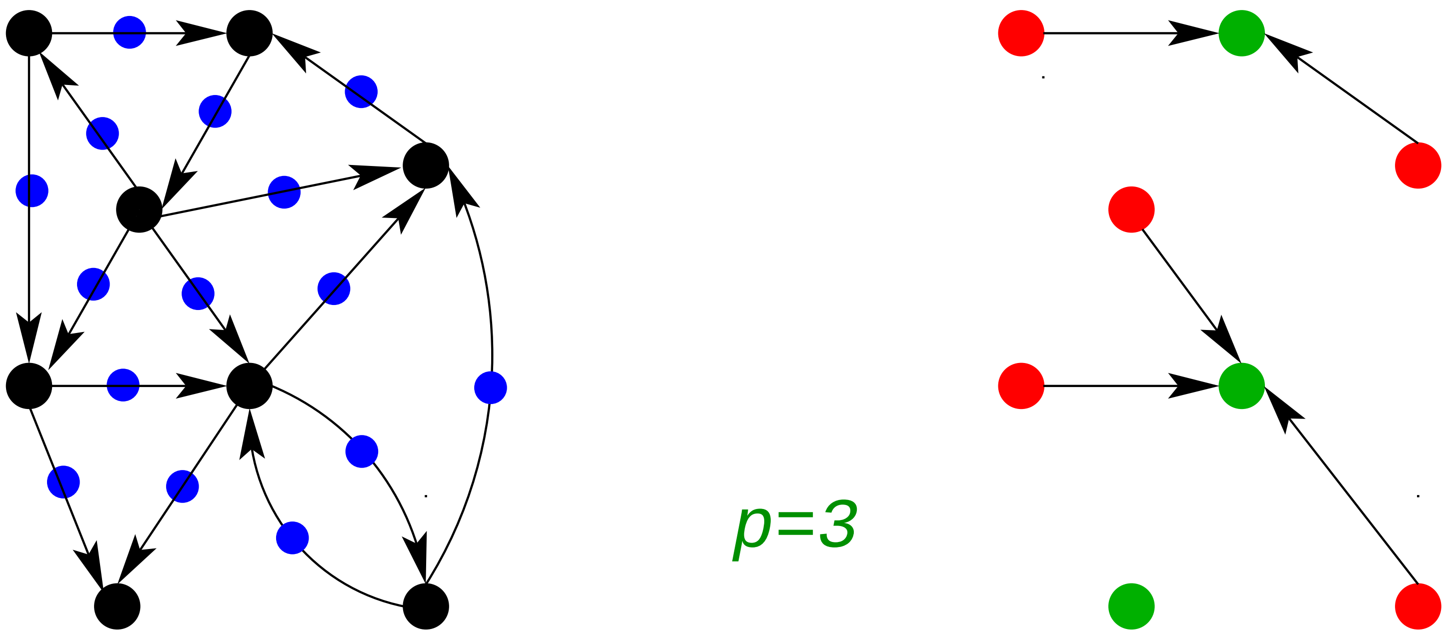
The relation with the stable set problem

An instance of the p-median problem : a directed graph $G = (V, A)$, a cost function c associatend with the arc set A and a fixed integer $p \leq |V|$.

- From $G = (V, A)$ define an undirected graph $I(G) = (A, E)$ called the *intersection graph of* G.
- The nodes of $I(G)$ are the arcs of G,
- The edges of $I(G)$ are defined as below:



- *The p-median problem reduces to find a stable set of size* $|V| - p$ *with minimum cost in* $I(G)$



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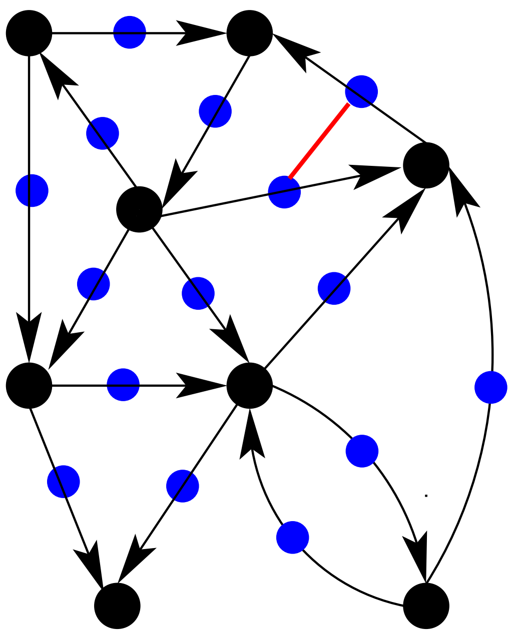

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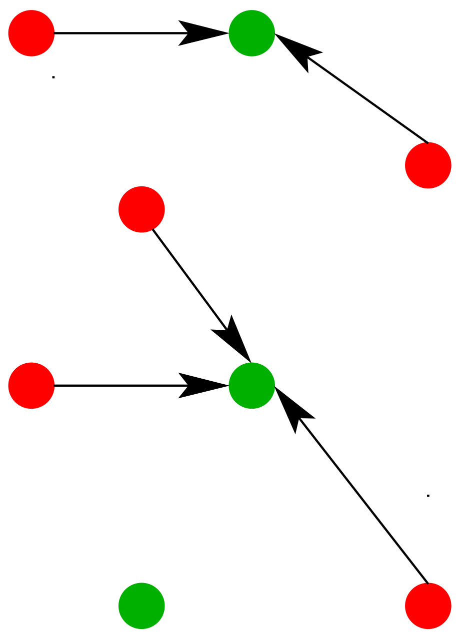
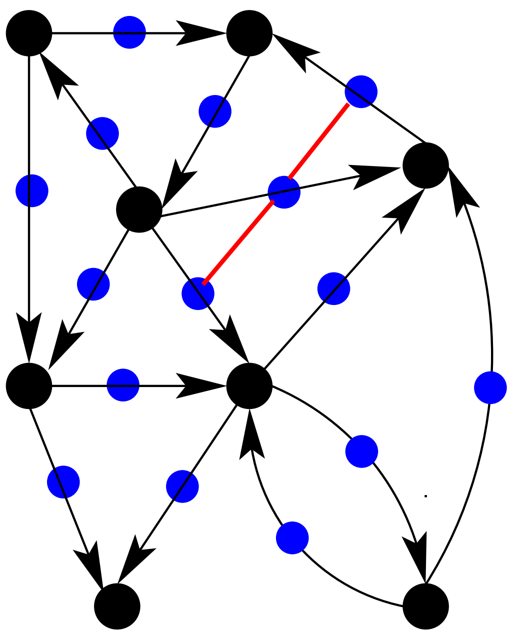


- *The p-median problem reduces to find a stable set of size $|V| - p$ with minimum cost in $I(G)$*



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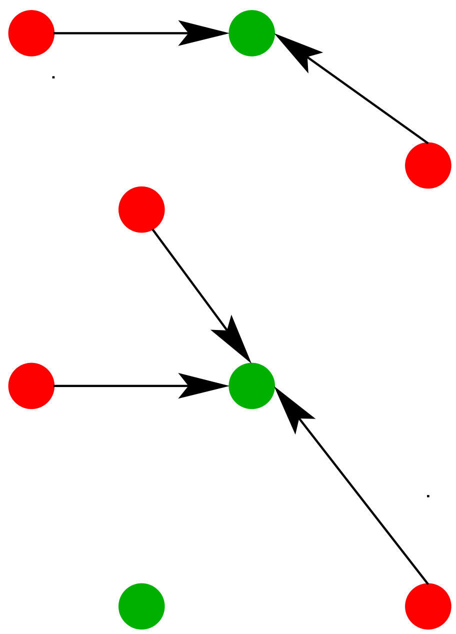

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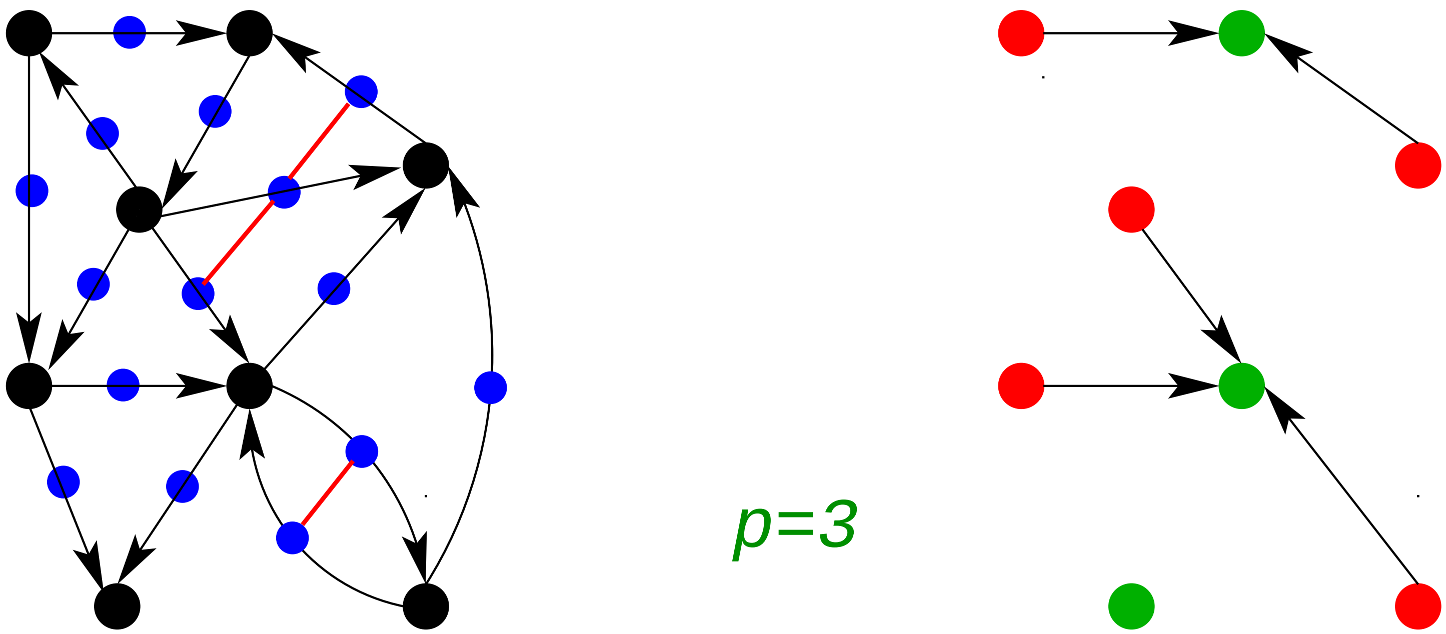

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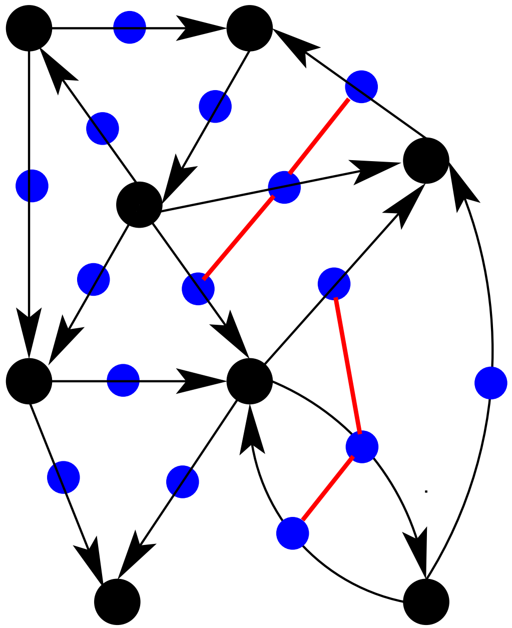

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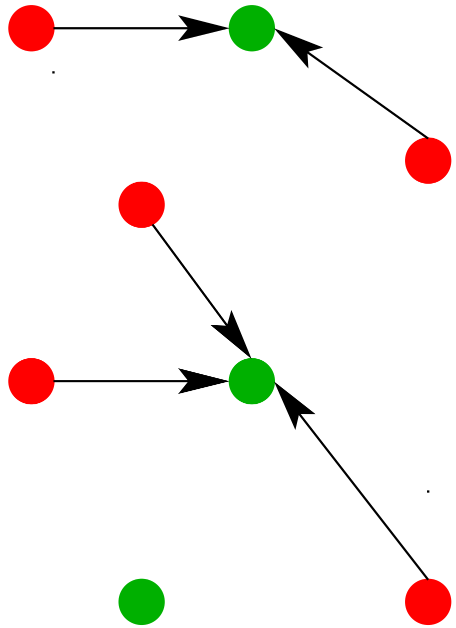

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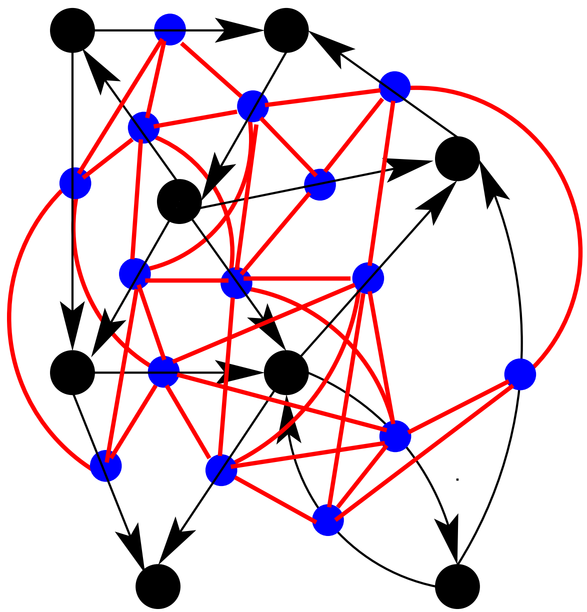

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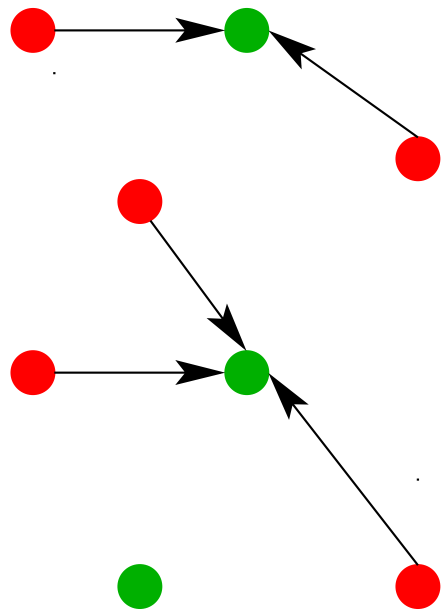

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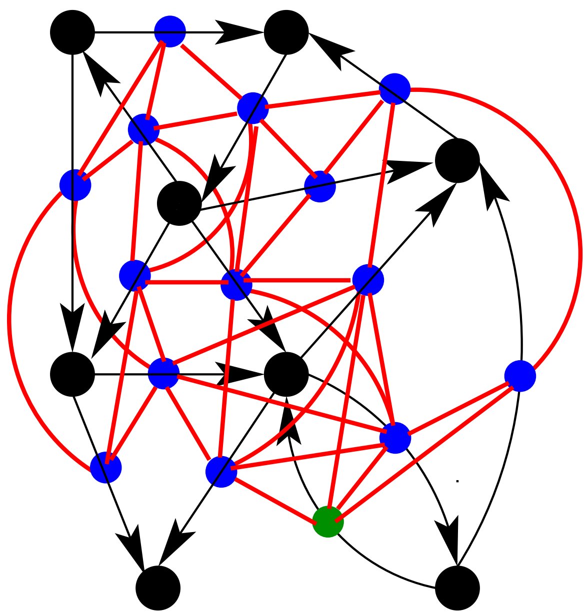

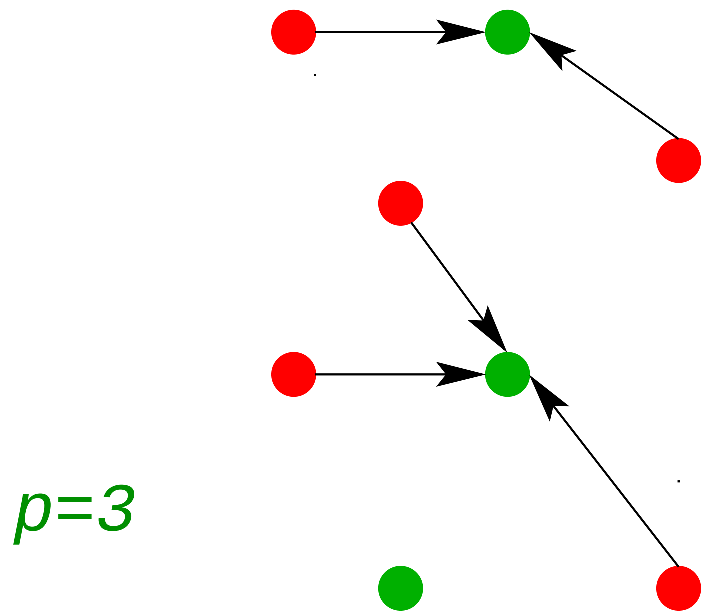

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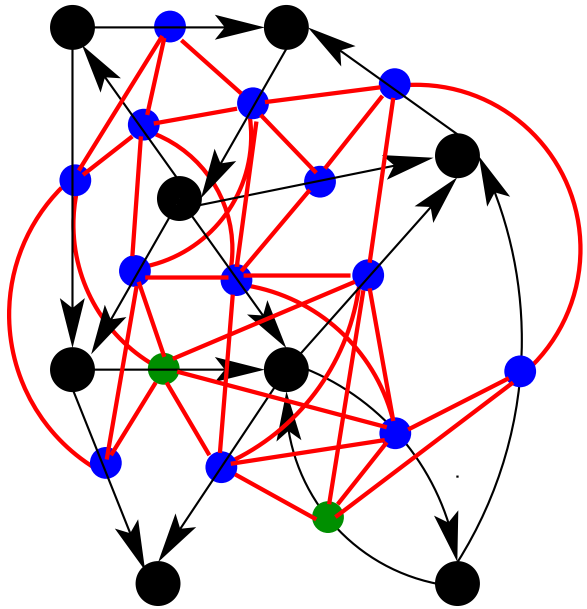

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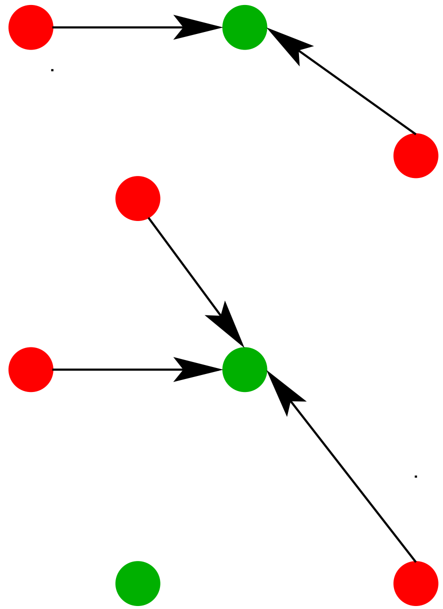

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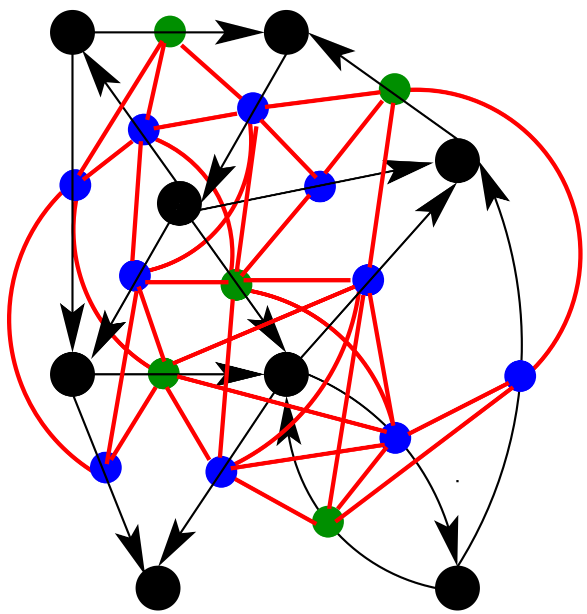

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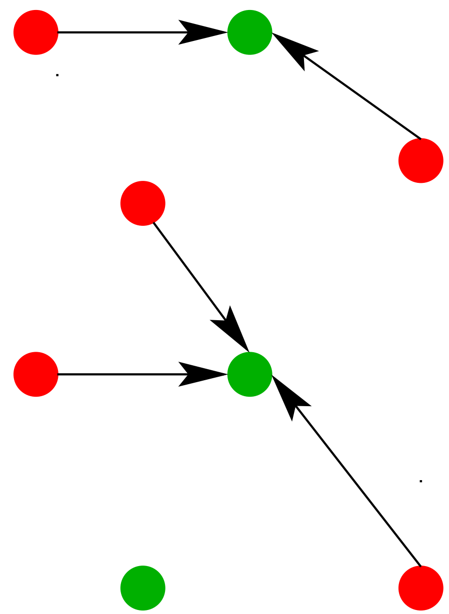

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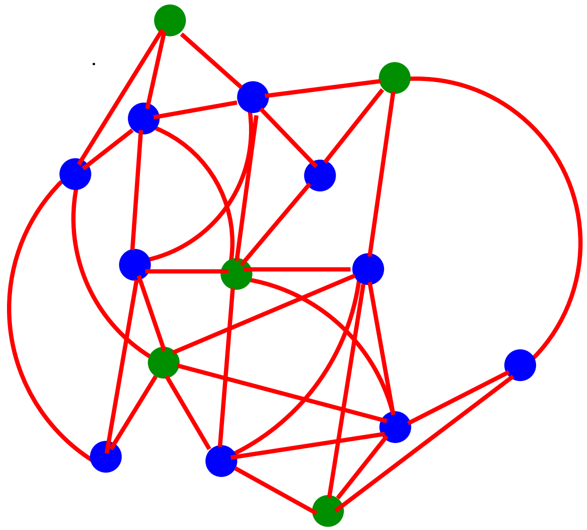

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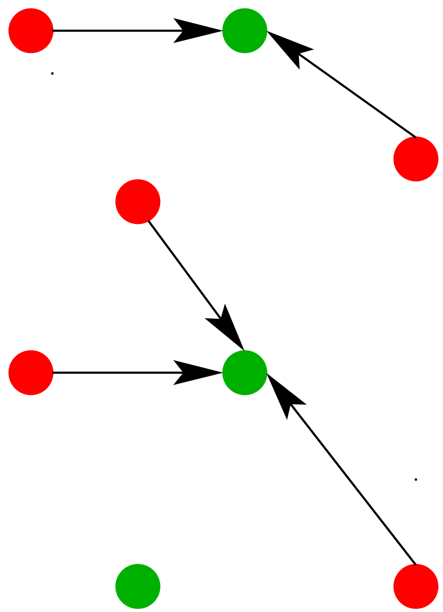


In green a stable set of size |V|-p=5

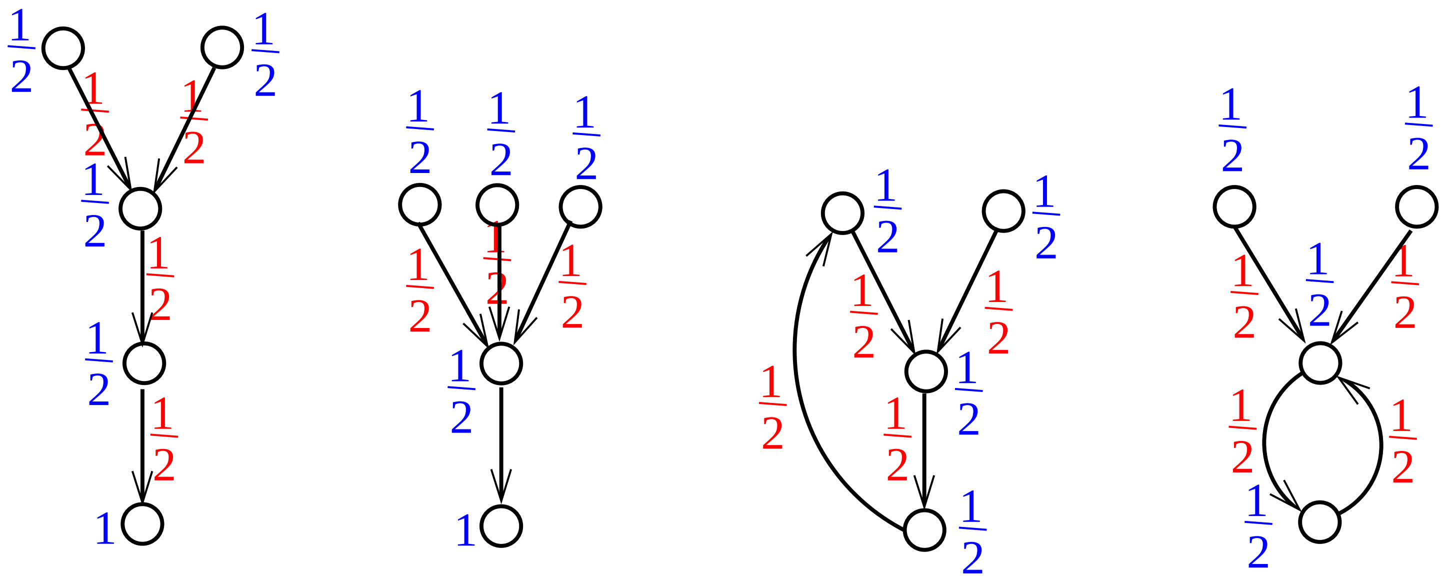
- The p-median problem may be formulated by the following linear integer program:

minimize $\sum_{(u,v)\in A} c(u,v)x(u,v)$

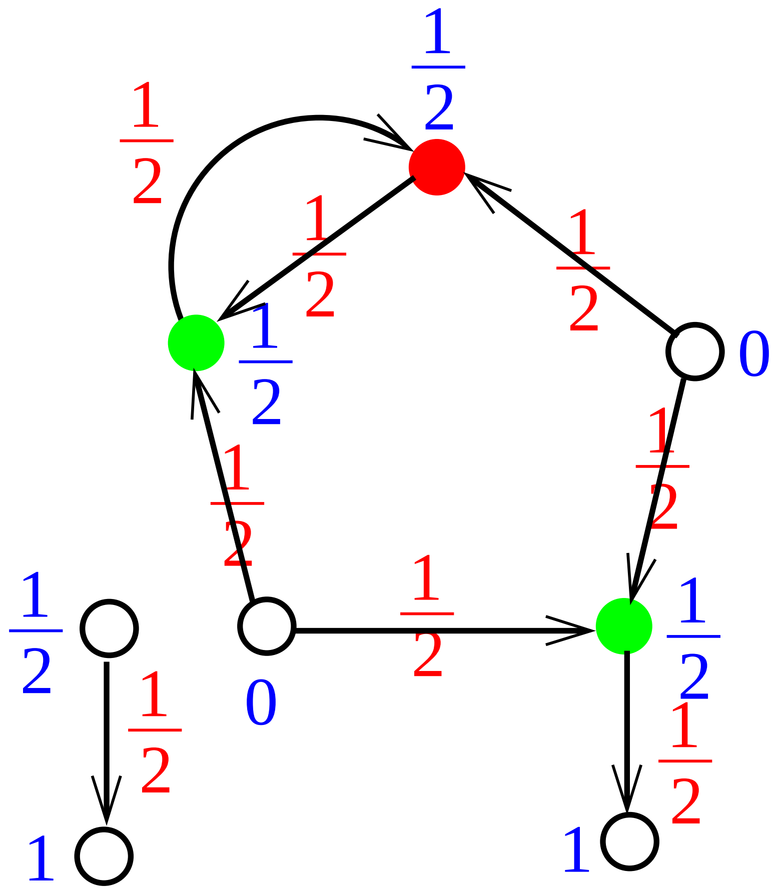
$$P_p(G) = \begin{cases} \sum_{v\in V} y(v) = p, & \\ \sum_{v:(u,v)\in A} x(u,v) = 1 - y(u) & \forall u \in V, \\ x(u,v) \le y(v) & \forall (u,v) \in A, \\ y(v) \le 1 & \forall v \in V, \\ x(u,v) \ge 0 & \forall (u,v) \in A, \end{cases}$$

x and y are integer.

- The following four figures give four fractional extreme points of $P_p(G)$.

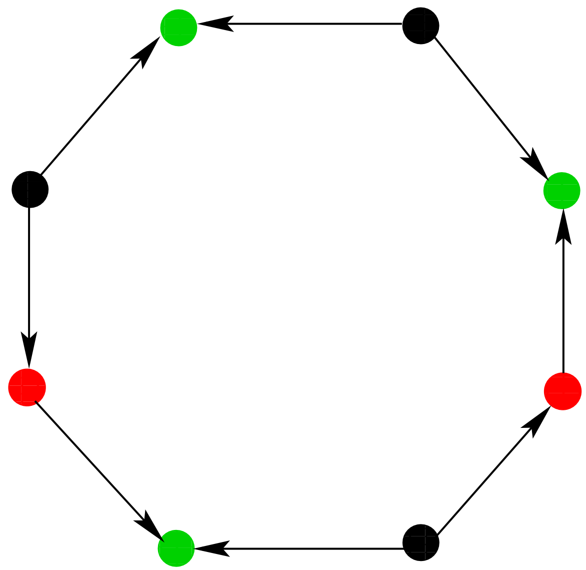


And also



Definitions

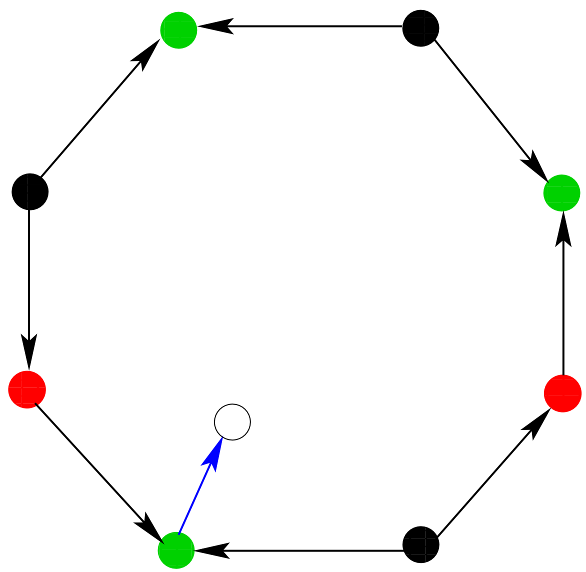
- The parity of a cycle is the number of green nodes+red nodes.



An Odd cycle

Definitions

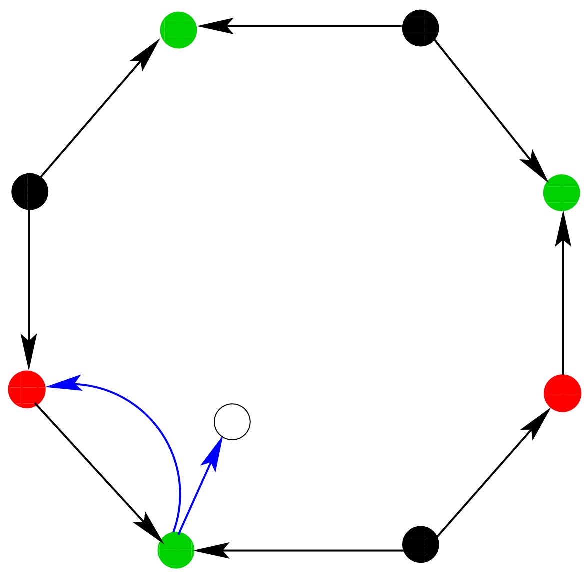
- The parity of a cycle is the number of green nodes+red nodes.



An Odd Y-cycle

Definitions

- The parity of a cycle is the number of green nodes+red nodes.



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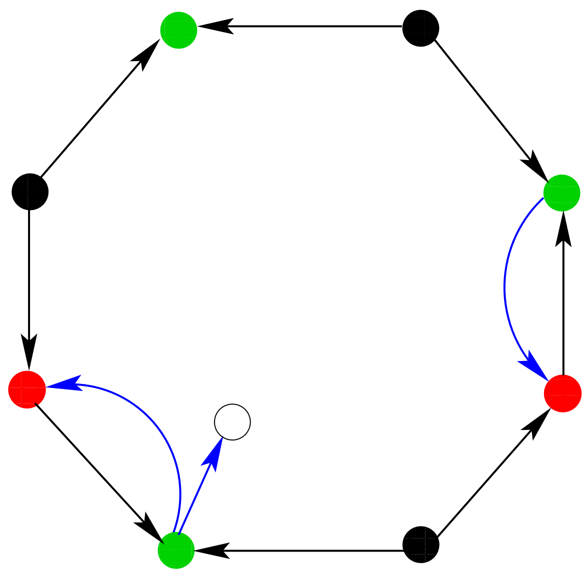

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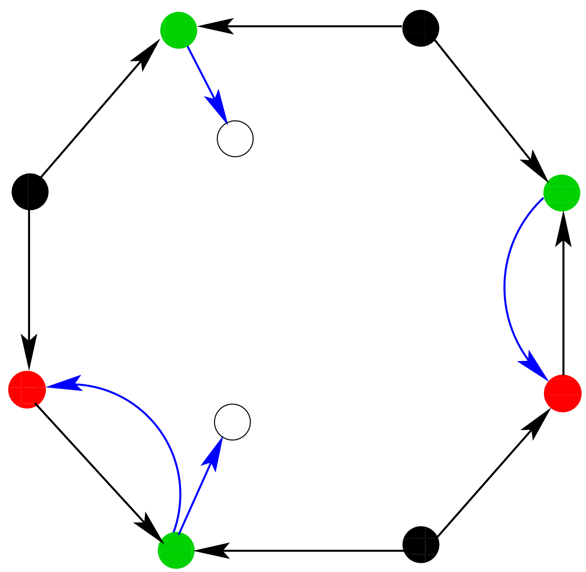

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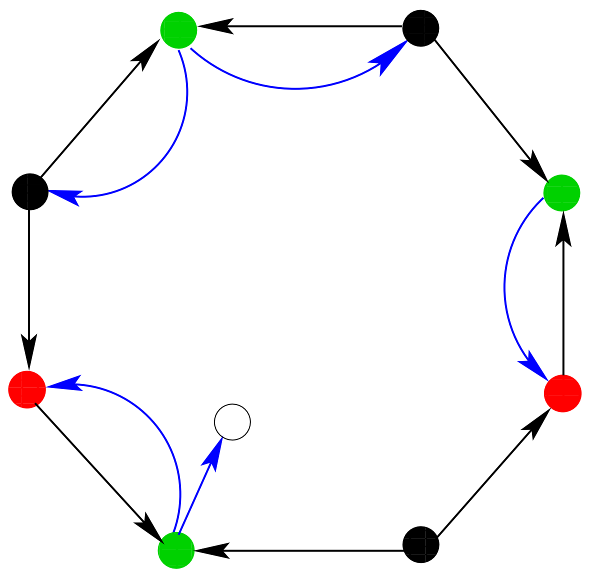
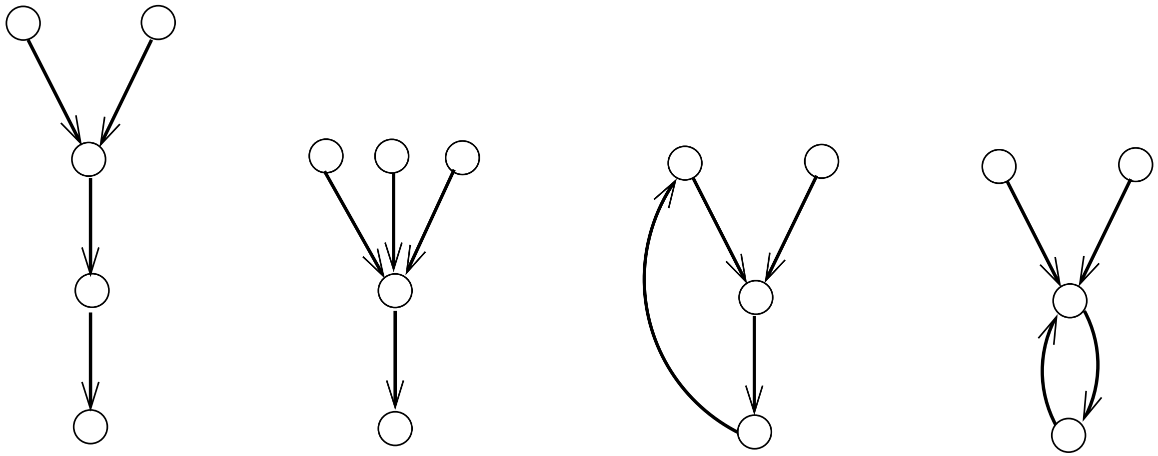


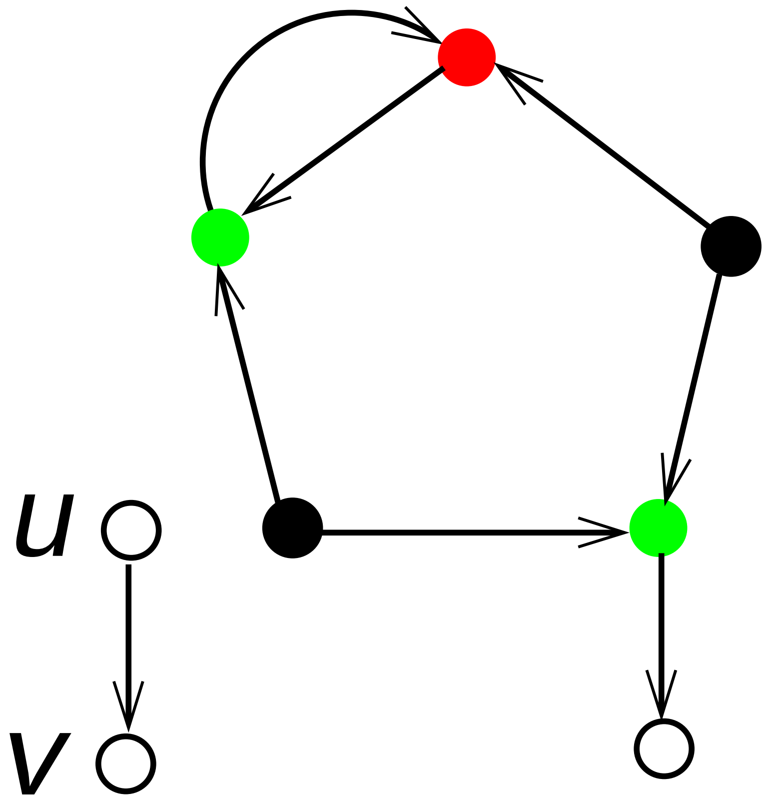
An Odd cycle which is not a Y-cycle

The main result

Theorem 2. *Let $G = (V, A)$ be a* **directed** *graph, then $P_p(G)$ is integral for any p if and only if it does not contain as a subgraph none of the graphs*

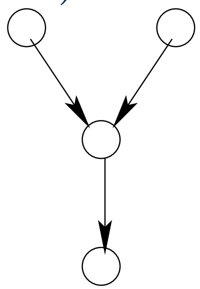


and does not contain an odd Y-cycle plus an arc (u, v) with u and v not in the cycle.


u
v

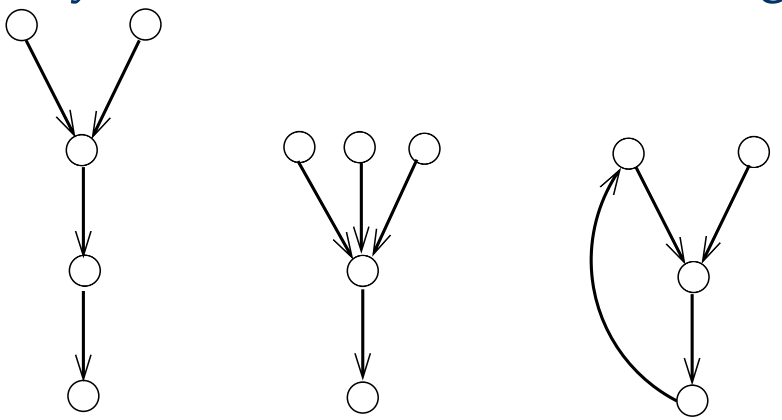
To prove Theorem 2, we need the following two results.

Theorem 3. *Let $G = (V, A)$ be an* **oriented** *graph. If G does not contain an odd directed cycle and*



as a subgraph, then $P_p(G)$ is integral, for any p.

Theorem 4. *Let $G = (V, A)$ be an* **oriented** *graph, then $P_p(G)$ is integral for any p if and only if it does not contain as subgraph one of the graphs*



and an odd Y-cycle plus an arc (u, v) with u and v not in the cycle.

A case in the proof of Theorem 2

- Let $G = (V, A)$ be a directed graph that does not contain the subgraphs of Theorem 2.

- Assume that (x, y) is an extreme fractional point of $P_p(G)$.

An extreme point is the **unique** *solution of a system of inequalities when replaced by equalities.*

To arrive to a contradiction, we plan to show that (x, y) is not the unique solution of the set of inequalities that are satisfied as equalities by (x, y).

- We can assume that $x(u, v) > 0$ for all $(u, v) \in A$.

- We may assume that $|\delta^-(v)| \leq 1$ for every pendent node v in G. (a node v is pendent if $\delta^+(v) = \emptyset$, there is no arc leaving v).

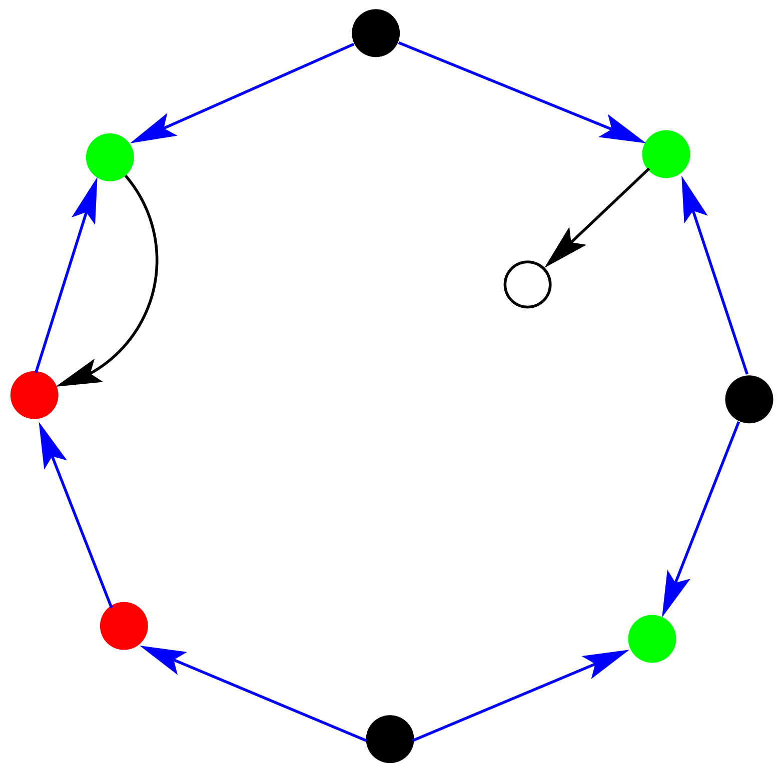
The case : G does not contain an odd Y-cycle.

Lemma 1. *$x(u, v) = y(v)$ for all (u, v) with v not a pendent node.*

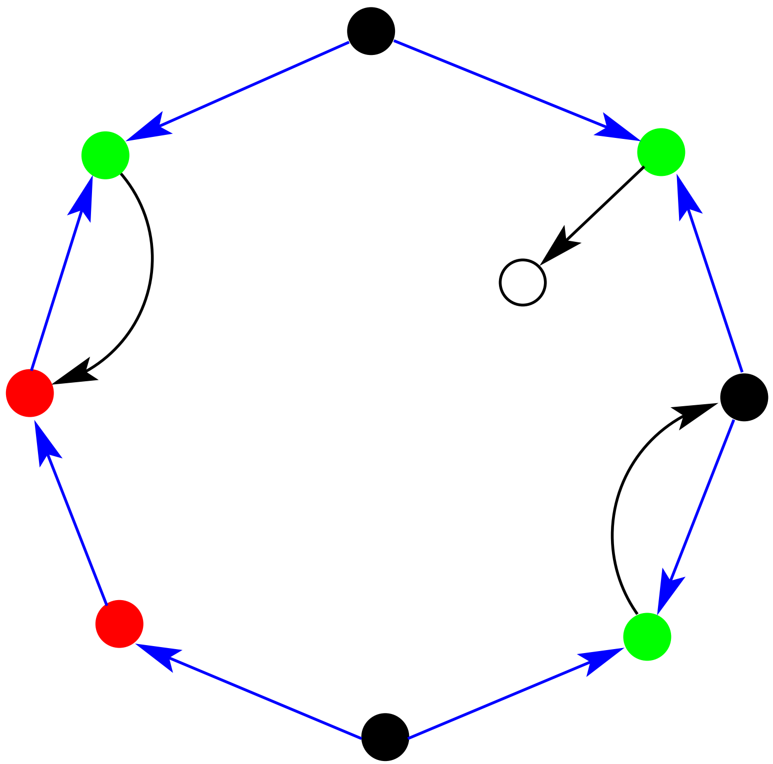
Lemma 2. *G does not contain a cycle.*

Proof:

- If G contains a cycle, then it contains a Y-cycle:



The cycle is in blue.



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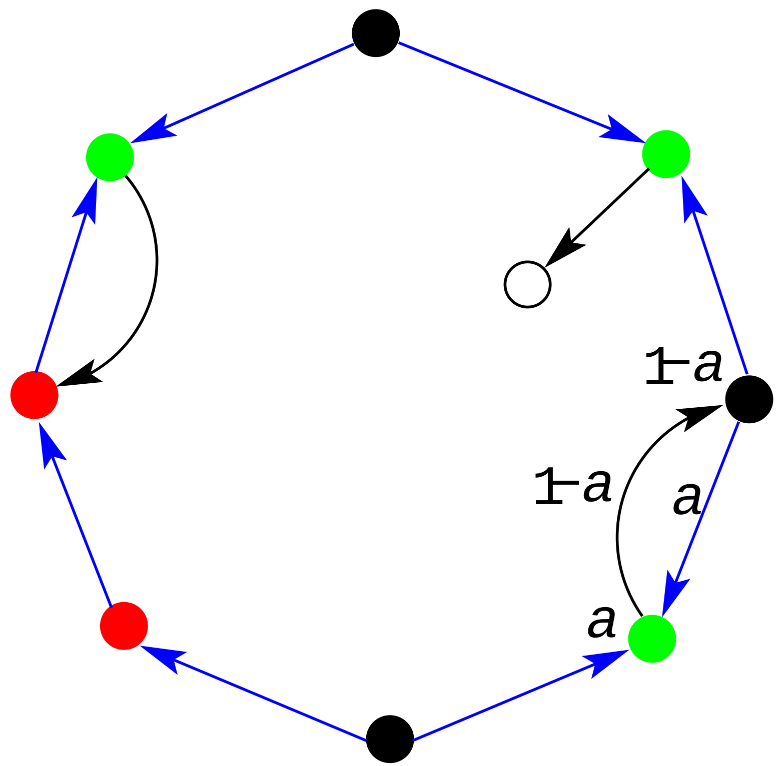

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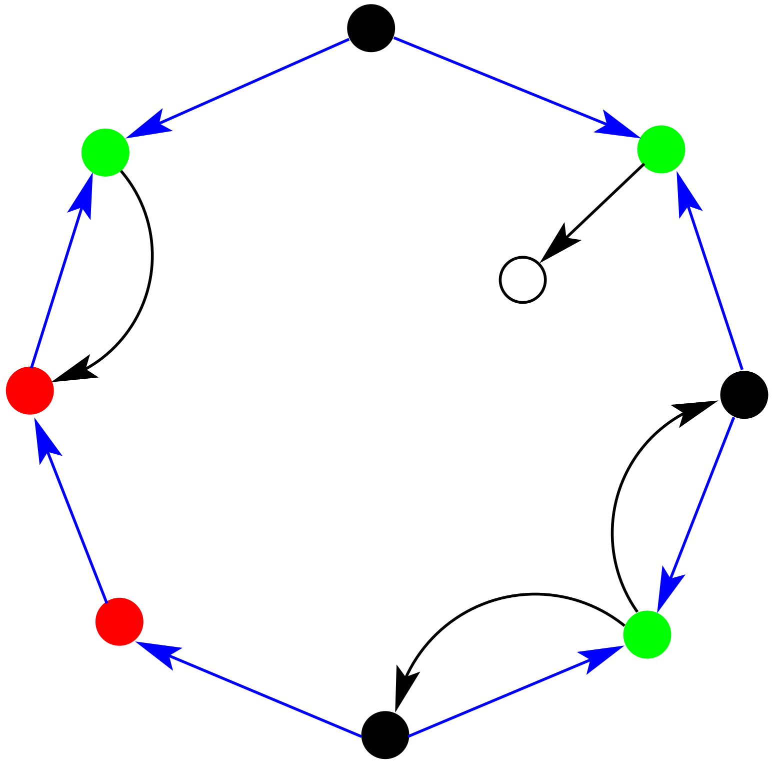

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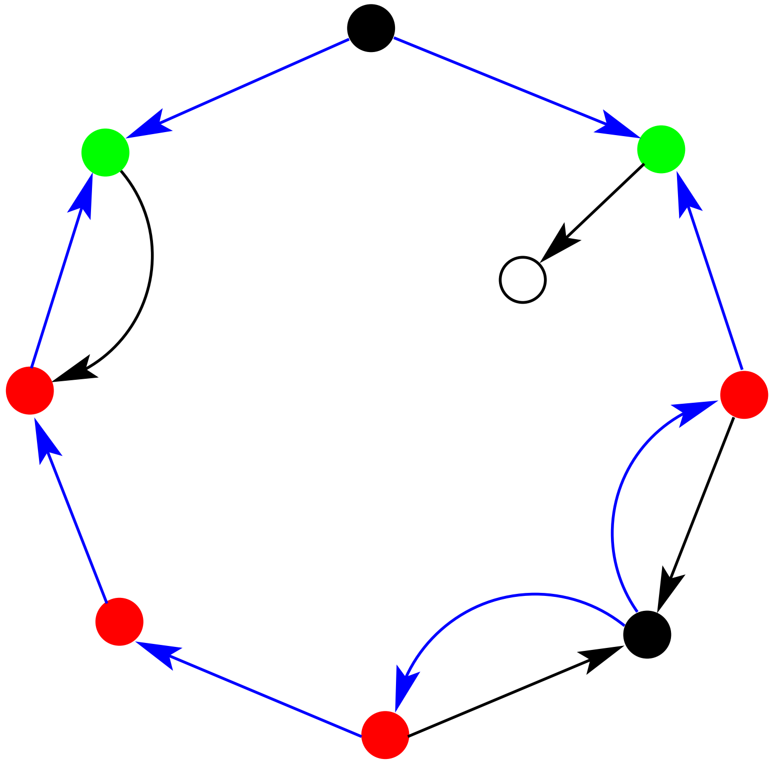
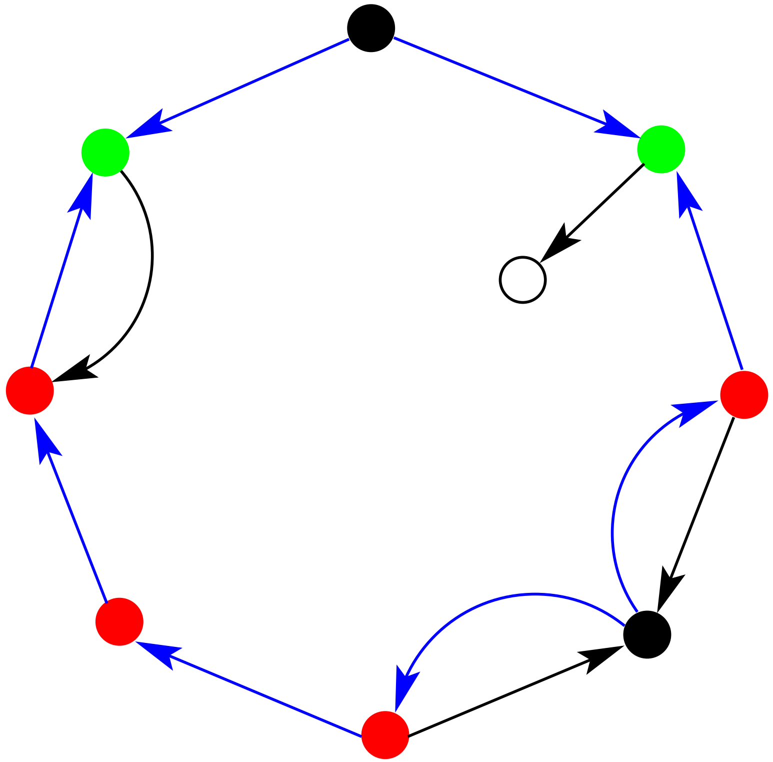


In blue: a Y-cycle.

A labeling function l assign to each node and variable the value +1, -1 or 0.

From a labeling function l and a solution (x, y) we define a new solution (x', y') as follows: $x'(u, v) = x(u, v) + l(u, v)\epsilon$ and $y'(u) = y(u) + l(u)\epsilon$, for each arc and node.



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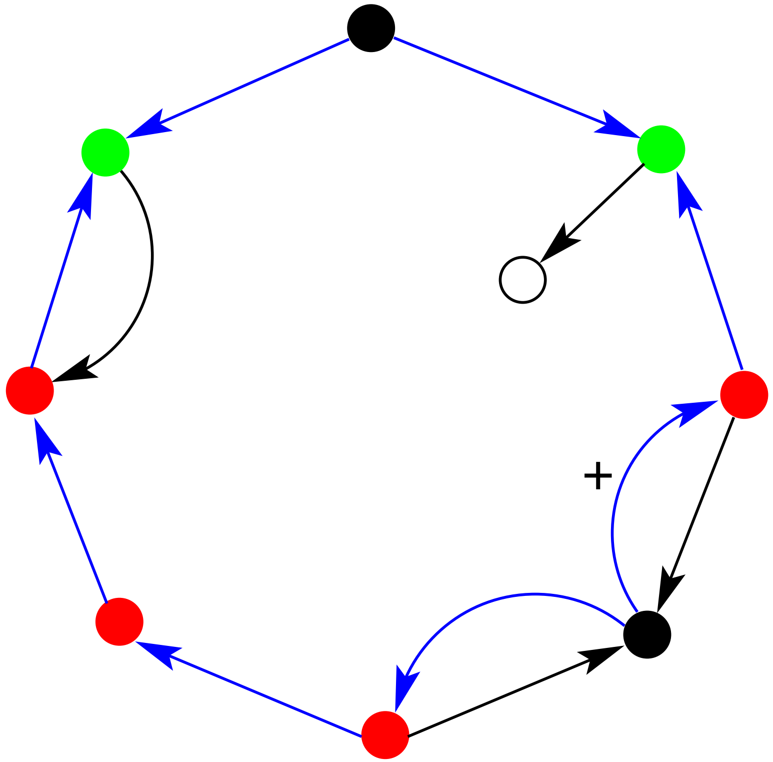
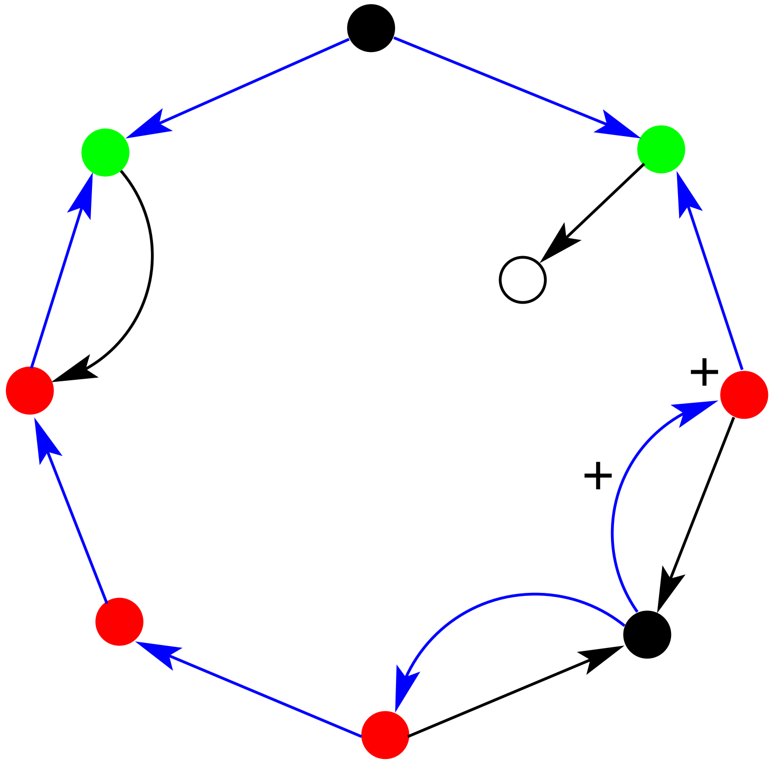


A labeling procedure for the cycle.



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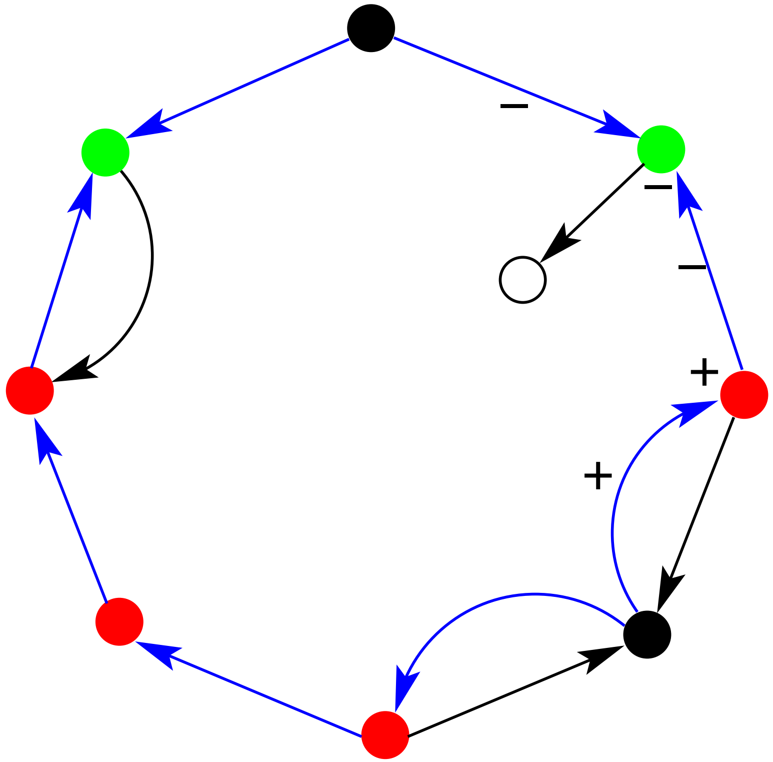

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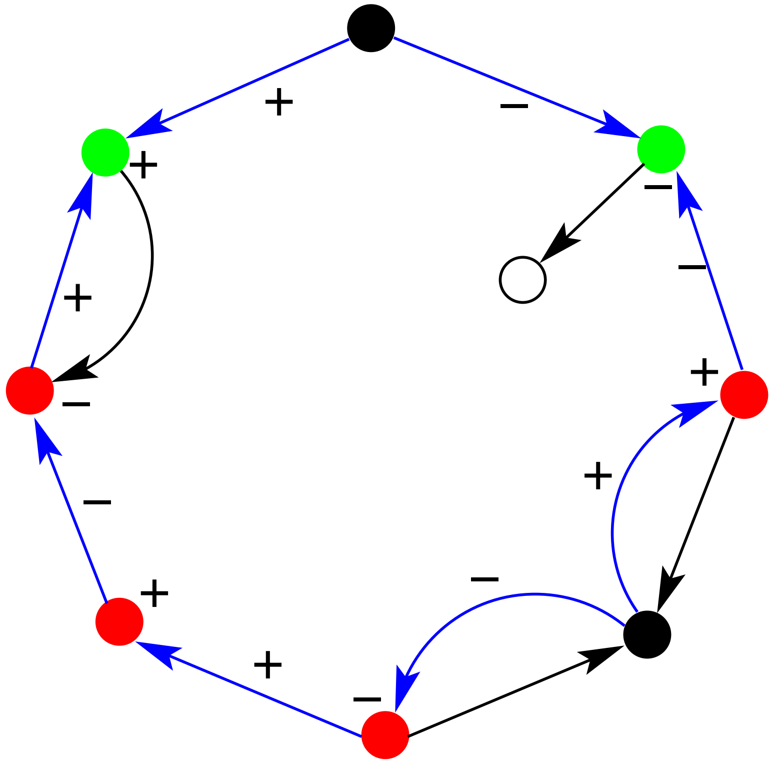
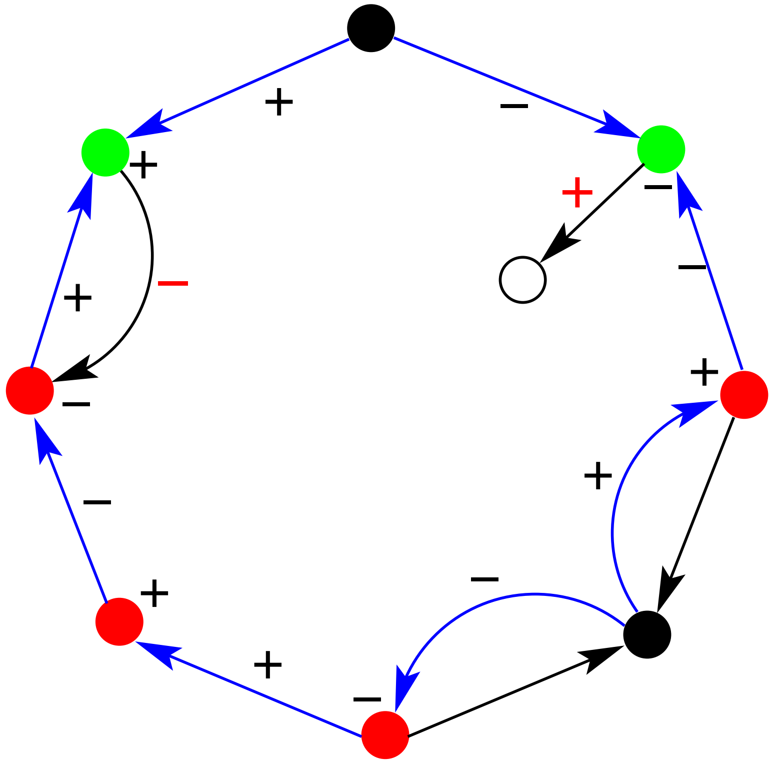


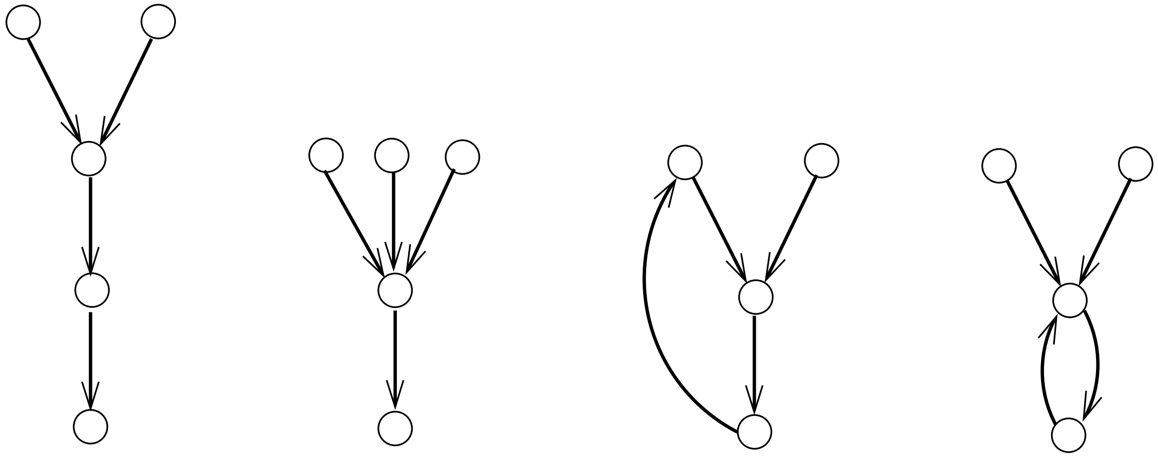
A labeling procedure for the cycle.



Extending the labeling procedure.

The end of the proof

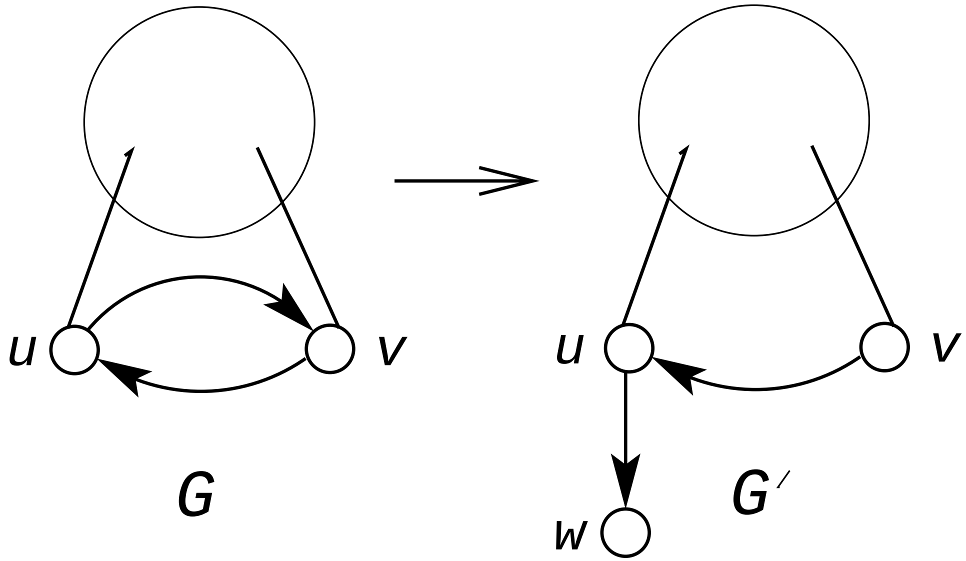
- Assume the contrary : G is a directed graph with no odd Y-cycle and does not contain none of the four following graphs as a subgraph:



- Denote by $Pair(G)$ the set of pair of nodes $\{u, v\}$ with (u, v) and (v, u) in A.

- The proof is by induction on $|Pair(G)|$. If $|Pair(G)| = 0$ then G is an *oriented* graph that satisfies the hypothesis of our theorem , so $P_p(G)$ is integral.

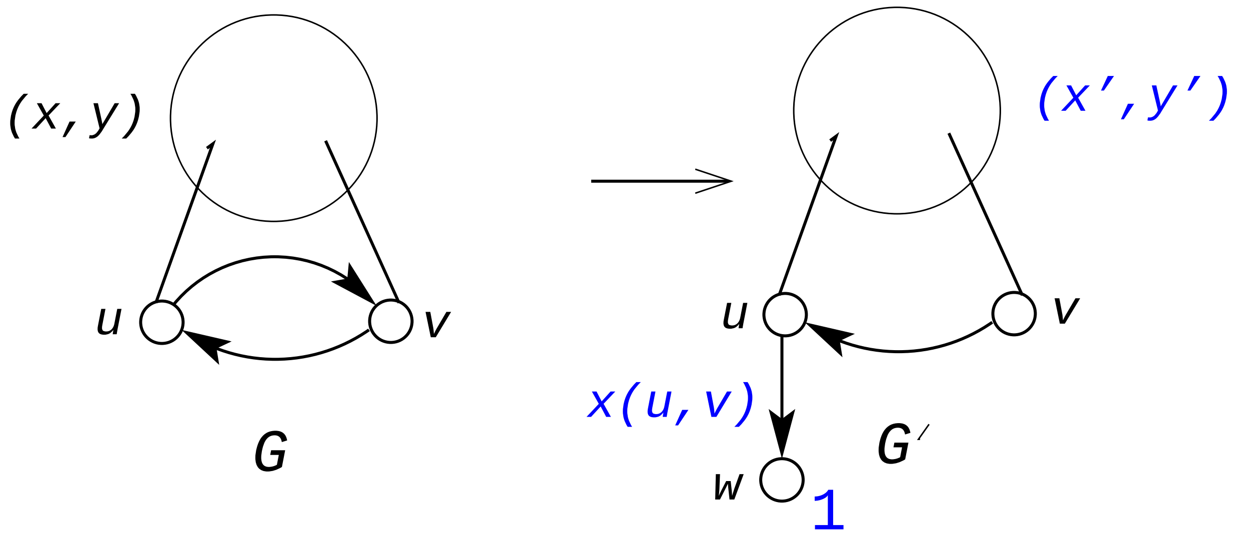
From G define the graph G' with $|Pair(G')| < |Pair(G)|$, as follows:



- It is clear that G' does not contain none of the forbiden subgraphs. Also since by Lemma 2 G does not contain a cycle, so G' too. Thus G' satisfies the induction hypothesis.

From the extreme point (x, y) of $P_p(G)$ define $(x', y') \in P_{p+1}(G')$ as follows:

- $x'(u, v) = x(u, v)$ for all arc (u, v) in G' different from (u, w),
- $y'(u) = y(u)$ for all node in G' different from w,
- $x'(u, w) = x(u, v)$ and $y'(w) = 1$



- By the induction hypothesis, $P_{p+1}(G')$ is integral, we have that *(x', y') is not an extreme point of $P_{p+1}(G')$.*

- Thus (x', y') is a convex combination of 0-1 vectors in $P_{p+1}(G')$, (x^i, y^i) $i = 1, \ldots, t$,

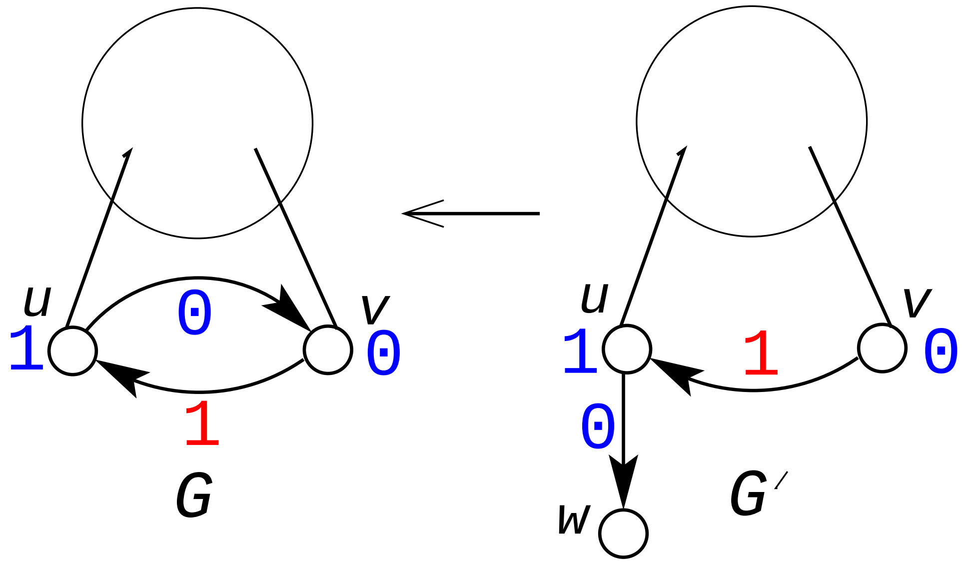
$$(x', y') = \sum_{i=1}^{t} \lambda_i (x^i, y^i),$$

$$\sum_{i=1}^{t} \lambda_i = 1,$$

$$\lambda_i \geq 0 \quad i = 1, \ldots, t.$$

- Each of the (x^i, y^i) satisfies the same equalities as (x', y'). Since $x'(v,u) > 0$, there is at least of these vectors, say (x^1, y^1), such that $x^1(v,u) = 1$.

- In the last step, as shown in the figure below, we construct from (x^1, y^1) a solution $(x^*, y^*) \in P_p(G)$ that satifies the same equalities as (x, y). This contradicts the fact that (x, y) is an extreme point of $P_p(G)$.



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