CONVOLUTION FILTERING AND MATHEMATICAL MORPHOLOGY ON AN IMAGE: A UNIFIED VIEW

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ABSTRACT

In this paper, we propose to show that a particular fuzzy extension of the mathematical morphology coincides with a non-additive extension of the linear filtering based on convolution.

***Index Terms*—** linear filtering, possibility theory, morphology

1. INTRODUCTION

In image processing, convolution filtering and mathematical morphology are generally distinguished.

Filtering by convolution on an image groups many techniques that originate from the signal processing domain. These techniques are based on the interpretation of a signal as a distribution in the sense of Schwartz [1]. This approach consists in convoluting the image to analyse by a special function called the convolution kernel. The convolution kernel is shifted on the whole set of pixels of the image in order to emphasize or to attenuate some features of the image. The modified features strongly depend on the choice of the convolution kernel.

Mathematical morphology on an image groups many techniques that originate from the pioneering works of Jean Serra [2]. These techniques are set operations based on Minkowski additions and subtractions. This approach consists in modifying the original image by the use of a pattern, called the structuring element. The structuring element is shifted on the whole set of pixels of the image in order to emphasize or to attenuate some features of the image. The modified features strongly depend on the choice of the structuring element.

This presentation is intentionally quite provocative. It aims at stressing an obvious analogy between linear filtering and mathematical morphology. This analogy was already noted in [3]:

> The structuring element is to mathematical morphology what the convolution kernel is to linear filter theory.

Until now, this analogy is just in the philosophy of these techniques but not in their mathematical expressions. Indeed, convolution filtering and mathematical morphology are generally considered as being complementary approaches but not compatible. In this paper, we propose a unified view of these two approaches. More precisely, we prove that the image obtained by dilation (resp. erosion) of an original image is the upper (resp. lower) bound of all the images that would have been obtained by filtering the original image with a particular family of convolution kernels. Filtering with a family of convolution kernels is not a straightforward extension of filtering with a unique convolution kernel. Performing all the filtering with all the kernels would not be satisfactory regarding the computational cost increase. In this paper, we show how the possibility theory can represent a particular family of convolution kernels and how the Choquet integral is used to extend the corresponding filtering.

This paper is organized as follows. Section 2 presents the possibilistic extension of the usual linear filtering. Section 3 presents a particular fuzzy extension of the usual mathematical morphology based on level cuts. Section 4 summarizes and concludes about the links of these methods.

2. KERNEL-BASED IMAGE FILTERING

2.1. Convolution kernel-based image filtering

Let I be a discrete image of N pixels. Let I_i be the measure of the illumination value associated with the i^{th} pixel and $\Omega = \{1, ..., N\}$ be the set of all the pixels. Filtering I by a filter, defined by its impulse response κ, mathematicaly corresponds to the discrete convolution of I by κ. That's the reason why κ can also be called the convolution kernel. $\hat{I}_n$, the illumination value of the filtered image associated with the n^{th} pixel of Ω is thus obtained by:

$$\hat{I}_n = \sum_{i=1}^{N} I_i \kappa_i^n. \quad (1)$$

where $\kappa^n = (\kappa_i^n)_{i=1,...,N}$ is the convolution kernel κ shifted at the n^{th} pixel of Ω.

In many applications like denoising, low-pass filtering or registrating, the convolution kernels that are used are positive and have a unitary gain, i.e. $\sum_{i=1}^{N} \kappa_i = 1$. In that case, the convolution kernel can be seen as a probability distribution. Therefore, a discrete convolution kernel κ induces a discrete probability measure P_κ, computed in this way:

$$\forall A \subseteq \Omega,\ P_\kappa(A) = \sum_{i \in A} \kappa_i, \tag{2}$$

For each pixel, its associated shifted convolution kernel κ^n is still a probability distribution. Thus, expression (1) is equivalent to computing the expected value $\hat{I}_n$ of the illumination I_n at the pixel n, considering the probability measure P_{κ^n} over the set of the measured illuminations I, i.e.:

$$\hat{I}_n = \mathbb{E}_{P_{\kappa^n}}(I). \tag{3}$$

2.2. Extension of image filtering to possibility theory

By writing the linear filtering with a unitary gain filter as an expectation according to a probability measure, we open new perspective to this approach by looking in the field of the new uncertainty theory. Instead of using an additive measure for each neighborhood of a pixel, i.e. a probability measure, we propose to use a non-additive confidence measure called a possibility measure [4]. Besides, we propose to use the Choquet integral that extends the usual linear filtering by extending the convolution operator to possibility measures in place of probability measures. This section presents and interprets this new filtering approach, based on possibility measures and Choquet integrals, that enables filtering an image with a family of convolution kernels.

2.2.1. A possibility measure is a family of filters

Since a possibility measure is non-additive, $\Pi(A) \neq 1 - \Pi(A^c)$, where A^c is the complementary set of A in Ω. Therefore, a possibility measure also defines a dual confidence measure, called a necessity measure, noted N and computed in this way:

$$\forall A \subseteq \Omega,\ N(A) = 1 - \Pi(A^c), \tag{4}$$

These two measures, Π and N, encode a family of probability measures, noted $\mathcal{M}(\Pi)$, and defined by:

$$\mathcal{M}(\Pi) = \{P \mid \forall A \subseteq \Omega, N(A) \leq P(A) \leq \Pi(A)\}.$$

This encoding property is due to the imprecise probability [5, 6] interpretation of the possibility theory.

A possibility measure can be defined from a possibility distribution π^n. Such a distribution is normalized in the sense that $\sup_{i \in \Omega} \pi_i^n = 1$. Its associated possibility measure is obtained by:

$$\forall A \subseteq \Omega,\ \Pi_{\pi^n}(A) = \sup_{i \in A} \pi_i^n, \tag{5}$$

Thus a unique possibility distribution π^n can encode a whole family of convolution kernels κ^n with unitary gain, noted $\mathcal{M}(\pi^n)$ and defined by:

$$\mathcal{M}(\pi^n) = \{\kappa^n \mid \forall A \subseteq \Omega, N_{\pi^n}(A) \leq P_{\kappa^n}(A) \leq \Pi_{\pi^n}(A)\}.$$

2.2.2. The possibilistic extension of the linear filtering

Since a possibility measure is non-additive, the conventional expectation operator cannot be used for linear filtering. The expectation operator must be replaced by its generalization, called the Choquet integral. Using a Choquet integral and a possibility distribution leads to an interval-valued expectation, instead of a single value, whose upper and lower bounds are computed in this way:

$$\overline{I}_n = \mathbb{C}_{\Pi_{\pi^n}}(I) = \sum_{i=1}^{N} \Pi_{\pi^n}(A_{(i)})(I_{(i)} - I_{(i-1)}), \tag{6}$$

$$\underline{I}_n = \mathbb{C}_{N_{\pi^n}}(I) = \sum_{i=1}^{N} N_{\pi^n}(A_{(i)})(I_{(i)} - I_{(i-1)}). \tag{7}$$

The index notation $(.)$ indicates a permutation that sorts the pixels such that $I_{(1)} \leq I_{(2)} \leq ... \leq I_{(N)}$ and $A_{(i)}$ is a set of pixels whose value is greater than $I_{(i)}$, i.e. $A_{(i)} = \{j \in \{1, ..., N\} | I_j > I_{(i)}\}$. By convention, $I_{(0)} = 0$.

The Choquet integral can be considered as a generalization of the conventional expectation operator since, when the used confidence measure is a probability measure, expressions (6) and (7) coincide and equal to the conventional expectation operator (3).

The key point of this approach is that the interval-valued expectation obtained by using a possibility distribution is the set of all the single-valued expectations obtained by using all the convolution kernels encoded by the considered possibility distribution. This property is shown in [7] and derived from a domination theorem proved by Denneberg [8], proposition 10.3 and Schmeidler [9], proposition 3:

Theorem 1 *Let π^n be a possibility distribution. For all I such that $\mathbb{C}_{\Pi_{\pi^n}}(|I|) < +\infty$, we have that*

$$\forall \kappa^n \in \mathcal{M}(\pi^n),\ \mathbb{C}_{N_{\pi^n}}(I) \leq \mathbb{E}_{P_{\kappa^n}}(I) \leq \mathbb{C}_{\Pi_{\pi^n}}(I). \tag{8}$$

Therefore, using a possibility distribution allows the modeling of a lack of knowledge on the proper convolution kernel to be used. Using the generalized expectation operator (6) and (7) directly impacts this ill-knowledge on the output. In [10], it is shown that the length of the interval $[\mathbb{C}_{N_{\pi^n}}(I), \mathbb{C}_{\Pi_{\pi^n}}(I)]$ depends on the specificity of π^n, i.e. on the size of the family of convolution kernels $\mathcal{M}(\pi^n)$. Obviously, the size of this family is a marker of the lack of knowledge on the convolution kernel to use and it is impacted on the size of the resulting interval by this approach.

3. FUZZY MORPHOLOGY BY LEVEL CUTS

Erosion and dilation are the two dual basic operations of the mathematical morphology. Their names come from the effects that these operators produce on binary images. Such a binary image is often obtained by classifying the pixels as belonging or not to a particular category of pixels. The result of this classification is a binary image I, i.e. $I_i \in \{0,1\}$, $\forall i \in \{1,...,N\}$. For instance, a medical image of a brain can be segmented by a nuclear medicine physician into a gray matter region and a white matter region. Similarly, the structuring element to be used for performing morphology operations is a binary pattern ν, shifted on the pixels $n \in \{1,...,N\}$. The structuring element is noted ν^n when it is shifted on the pixel n.

The dilation of a binary (classified) image I is the classified image:

$$DI_n = \mathbb{1}_{\{j|\nu_j^n=1\}\cap\{j|I_j=1\}\neq\emptyset},\ \forall n \in \{1,...,N\}. \quad (9)$$

The value at the n^{th} pixel of the dilated image DI equals 1 if there exist a pixel j of the neighborhood defined by shifting the structuring element in n, that is included in the set of the pixels of the image I whose value equals 1. The value of a pixel n of the dilated image DI equals 0 otherwise.

The erosion of a binary (classified) image I results in the eroded binary (classified) image:

$$EI_n = \mathbb{1}_{\{j|\nu_j^n=1\}\subseteq\{j|I_j=1\}},\ \forall n \in \{1,...,N\}. \quad (10)$$

The value at the n^{th} pixel of the eroded image EI equals 1 if all the pixels of the neighborhood defined by shifting the structuring element in n, are included in the set of the pixels of the image I whose value equals 1. The value of a pixel n of the eroded image EI equals 0 otherwise.

The considered extension of these operators is called a fuzzy morphology, because the structuring element and the pixels are no more seen as crisp binary elements but as fuzzy sets. This approach (definition 1 in [11]) makes an extensive use of the level cuts of these fuzzy sets and of the underlying image.

The image I is no more binary but is grey-leveled, i.e. $\forall i \in \{1,...,N\}$, $I_i \in \mathbb{R}^+$. For extending the usual binary approach, binary images are extracted form this grey-leveled image, by using the γ-cuts of I. These binary images, noted I^γ, are defined by:

$$I_i^\gamma = 1, \text{ if } I_i \geq \gamma,$$
$$I_i^\gamma = 0, \text{ otherwise.}$$

As for the partitionning of the image, the usual binary approach is based on punctual pixels. Mathematically, every pixel is modeled by a Dirac distribution on the center of the pixel. In the fuzzy approach, for each pixel i, a fuzzy subset ς^i is defined that quantifies the membership of an element ω of the underlying infinite 2D domain of Ω to the neighborhood ς^i of the pixel i. Its α-cuts are defined by $\varsigma_\alpha^i = \{\omega \in \Omega | \varsigma^i(\omega) \geq \alpha\}$ for each neighborhood of pixel.

In the same way, the structuring element, positioned on a pixel n is modeled by a fuzzy subset ν^n that corresponds to the membership of an element ω of the underlying infinite 2D domain of Ω to the structuring element ν^n shifted on the pixel n. Its α-cuts are defined by $\nu_\alpha^n = \{\omega \in \Omega | \nu^n(\omega) \geq \alpha\}$. The range of α for ν^n and ς^i is $[0,1]$ because they are membership functions of fuzzy subsets [12].

The fuzzy generalization of the dilation that we consider here first applies the usual binary morphology operation to given levels α and γ. It checks if for the α-level cut of the structuring element ν_α^n and the γ-level cut of the image I^γ have a non empty intersection. In that case, the value of the dilated image on n for the levels α and γ equals 1.

Mathematically, $\forall n \in \{1,...,N\}$,

$$DI_n^{\gamma\alpha} = \sup_{i=1,...,N} \Big[\sup_{\omega\in\Omega} \big[\mathbb{1}_{\varsigma_\alpha^i}(\omega)\mathbb{1}_{\nu_\alpha^n}(\omega)\big] I_i^\gamma \Big]. \quad (11)$$

The integration of all these values $DI_n^{\gamma\alpha}$ (for all the levels γ and α, for each pixel $n \in \{1,...,N\}$) leads to the following fuzzy extension:

$$\begin{aligned} DI_n &= \int_0^{+\infty}\int_0^1 DI_n^{\gamma\alpha} d\alpha d\gamma, \\ &= \int_0^{+\infty}\int_0^1 \sup_{i=1,...,N} \Big[\sup_{\omega\in\Omega} \big[\mathbb{1}_{\varsigma_\alpha^i}(\omega)\mathbb{1}_{\nu_\alpha^n}(\omega)\big] I_i^\gamma \Big] d\alpha d\gamma, \\ &= \int_0^{+\infty} \sup_{i=1,...,N} \Big[\sup_{\omega\in\Omega} \big[\int_0^1 \mathbb{1}_{\varsigma_\alpha^i}(\omega)\mathbb{1}_{\nu_\alpha^n}(\omega) d\alpha \big] I_i^\gamma \Big] d\gamma. \end{aligned}$$

We can see that $\forall \alpha \in [0,...,1]$ and $\forall \omega \in \Omega$, $\mathbb{1}_{\varsigma_\alpha^i}(\omega)\mathbb{1}_{\nu_\alpha^n}(\omega) = \mathbb{1}_{\varsigma_\alpha^i \cap \nu_\alpha^n}(\omega)$. This characteristic function equals 1 until a level α_0, because the level cuts of the maxitive kernels ς^i and ν^n are stacked. α_0 can equal 0 if ω is not in the support of the intersection of the maxitive kernels ν^n and ς^i. Thus, $\int_0^1 \mathbb{1}_{\varsigma_\alpha^i \cap \nu_\alpha^n}(\omega) d\alpha = \int_0^{\alpha_0} \mathbb{1}_{\varsigma_\alpha^i \cap \nu_\alpha^n}(\omega) d\alpha = \alpha_0 = \sup\{\alpha | \omega \in \varsigma_\alpha^i \cap \nu_\alpha^n\}$. Such an expression is identified to the membership function of a fuzzy subset (see [4]) defined by $min(\varsigma^i(\omega), \nu^n(\omega))$. To sum up:

$$\int_0^1 \mathbb{1}_{\varsigma_\alpha^i}(\omega)\mathbb{1}_{\nu_\alpha^n}(\omega) d\alpha = min(\varsigma^i(\omega), \nu^n(\omega)).$$

Therefore, we obtain:

$$DI_n = \int_0^{+\infty} \sup_{i=1,...,N} \Big[\sup_{\omega\in\Omega} min(\varsigma^i(\omega), \nu^n(\omega)) I_i^\gamma \Big] d\gamma.$$

Let's define

$$\pi_i^n = \sup_{\omega\in\Omega} min(\varsigma^i(\omega), \nu^n(\omega)).$$

π_i^n is a possibility distribution. Indeed, $\pi_n^n = 1$. The associated possibility measure can be written as: $\forall A \subseteq \{1,...,N\}$,

$$\Pi_{\pi^n}(A) = \sup_{i\in A} \pi_i^n = \sup_{i=1,...,N} \mathbb{1}_A(i)\pi_i^n.$$

Therefore, from the definition of I^γ, we have:

$$DI_n = \int_0^{+\infty} \Pi_{\pi^n}(\{j \in \{1,...,N\} \mid I_j \geq \gamma\}) d\gamma.$$

Reminded that the index notation $(.)$ indicates a permutation that sorts the pixels such that $I_{(1)} \leq I_{(2)} \leq ... \leq I_{(N)}$ and $A_{(i)}$ is a set of pixels whose value is greater than $I_{(i)}$, i.e. $A_{(i)} = \{j \in \{1,...,N\} | I_j > I_{(i)}\}$, we have that $\forall \gamma \in [0, \infty[$, there exists $i \in \{1,...,N\}$, such that $I_{(i-1)} < \gamma \leq I_{(i)}$. Therefore, for a given $\gamma \geq 0$, $\exists i \in \{1,...,N\}$, such that $\{j \in \{1,...,N\} | I_j \geq \gamma\} = \{(i),...,(n)\} = A_{(i)}$. Therefore,

$$DI_n = \sum_{i=1}^{N} \int_{I_{(i-1)}}^{I_{(i)}} \Pi_{\pi^n}(A_{(i)}) d\gamma, \text{ thus,}$$

$$DI_n = \sum_{i=1}^{N} (I_{(i)} - I_{(i-1)}) \Pi_{\pi^n}(A_{(i)}). \quad (12)$$

With the same calculus, the fuzzy erosion can be obtained by:

$$EI_n = \sum_{i=1}^{N} (I_{(i)} - I_{(i-1)}) N_{\pi^n}(A_{(i)}). \quad (13)$$

4. CONCLUSION

The obvious conclusion of this paper is that the fuzzy dilation and erosion are respectively equivalent to the upper and lower bound of a maxitive kernel based filtering. Indeed, expressions (12) and (6) are equal, as well as expressions (13) (7). According to our knowledge, this is the first time that the mathematical morphology and the linear filtering are meaningfully linked. This new insight could be the basis of numerous further development for each approach. It could be possible to use one operator in place of the other. As a clue of this possible interplay is the edge detection: Both the linear filtering and the mathematical morphology are used for detecting edges on an image. By the way, in a previous paper [13], we proposed an edge detector based on the extension of the filtering with possibility theory.

5. REFERENCES

[1] L. Schwartz, *Théorie des distributions*, vol. 1, Hermann, Paris, 1950.

[2] J. Serra, *Image analysis and mathematical morpholgy*, Academic Press, inc., London, 1982.

[3] A. Guillen, Ph. Calcagno, G. Courrioux, A. Joly, and P. Ledru, "Geological modelling from field data and geological knowledge: Part ii. modelling validation using gravity and magnetic data inversion," *Physics of the Earth and Planetary Interiors*, vol. 171, pp. 158–169, 2008.

[4] D. Dubois and H. Prade, *Théorie des possibilités : Applications à la représentation des connaissances en informatique*, Masson, 1988.

[5] P. Walley, *Statistical Reasoning with Imprecise Probabilities*, Chapman and Hall, New York, 1991.

[6] P. Walley, "Towards a unified theory of imprecise probability.," *Int. J. Approx. Reasoning*, vol. 24, no. 2-3, pp. 125–148, 2000.

[7] K. Loquin and O. Strauss, "Imprecise functional estimation: the cumulative distribution case," in *SMPS08, Soft Methods in Probability and Statistic*, Toulouse, September 2008.

[8] D. Denneberg, *Non Additive Measure and Integral*, Kluwer Academic Publishers, Dordrecht, 1994.

[9] D. Schmeidler, "Integral representation without additivity," in *Proceedings of the American Mathematical Society*, June 1986, vol. 97 of *2*, pp. 255–261.

[10] K. Loquin, *De l'Utilisation des Noyaux Maxitifs en Traitement de l'Information*, Ph.D. thesis, LIRMM - UM2, 2008.

[11] I. Bloch and H. Maitre, "Fuzzy mathematical morphologies: A comparative study," *Pattern Recognition*, vol. 28, no. 9, pp. 1341–1387, Sept. 1995.

[12] L.A. Zadeh, "Fuzzy sets as a basis for a theory of possibility," *Fuzzy Sets and Systems*, vol. 1, pp. 3–28, 1978.

[13] F. Jacquey, K. Loquin, F. Comby, and O. Strauss, "Non-additive approach for gradient-based edge detection," in *ICIP'07 International Conference on Image Processing*, San Antonio, Texas, USA, September 2007, pp. III: 49–52.