A reduced polytopic LPV synthesis for a sampling varying controller: experimentation with a $T$ inverted pendulum

*ECC’07@Kos – SafeNECS invited session*

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Summary

• Context and objectives
• Design of variable sampling controllers
  ◦ LPV polytopic model
  ◦ period dependent performance specification
  ◦ $H_\infty$ robust control
• Experimental assessment
  ◦ T inverted pendulum
  ◦ Control design
  ◦ Simulations
  ◦ Experiments
• Conclusion and future directions
Context and objectives

Context: Networked control nodes

- Digital control of a continuous system
- Shared execution resource (computer and/or field-bus)
- Varying available resources (energy aware nodes)

Traditional implementation:

- separation of concerns control/real-time
- based on worst cases
- not control aware

$\Rightarrow$ neither efficient nor flexible
Context and objectives

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on-line adaptive scheduling parameters
w.r.t. availability of execution resources
Context and objectives

Context: Networked control nodes

- Digital control of a continuous system
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on-line adaptive scheduling parameters w.r.t. availability of execution resources

\[
\text{CPU load} = \frac{c}{h} \quad \text{with } c : \text{execution time, } h : \text{period}
\]

\[
\text{Bandwidth} = \frac{d}{h} \quad d : \text{message size}
\]
Computer load aware control

Method: Design of a controller

- Which period may vary in $[h_{\text{min}}, \ldots, h_{\text{max}}]$
- Preserving of the stability during timing change
- Meeting a specified performance
Computer load aware control

Method: Design of a controller

- Which period may vary in $[h_{\text{min}}, ..., h_{\text{max}}]$
- Preserving of the stability during timing change
- Meeting a specified performance

Possible solutions:
Bank of $n$ correctors for $n$ periods:

- Tedious when $n$ large
- $a$ posteriori study of the transitions between controllers
- Cost of the implementation
Computer load aware control

Method: Design of a controller
- Which period may vary in \([h_{\text{min}}, \ldots, h_{\text{max}}]\)
- Preserving of the stability during timing change
- Meeting a specified performance

Paper objective:
Synthesis of a variable sampling rate controller using a Linear Parameters Varying (LPV) method.
Recall: LPV design methods

LPV system: discrete time system with $\theta_k \in \Theta \subset \mathbb{R}^n$ vector of varying parameters:

$$
\begin{align*}
    x_{k+1} &= A(\theta_k)x_k + B(\theta_k)u_k \\
    y_k &= C(\theta_k)x_k + D(\theta_k)u_k
\end{align*}
$$
Recall : LPV design methods

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$$

LPV method : synthesis of a controller parametrised by $\theta_k$

[Apkarian et Gahinet (1995)]

• Consider the period $h$ as the varying parameter ;
• Continuous linear system $\Rightarrow$ discrete time LPV model ;
• Synthesis of a $h$-dependent controller.
Parametrised discretization

Continuous time system:

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}
\]

Discrete time system:

\[
\begin{align*}
x_{k+1} &= A_d(h) x_k + B_d(h) u_k \\
y_k &= C_d x_k + D_d u_k
\end{align*}
\]
Parametrised discretization

Continuous time system:

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y_k &= C_d x_k + D_d u_k
\end{aligned}
\]

with

\[
\begin{aligned}
A_d(h) &= A_{h_0} A_{\delta h} \\
B_d(h) &= B_{h_0} + A_{h_0} B_{\delta h} \\
h &= h_0 + \delta h
\end{aligned}
\]

Methodology: Develop $A_d(h)$ and $B_d(h)$ in a polytopic way.
Recall : Polytopic model

Discrete time LPV polytopic model :

\[
\begin{align*}
    x_{k+1} &= A(\theta_k) x_k + B(\theta_k) u_k \\
    y_k &= C(\theta_k) x_k + D(\theta_k) u_k
\end{align*}
\]

with :

- \( A(\theta_k), B(\theta_k), C(\theta_k), D(\theta_k) \) affine en \( \theta_k \in \Theta \)
- \( \Theta \) is a polytope with \( k \) vertices \( \omega_i \)
- \( \omega_i = \theta_k^{\min}, \theta_k^{\max} \)
- \( \theta_k = \sum_{i=1}^{k} \alpha_i(k) \omega_i \) with \( \alpha_i(k) \geq 0, \sum_{i=1}^{k} \alpha_i(k) = 1 \)

\[
\begin{pmatrix}
    A(\theta_k) & B(\theta_k) \\
    C(\theta_k) & D(\theta_k)
\end{pmatrix}
= \sum_{i=1}^{k} \alpha_i(k)
\begin{pmatrix}
    A_i & B_i \\
    C_i & D_i
\end{pmatrix}
\]
Polytopic model: Taylor’s expansion of $\exp(Ah)$

Affine form via a Taylor’s expansion:

Affine form in $\delta^i \delta_h$, $i = 1 \ldots N$:

\[
A_d(h) = A_{h_0} A_{\delta_h} = A_{h_0} e^{A \delta_h} \approx A_{h_0} \left( \sum_{i=1}^{N} \frac{A^i}{i!} \delta^i_h \right)
\]

\[
B_d(h) = B_{h_0} + A_{h_0} B_{\delta_h} = B_{h_0} + A_{h_0} \left( \sum_{i=1}^{N} \frac{A^{i-1} B}{i!} \delta^i_h \right)
\]

with $h = h_0 + \delta_h$

N parameters $\Rightarrow 2^N$ vertices!
Approximation error

Study of the order $N$ for the series:

- Error criterion $J_M(h) = \|G_d(h, z) - \hat{G}_{d_N}(h, z)\|_\infty$
- Example on a stable 2nd order system
Polytope size reduction

Transform into a polytopic model with:

- $\theta_k = (\delta_h, \delta^2_h, \ldots, \delta^N_h)^T$
- State matrices affine in $\theta_k$

Exploiting the parameters dependency to reduce the polytope:
Polytope size reduction

Transform into a polytopic model with:
- \( \theta_k = (\delta_1, \delta_2^2, \ldots, \delta_N^N)^T \)
- State matrices affine in \( \theta_k \)

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\( N=2 \)
Polytope size reduction

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Exploiting the parameters dependency to reduce the polytope:

\( N=2 \)
Polytope size reduction: general case

\[ h = h_{\text{min}} + \delta, \quad 0 \leq \delta \leq \delta_{\text{max}}, \quad \delta_{\text{max}} = h_{\text{max}} - h_{\text{min}} \]

\[(0, 0, 0, \ldots, 0)\]
\[(\delta_{\text{max}}, 0, 0, \ldots, 0)\]
\[(\delta_{\text{max}}, \delta_{\text{max}}^2, 0, \ldots, 0)\]
\[\vdots\]
\[(\delta_{\text{max}}, \delta_{\text{max}}^2, \delta_{\text{max}}^3, \ldots, \delta_{\text{max}}^N)\]

\(N + 1\) vertices rather than \(2^N\)!

\(k\) influences the conservatism and complexity
- of the synthesis \((2k + 1)\) LMIs, \(k = N + 1\) or \(2^N\)
- of the real-time implementation: convex combination of \(k\) controllers
Recall: Polytopic LPV controller


- Internal stability of the closed loop
- $\| z \|_2 < \gamma \| w \|_2$

for all trajectory $h$ in $H$

**Polytopic controller:**

$$K(H): \begin{cases} \tilde{x}_{k+1} = A_K(H)\tilde{x}_k + B_K(H)y_k \\ u_k = C_K(H)\tilde{x}_k + D_K(H)y_k \end{cases}$$

with:

$$\begin{pmatrix} A_K(H) & B_K(H) \\ C_K(H) & D_K(H) \end{pmatrix} = \sum_{i=1}^{N} \alpha_i(k) \begin{pmatrix} A_{K_i} & B_{K_i} \\ C_{K_i} & D_{K_i} \end{pmatrix}$$
Performance specification

- $H_\infty$ framework
- find a controller $K$ such internal stability is achieved
- $\|\tilde{z}\|_2 < \gamma \|\tilde{w}\|_2$, where $\gamma$ is the $H_\infty$ attenuation level
Performance specification

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- find a controller $K$ such internal stability is achieved
- $\|\tilde{z}\|_2 < \gamma \|\tilde{w}\|_2$, where $\gamma$ is the $H_\infty$ attenuation level

![Diagram of control system]

- closed-loop performance strongly depends on the sampling period → make the performance specification vary with the period
- weighting functions are sampling rate dependent $W_i(h), W_o(h)$
- smaller the period, faster the system: $\omega_S = \alpha/h$
Discrete LPV synthesis

- Solving \((2k + 1)\) LMI to synthesise \(k\) fixed gains controllers vertex controllers (off-line)

- On-line convex combination of the vertex controllers giving:

\[
K(h) = K(\alpha_1, \ldots, \alpha_k) = \sum_{j=1}^{k} \alpha_j K_j
\]

- \(\alpha_i(h)\) polytopic coordinates (explicit recursive solution for the reduced polytope)

- \text{sizeof } K(h) = \text{sizeof } K_j
Experiment

open-loop unstable, non-minimum phase, under-actuated

$\Rightarrow$ difficult to control
Modelling

\[
\begin{pmatrix}
m_1 & m_1 l_0 \\
\dot{m}_1 l_0 & \ddot{J}
\end{pmatrix}
\begin{pmatrix}
\ddot{z} \\
\ddot{\theta}
\end{pmatrix}
+ \begin{pmatrix}
-f_{vz} & -m_1 z \dot{\theta} \\
2m_1 z \dot{\theta} & 0
\end{pmatrix}
\begin{pmatrix}
\ddot{z} \\
\ddot{\theta}
\end{pmatrix}
+ \begin{pmatrix}
-m_1 \sin \theta \\
-(m_1 l_0 + m_2 l_c) \sin \theta - m_1 z \cos \theta
\end{pmatrix}
g = \begin{pmatrix}
u \\
0
\end{pmatrix}
\] (1)

<table>
<thead>
<tr>
<th>Name</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_1$</td>
<td>0.217 kg</td>
<td>horizontal sliding rod mass</td>
</tr>
<tr>
<td>$m_2$</td>
<td>1.795 kg</td>
<td>vertical rod mass</td>
</tr>
<tr>
<td>$l_0$</td>
<td>0.33</td>
<td>vertical rod length</td>
</tr>
<tr>
<td>$l_c$</td>
<td>-0.032 m</td>
<td>vertical rod position of the centre of gravity</td>
</tr>
<tr>
<td>$g$</td>
<td>9.81 m.s$^{-2}$</td>
<td>gravity acceleration</td>
</tr>
<tr>
<td>$\ddot{J}$</td>
<td>0.061 Nm$^2$</td>
<td>Nominal inertia</td>
</tr>
<tr>
<td>$f_{vz}$</td>
<td>0.1 kg.s$^{-1}$</td>
<td>viscous friction</td>
</tr>
</tbody>
</table>
Modelling

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -l_0 \dot{x}_4 + x_1 x_4^2 + g \sin x_3 - \frac{f_{vz}}{m_1} x_2 + \frac{u}{m_1} \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{1}{J_0(x_1) - m_1 l_0^2} ( + g (m_1 x_1 \cos x_3 + m_2 l_c \sin x_3) \\
&\quad - m_1 (l_0 x_4 + 2 x_2) x_1 x_4 - l_0 u )
\end{align*}
\]

\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 \\
\frac{-l_0 g m_1}{J-m_1 l_0^2} & -\frac{f_{vz}}{m_1} & \frac{-l_0 g m_2 l_c}{J-m_1 l_0^2} + g & 0 \\
0 & 0 & \frac{g m_1}{J-m_1 l_0} & 0 \\
\frac{gm_1}{J-m_1 l_0} & 0 & \frac{g m_2 l_c}{J-m_1 l_0^2} & 0 \\
\end{pmatrix},
B = \begin{pmatrix}
0 \\
\frac{l_0^2}{J-m_1 l_0^2} + \frac{1}{m_1} \\
0 \\
\frac{-l_0}{J-m_1 l_0^2} \\
\end{pmatrix}
\]
Performance specification

General control configuration:

\[ r \rightarrow K \rightarrow G \rightarrow y \]

\[ W_u \rightarrow \tilde{u} \rightarrow \tilde{e} \]

Sampling period range: [1,3] msecs
Performance specification

control objectives:

\[ W_e(p, f) = \frac{p M_S + \omega_S(f)}{p + \omega_S \epsilon_S} \quad \omega_S(f) = h_{\min} \omega_{S_{\max}} f \]

\[ W_u(p, f) = \frac{1}{M_U} \]

where \( f = 1/h \), \( \omega_{S_{\max}} = 1.5 \text{ rad/s} \), \( M_S = 2 \), \( \epsilon_S = 0.01 \) and \( M_U = 5 \).

discretization leads to constant weighting functions...

discrete time mixed sensitivity problem:

\[ \left\| \begin{bmatrix} W_e(I - M S_y G K_1) & W_e M S_y G \\ W_u S_u K_1 & W_u T_u \end{bmatrix} \right\|_\infty \leq \gamma \]
Taylor expansion errors

Approximation error estimate: \( J_M(h) = \| G_d(z, h) - \tilde{G}_{dM}(z, h) \|_\infty \)
### LPV/$H_\infty$ design

<table>
<thead>
<tr>
<th>Polytope</th>
<th>Nb vertices</th>
<th>$\gamma_{opt}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Taylor order N=2</td>
<td>full</td>
<td>4</td>
</tr>
<tr>
<td>Taylor order N=2</td>
<td>reduced</td>
<td>3</td>
</tr>
<tr>
<td>Taylor order N=4</td>
<td>full</td>
<td>16</td>
</tr>
<tr>
<td>Taylor order N=4</td>
<td>reduced</td>
<td>5</td>
</tr>
</tbody>
</table>
LPV/$H_\infty$ design

Sensitivity functions (N=2, reduced polytope, 10 frozen values of $h$)
Response time of the non-linear process
Simulation

[Graphs showing the angle of a pendulum and system response over time]

Angle du pendule

Commande

Période d'échantillonnage

ECC'07@Kos – SafeNeCS invited session – p.21/23
Matlab/Simulink using the Real-time Workshop and xPC Target

Pendulum angle

Control input

Sampling period

Time [s]

θ [rad]

u [v]

$h$ [s]
Summary and ongoing work

• Design of a sampling varying controller using the LPV framework
  ◦ Efficient polytopic approach
  ◦ Sampling rate dependent performance

• Advantages :
  + Unique synthesis w.r.t. specified performance
  + Stability is guaranteed for all period transitions
  + Moderate complexity and experimental feasibility

• Further use in control/scheduling architectures
  ◦ Compliant with simple feedback-schedulers
  ◦ Ongoing investigate robustness w.r.t. timing uncertainties