# Scarf's Lemma and the Stable Paths Problem

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# Scarf's Lemma

Let $A$ be an $m \times t$ matrix where $m \leq t$. A set of columns $S$ is called dominating if

for every column $c$ there exists a row $r$ such that $a_{rc} \leq a_{rs}$ for EVERY column $s \in S$.

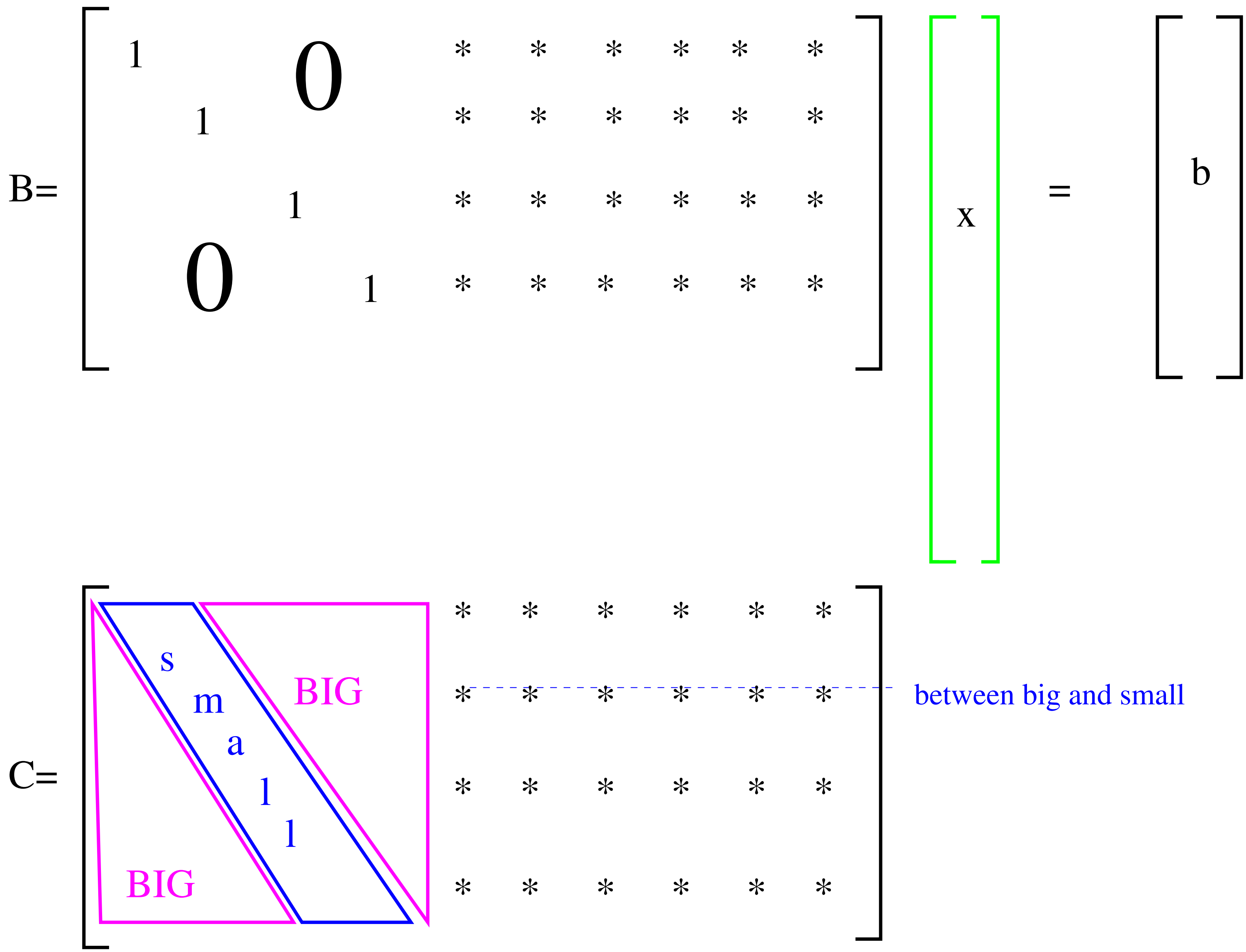
$$\begin{bmatrix} 2 & 1 & 7 & 4 & 5 & 2 \\ 8 & 2 & 2 & 3 & 6 & 7 \\ 1 & 3 & 2 & 2 & 1 & 4 \\ 4 & 4 & 8 & 1 & 5 & 5 \end{bmatrix}$$

# Scarf's Lemma

Let $B$ and $C$ be $m \times t$ matrices where $m < t$, and let $b \in R_+^m$ with the following properties:

- The first $m$ columns of $B$ form an identity matrix,
- the set of non-negative solutions $x \in R_+^t$ to $Bx = b$ is bounded,
- $c_{ii} < c_{ik} < c_{ij}$ for each $k > m$ and $j \leq m$, $i \neq j$.

Then the number of non-negative solutions $x$ to $Bx = b$ whose support $supp(x)$ is dominating in $C$ (and is an independent $m$-set of columns of $B$) is ODD. In particular, there is at least one.


B=
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1
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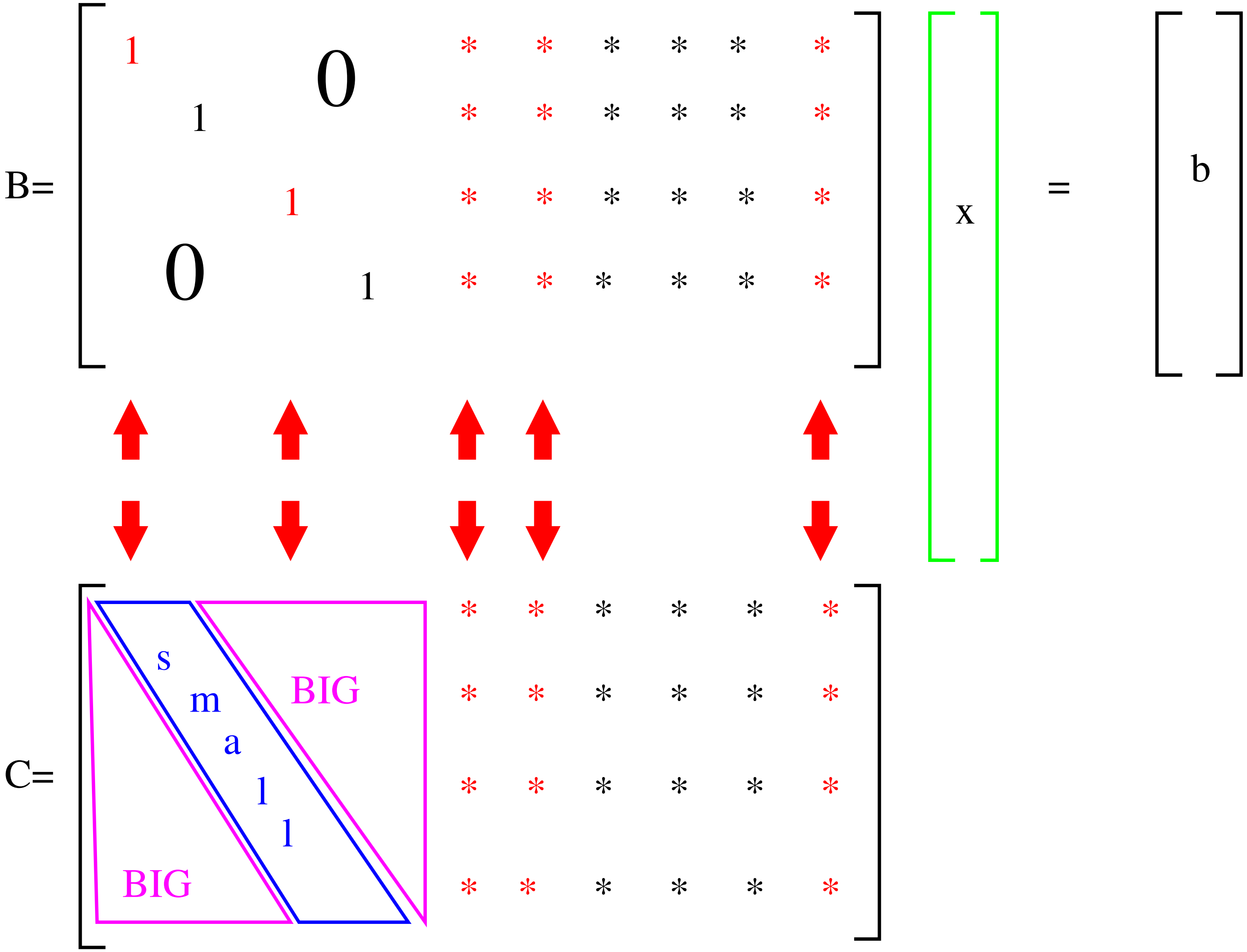
BIG
between big and small



B=
1
1
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x
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b
C=
s
m
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l
BIG
BIG

Note that if $b$ has all positive entries then the set consisting of first $m$ columns IS the support of a solution $x$ to $Bx = b$, but is NOT a dominating set in $C$.

An example of a dominating set of columns in $C$: the first $m-1$ columns, together with the column indexed $k > m$ with the LARGEST entry in row $m$.

# Scarf's Lemma

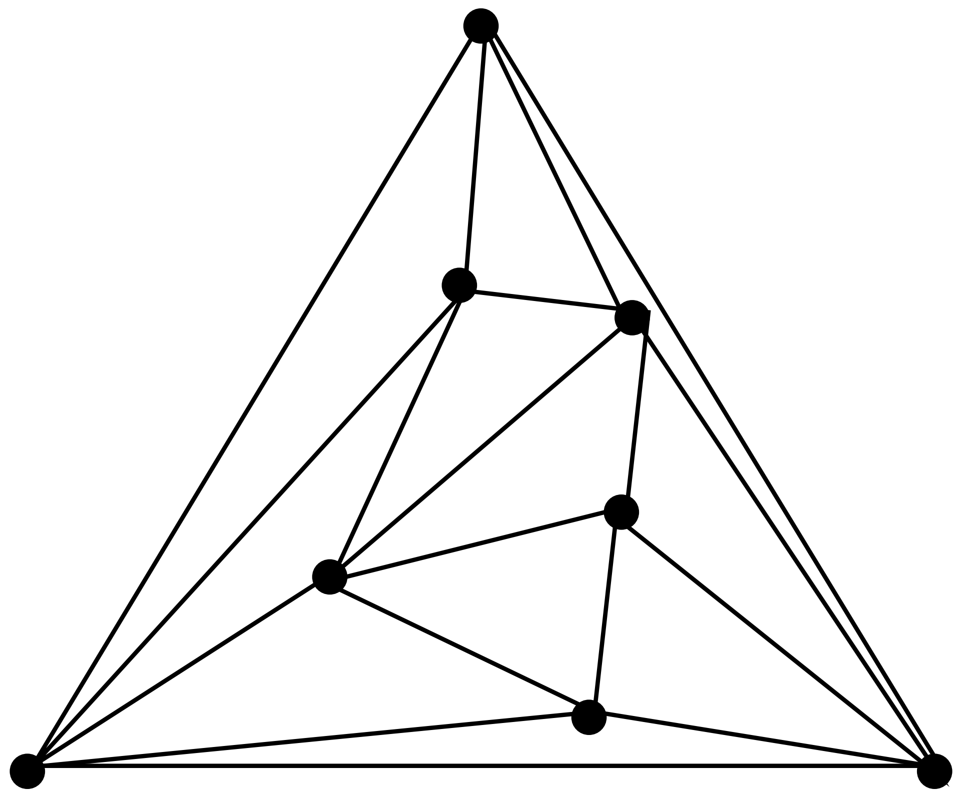
Let $B$ and $C$ be matrices satisfying the conditions of Scarf's Lemma. Let $V$ denote the set of column indices of $B$ and $C$. We define two $m$-uniform hypergraphs on the vertex set $V$ as follows:

$\mathcal{B}$: those $m$-sets of columns that are the support of a solution $x$ to $Bx = b$

$\mathcal{C}$: those $m$-sets of columns that are dominating in $C$ TOGETHER WITH the single $m$-set consisting of the first $m$ columns.

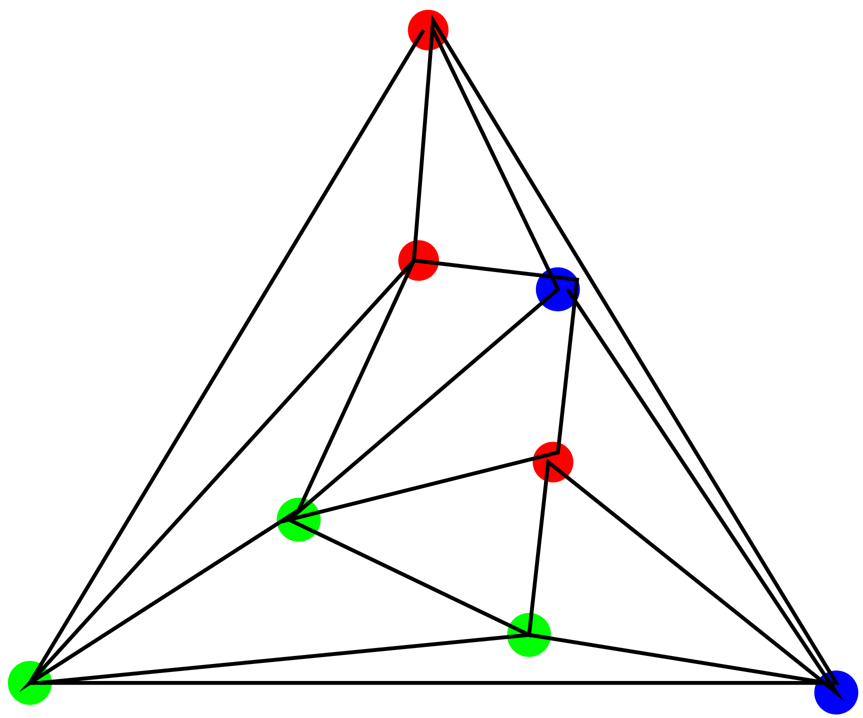
Then $|\mathcal{B} \cap \mathcal{C}|$ is EVEN.

# Restricted Triangulations



# Sperner's Lemma for Restricted Triangulations

The number of multicoloured elementary simplices in an $(n+1)$-colouring of a nontrivial restricted triangulation is EVEN.

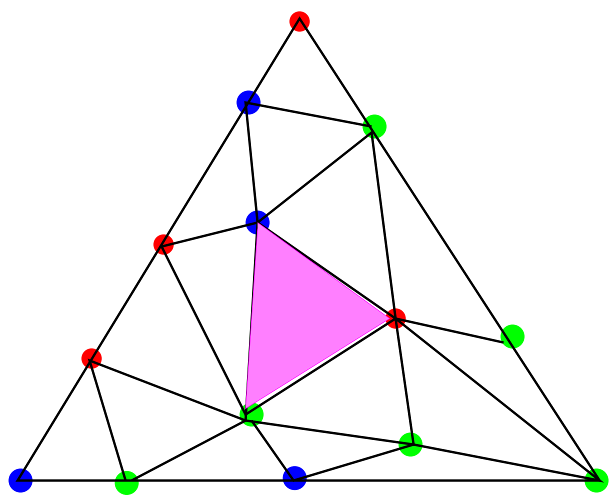


# Sperner's Lemma

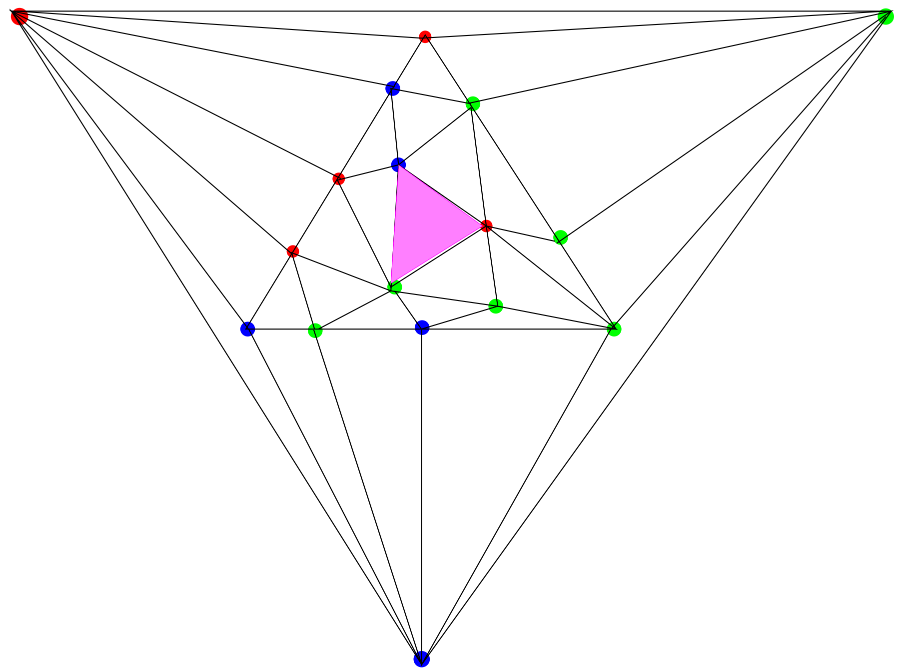
Consider a triangulation of the $n$-dimensional simplex $T$.

Consider any Sperner colouring of the points with $n+1$ colours.

Then the number of multicoloured elementary simplices is ODD. (In particular there is at least one.)



Sperner’s Lemma for restricted triangulations implies the usual formulation of Sperner’s Lemma:



The Sperner colouring guarantees that the only “extra” multicoloured simplex added by this construction is the exterior one.

# Sperner's Lemma for Restricted Triangulations

Let $m = (n+1)$, and let an $m$-coloured restricted triangulation $T$ of an $n$-simplex be given. Let $V$ denote the set of points of $T$. We define two $m$-uniform hypergraphs on the vertex set $V$ as follows:

$\mathcal{B}$: those $m$-sets that are multicoloured

$\mathcal{C}$: those $m$-sets that are the vertices of an elementary simplex.

Then $|\mathcal{B} \cap \mathcal{C}|$ is EVEN.

# A General Theorem

THEOREM: Let $\mathcal{B}$ and $\mathcal{C}$ be $m$-uniform hypergraphs on the same vertex set $V$. Suppose

- $\mathcal{B}$ has the UNIQUE ADD property: for each $B \in \mathcal{B}$ and each $u \notin B$ there exists a unique $v \in B$ such that $B \setminus \{v\} \cup \{u\} \in \mathcal{B}$.

- $\mathcal{C}$ has the UNIQUE REMOVE property: for each $C \in \mathcal{C}$ and each $v \in C$ there exists a unique $u \notin C$ such that $C \setminus \{v\} \cup \{u\} \in \mathcal{C}$.

Then $|\mathcal{B} \cap \mathcal{C}|$ is EVEN.

Implicit in the proofs of Sperner and Scarf, this link also interpreted/rediscovered by several authors in different formulations, including Kuhn (1968), Aharoni and Fleiner (2003).

# Sperner's Lemma for Restricted Triangulations

Let $m = (n+1)$, and let an $m$-coloured restricted triangulation $T$ of an $n$-simplex be given. Let $V$ denote the set of points of $T$. We define two $m$-uniform hypergraphs on the vertex set $V$ as follows:

$\mathcal{B}$: those $m$-sets that are multicoloured. Then $\mathcal{B}$ has the UNIQUE ADD property.

$\mathcal{C}$: those $m$-sets that are the vertices of an elementary simplex. Then $\mathcal{C}$ has the UNIQUE REMOVE property.

Therefore $|\mathcal{B} \cap \mathcal{C}|$ is EVEN.

## Scarf's Lemma

Let $B$ and $C$ be matrices satisfying the conditions of Scarf's Lemma. Let $V$ denote the set of column indices of $B$ and $C$. We define two $m$-uniform hypergraphs on the vertex set $V$ as follows:

$\mathcal{B}$: those $m$-sets of columns that are the support of some solution $x$ to $Bx = b$. Then $\mathcal{B}$ has the UNIQUE ADD property.

$\mathcal{C}$: those $m$-sets of columns that are dominating in $C$ TOGETHER WITH the single $m$-set consisting of the first $m$ columns. Then $\mathcal{C}$ has the UNIQUE REMOVE property.

Therefore $|\mathcal{B} \cap \mathcal{C}|$ is EVEN.

## The General Theorem

THEOREM: Let $\mathcal{B}$ and $\mathcal{C}$ be $m$-uniform hypergraphs on the same vertex set $V$. Suppose

- $\mathcal{B}$ has the UNIQUE ADD property,
- $\mathcal{C}$ has the UNIQUE REMOVE property.

Then $|\mathcal{B} \cap \mathcal{C}|$ is EVEN.

FOR EXAMPLE for graphs ($m = 2$): Any complete bipartite graph has the UNIQUE ADD property. Any 2-regular graph has the UNIQUE REMOVE property. Therefore

"Any 2-regular graph has an even number of edges crossing any cut."

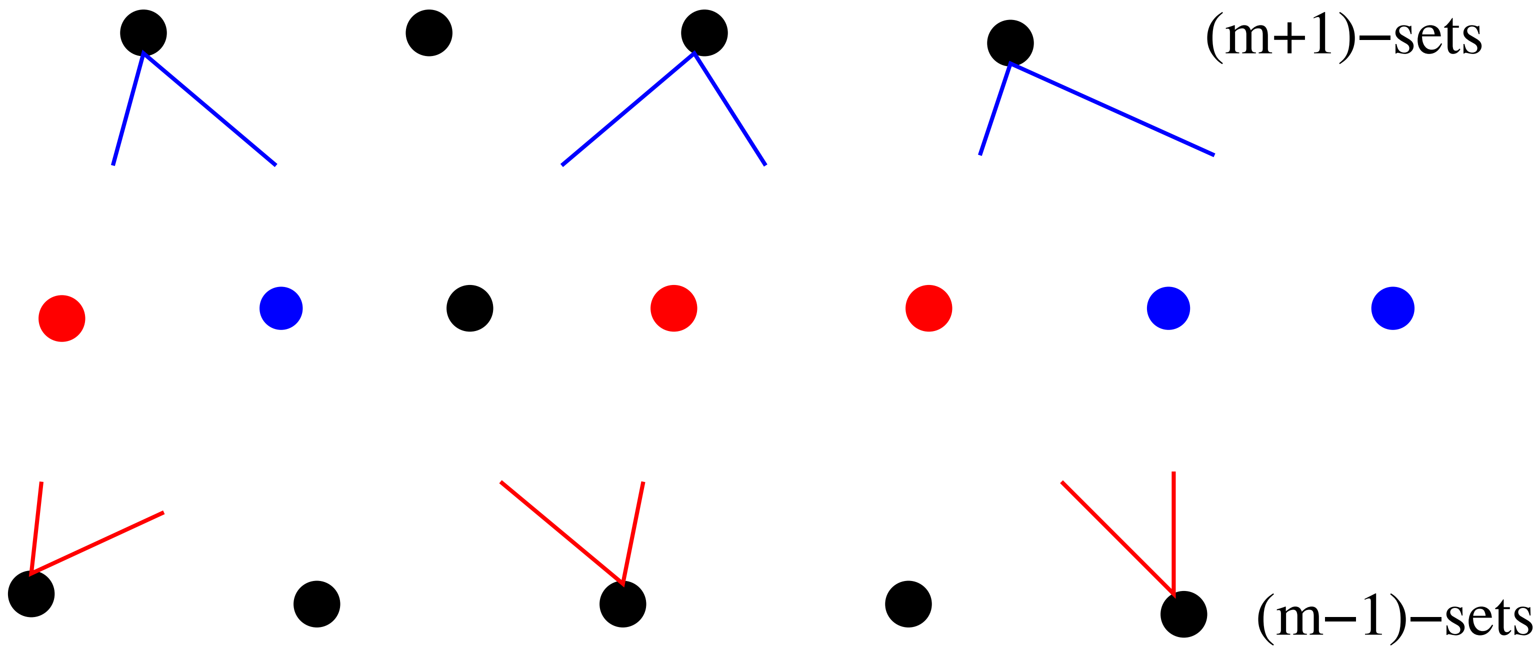
FOR EXAMPLE for $m = 3$:

- Any complete tripartite 3-uniform hypergraph $W$ has the UNIQUE ADD property.

- Any disjoint union $U$ of two Steiner triple systems has the UNIQUE REMOVE property.

Therefore the intersection of any such $U$ and any such $W$ has even size.

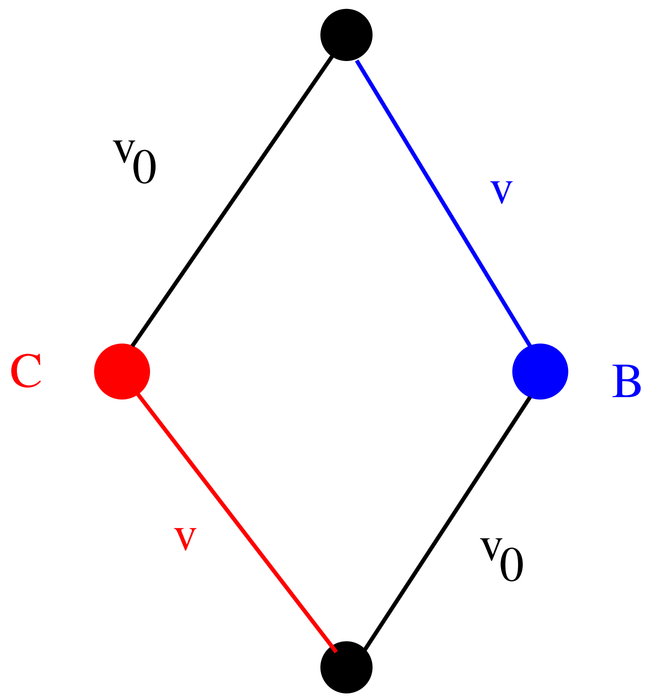
# Proof




Every $(m+1)$-set has degree 0 or 2 into $\mathcal{B}$.

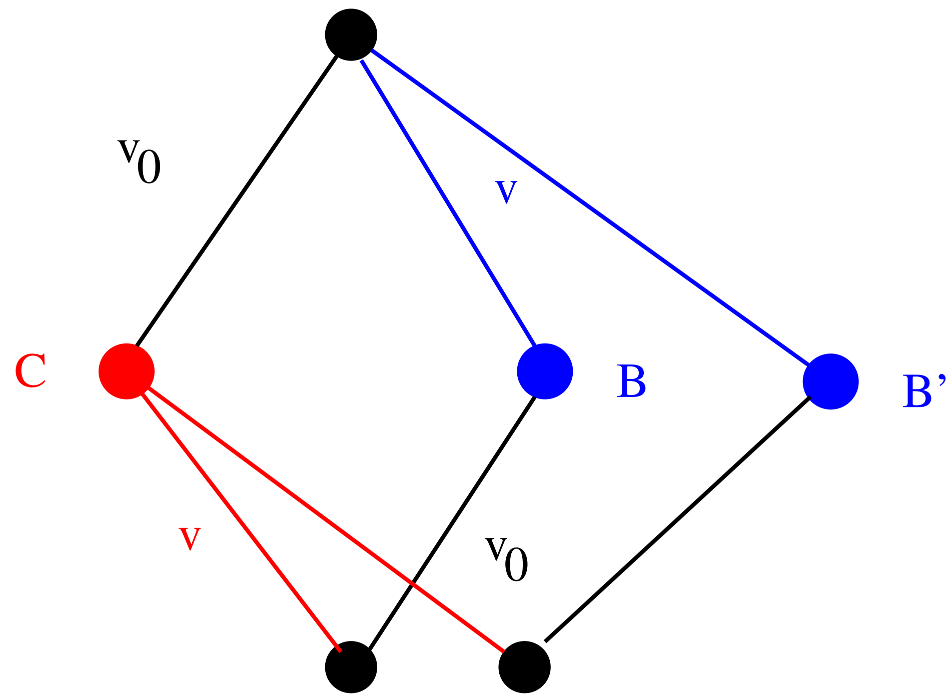
Every $(m-1)$-set has degree 0 or 2 into $\mathcal{C}$.

Fix a vertex $v_0 \in V$. We define a directed graph $D$ with vertex set $\mathcal{B} \cup \mathcal{C}$ as follows: put an arc from $C$ to $B$ if there exists $v \in C$, $v \neq v_0$ such that $C \setminus \{v\} = B \setminus \{v_0\}$.

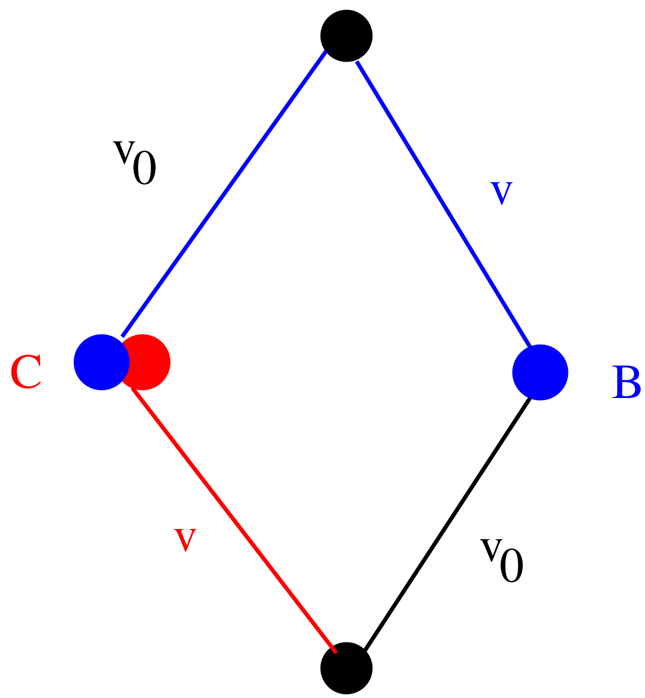


For $C \in \mathcal{C}$: if $v_0 \in C$ then $d^+(C) = 0$.

If $v_0 \notin C$ then $d^+(C) = 0$ or 2,

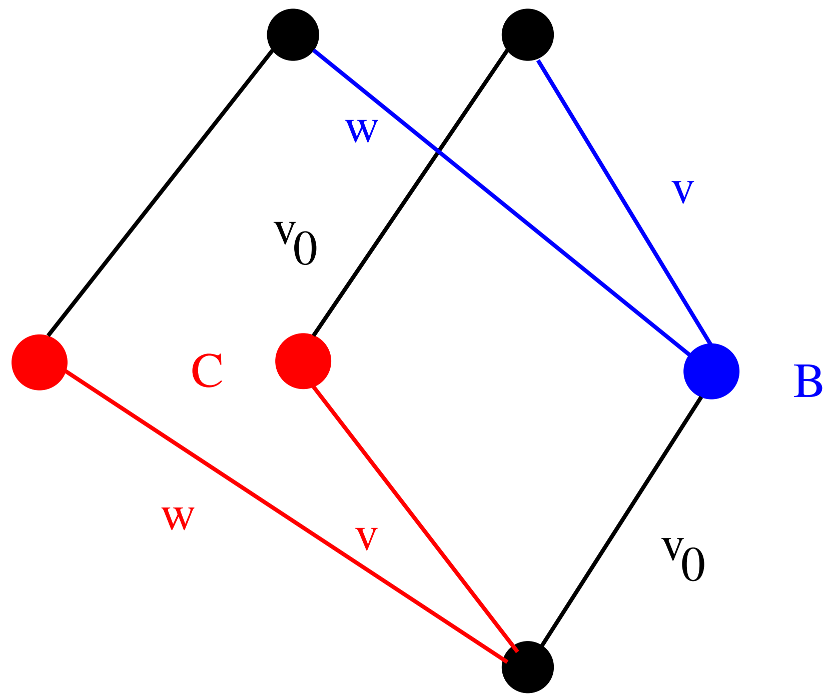



unless $C \in \mathcal{B} \cap \mathcal{C}$, in which case $d^+(C) = 1$.

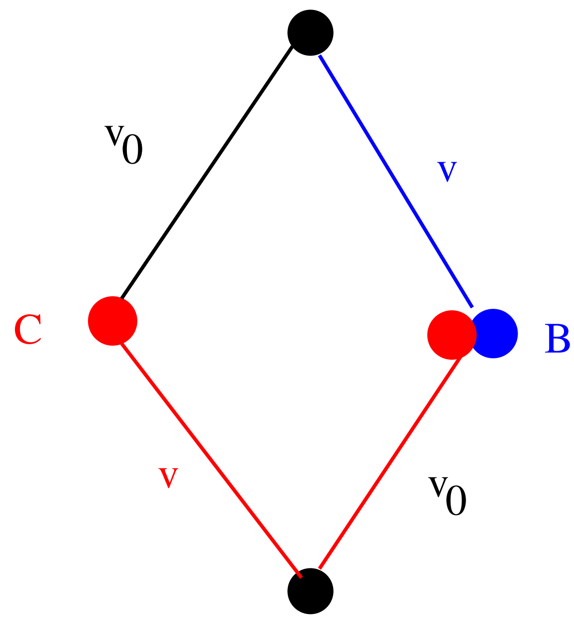


For $B \in \mathcal{B}$: if $v_0 \notin B$ then $d^-(B) = 0$.

If $v_0 \in B$ then $d^-(B) = 0$ or 2,




unless $B \in \mathcal{B} \cap \mathcal{C}$, in which case $d^{-}(B) = 1$.




Thus in the underlying graph of $D$, the vertices of degree 1 are those in $\mathcal{B} \cap \mathcal{C}$, all other vertices have degree 0 or 2.

Therefore $|\mathcal{B} \cap \mathcal{C}|$ is EVEN.

## More generally

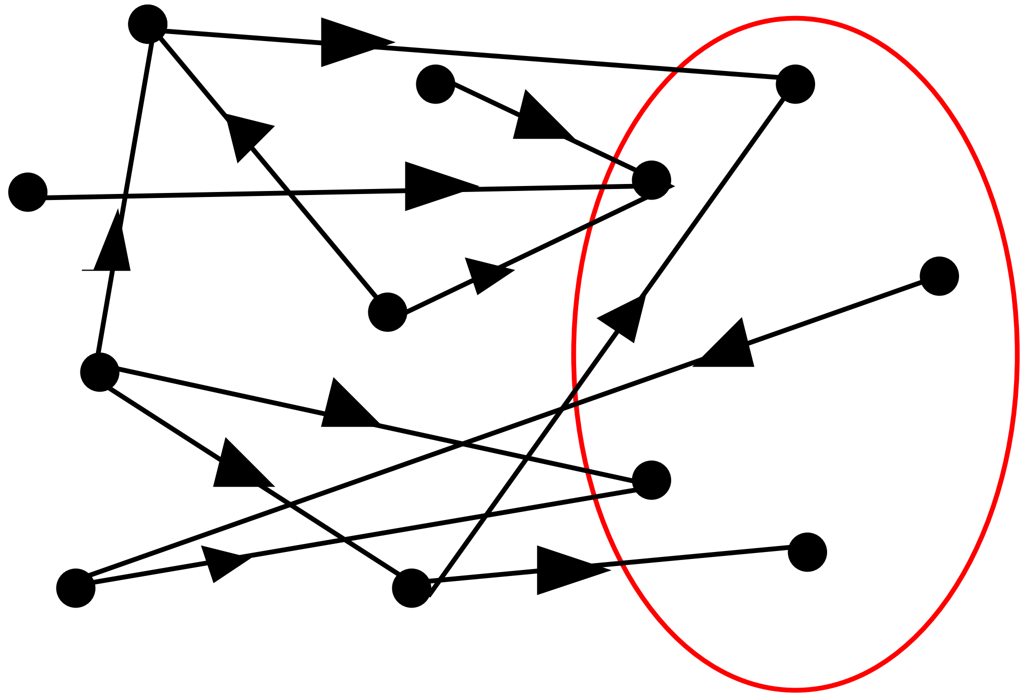
THEOREM: Let $\mathcal{B}$ and $\mathcal{C}$ be $m$-uniform hypergraphs on the same vertex set $V$. Suppose

- each $(m+1)$-set contains an EVEN number of elements of $\mathcal{B}$.

- each $(m-1)$-set is contained in an EVEN number of elements of $\mathcal{C}$.

Then $|\mathcal{B} \cap \mathcal{C}|$ is EVEN.

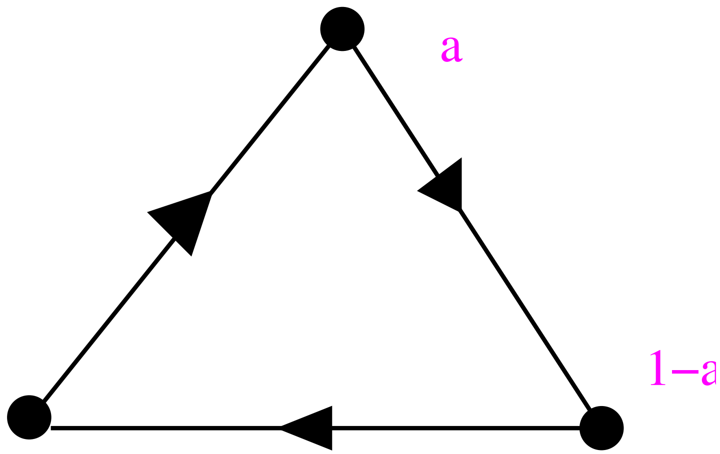
# Application of Scarf's Lemma: Fractional Kernels



A fractional kernel in $D$ is a non-negative function $f$ on the vertices of a directed graph, such that

- $\sum_{u \in N(v)} f(u) \geq 1$ for every vertex $v$, where $N(v)$ denotes the set $\{v\} \cup \{u : (v, u) \in D\}$ ($f$ is fractionally absorbing)

- $\sum_{u \in K} f(u) \leq 1$ for each clique $K$ ($f$ is fractionally independent).




THEOREM (Aharoni, Holzman 1995): Every clique-acyclic directed graph has a fractional kernel.

PROOF: Uses Scarf's Lemma.

The matrix $B$ encodes fractional independence

The matrix $C$ encodes fractional absorption

## Application: The Stable Paths problem

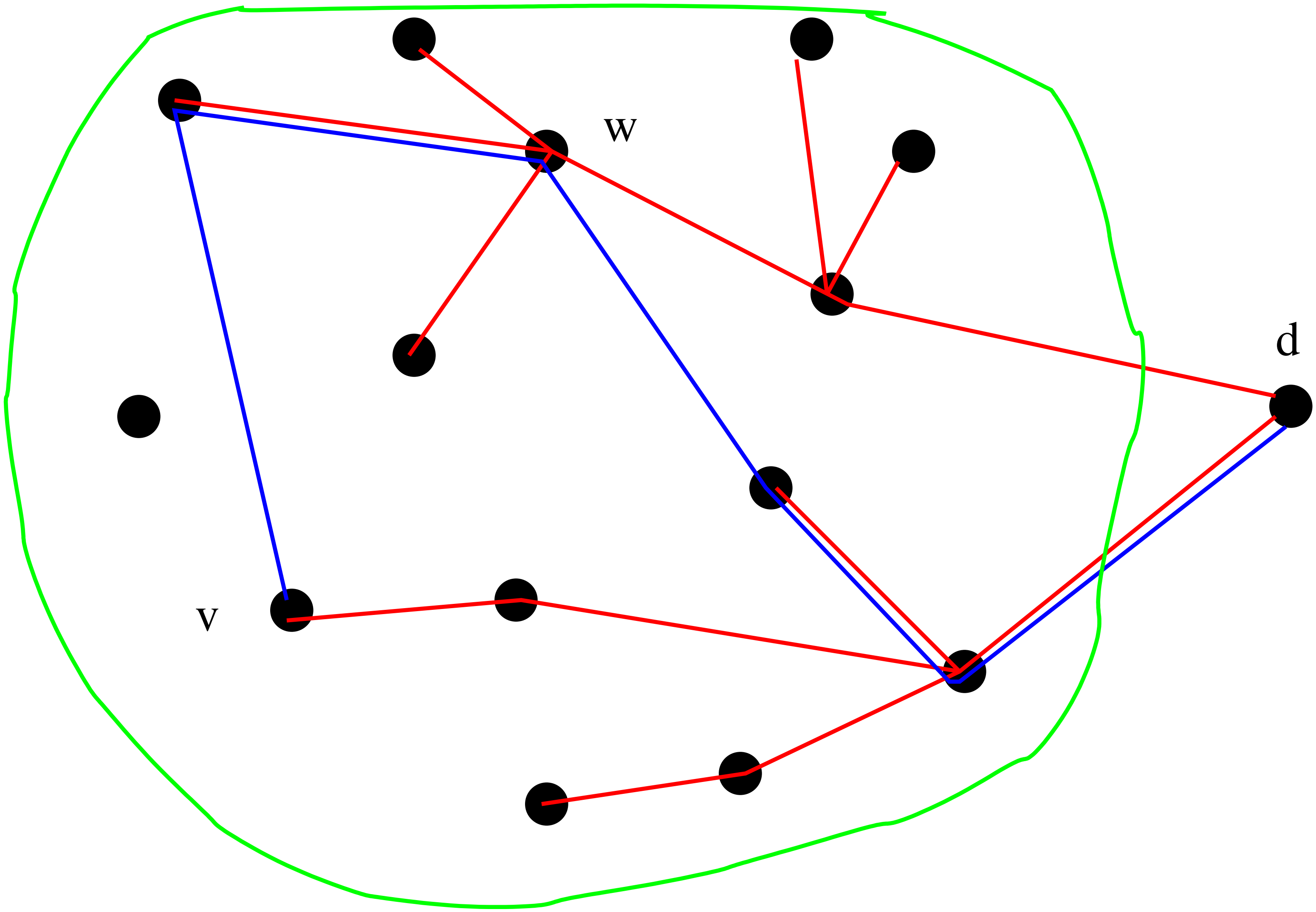
INSTANCE:

- A graph $G$ with a distinguished vertex $d$ (the destination),
- an ordered list $\pi(v)$ of paths from $v$ to $d$ for each vertex $v$ (the preference list of $v$).

SOLUTION: a tree $T$ in $G$, rooted at $d$, such that for every vertex $v$ and path $P \in \pi(v)$, either

- $v$ prefers its path in $T$ to $P$, or
- there is a PROPER final segment of $P$ that is not contained in $T$.

Motivation: internet routing protocols (Border Gateway Protocol)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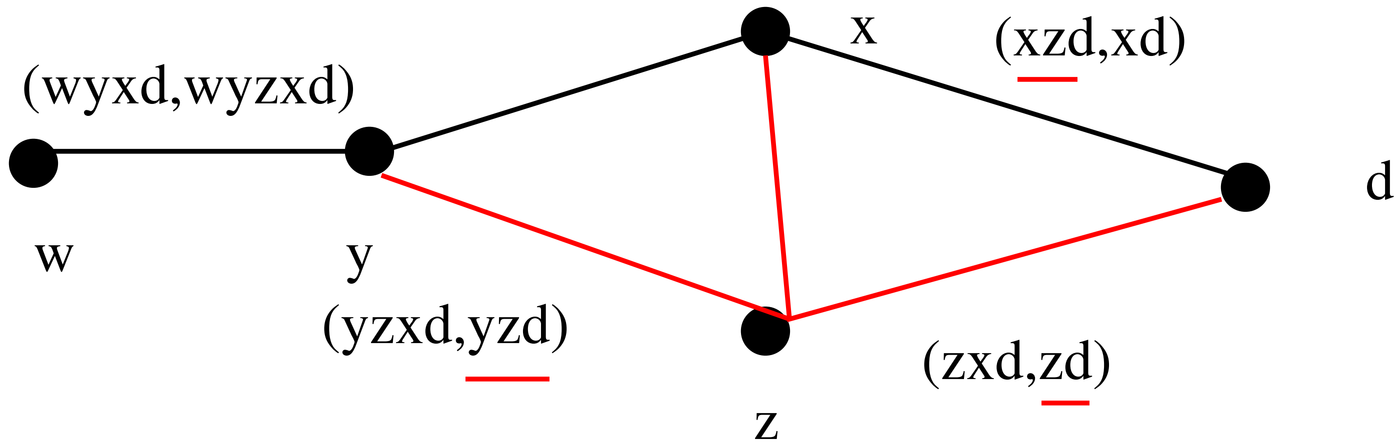
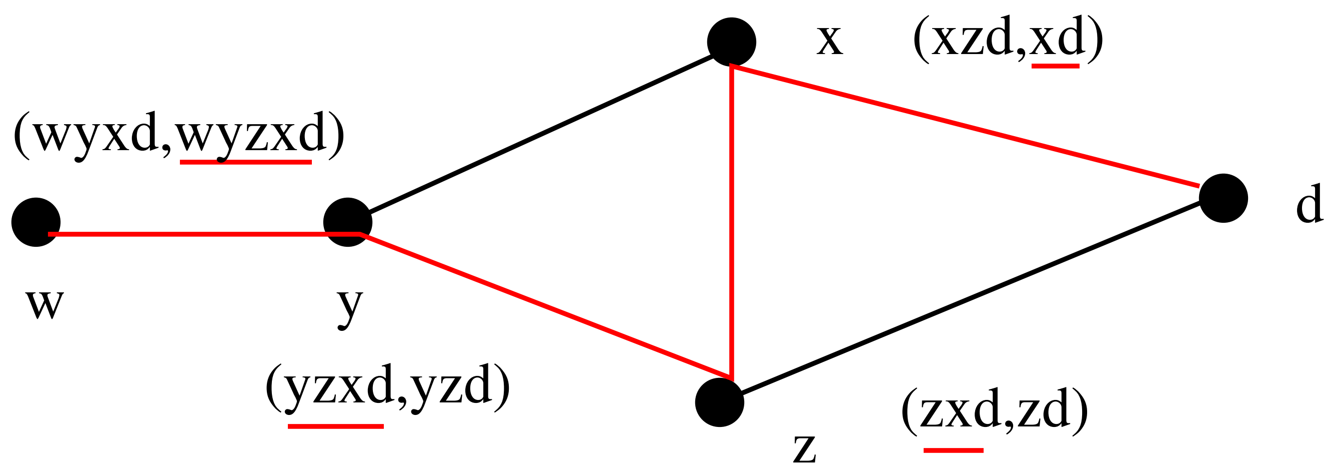
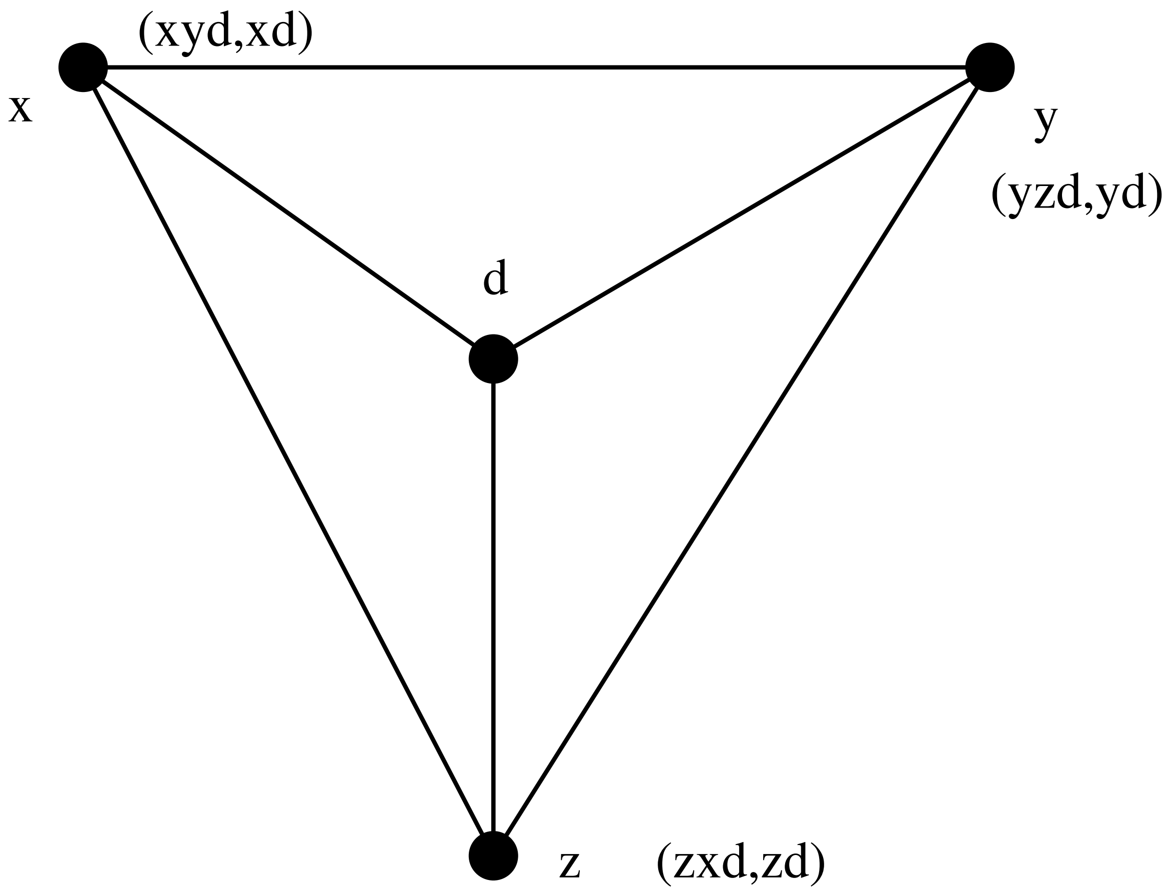
d
v


x
(xzd,xd)
(wyxd,wyzxd)
d
w
y
(yzxd,yzd)
(zxd,zd)
z


x
(xzd,xd)
(wyxd,wyzxd)
d
w
y
(yzxd,yzd)
(zxd,zd)
z

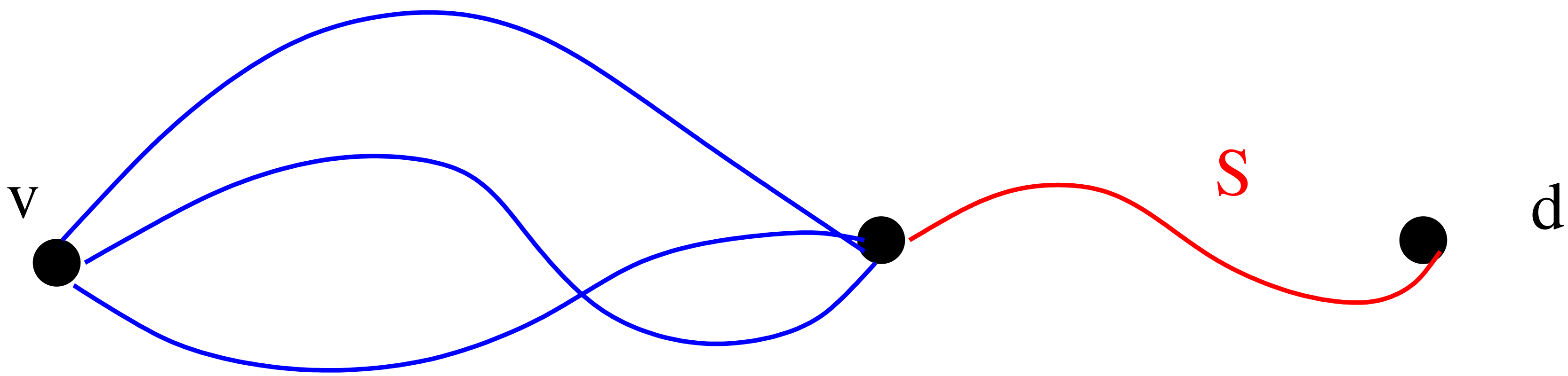
NOT EVERY instance of SPP has a solution:



# A fractional version

SOLUTION: A function that assigns a weight $w(P)$ to each path $P \in \cup_v \pi(v)$ such that

- for each $v$, $\sum_{P \in \pi(v)} w(P) \leq 1$,
- (tree condition) for each vertex $v$ and path $S$, $\sum_{P \in \pi(v,S)} w(P) \leq w(S)$, where $\pi(v,S)$ denotes the set of paths in $\pi(v)$ that end with the segment $S$,



(stability condition) for each $v$ and each $P \in \pi(v)$, either

- $\sum_{Q \in \pi(v)} w(Q) = 1$ AND $v$ prefers ALL its paths $Q \in \pi(v)$ for which $w(Q) > 0$ to $P$, or

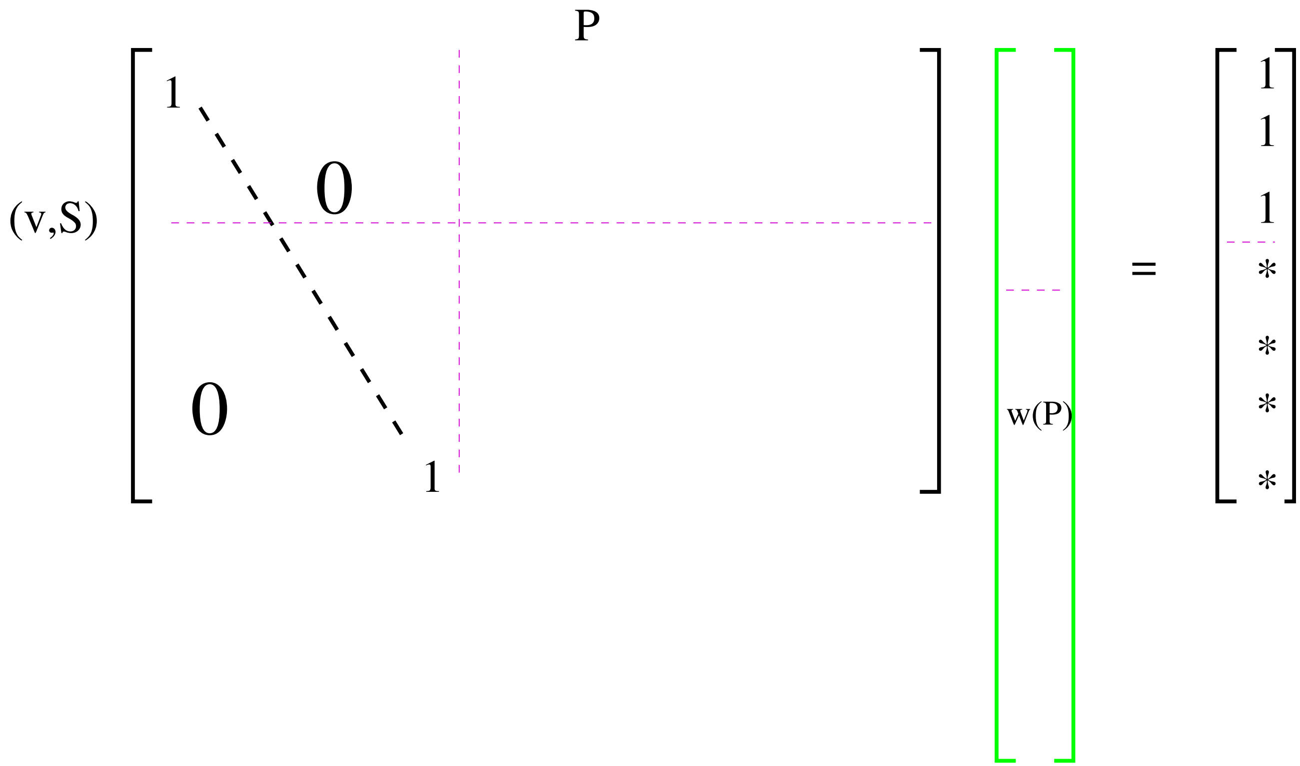
- there exists a PROPER final segment $S$ of $P$ such that $\sum_{Q \in \pi(v,S)} w(Q) = w(S)$ AND $v$ prefers ALL paths $Q \in \pi(v, S)$ for which $w(Q) > 0$ to $P$.

THEOREM (PH, Wilfong): Every instance of SPP has a fractional solution.

PROOF: Uses Scarf's Lemma.

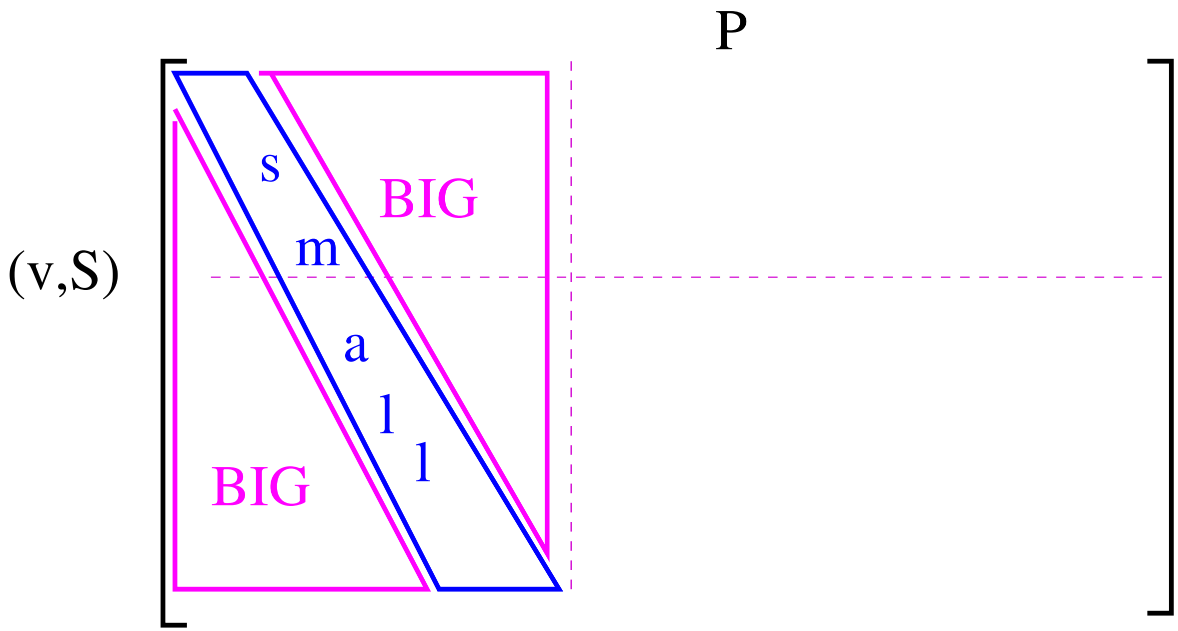
- The matrix $B$ encodes the tree condition.
- The matrix $C$ encodes the stability condition.
- The solution $x$ gives the weight function on paths that is the fractional solution to SPP.

# The matrix B




The $((v, S), P)$ entry is $-1$ if $P = S$, 1 if $P \in \pi(v, S)$, and 0 otherwise.

# The matrix C




The $((v,S),P)$ entry is the rank of $P$ in $\pi(v,S)$, if $P \in \pi(v,S)$, and $M$ otherwise, where $M$ is larger than any rank.

The solution $x$ from Scarf's Lemma gives a weight function $w$ on all paths in $\cup_v \pi(v)$.

The matrix $B$ ensures that the tree condition is satisfied.

The dominating property of $supp(x)$ ensures that the stability condition is satisfied.