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International Journal of Approximate Reasoning xxx (2010) xxx–xxx

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

journal homepage: www.elsevier.com/locate/ijar

Imprecise expectations for imprecise linear filtering

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ARTICLE INFO

Article history:
Received 9 March 2009
Received in revised form 14 June 2010
Accepted 25 June 2010
Available online xxxx

Keywords:
Linear time-invariant filtering
Expectation
Interval valued signal
Choquet integral
Non-additive confidence measure

ABSTRACT

In most sensor measure based applications, the raw sensor signal has to be processed by an appropriate filter to increase the signal-to-noise ratio or simply to recover the signal to be measured. In both cases, the filter output is obtained by convoluting the sensor signal with a supposedly known appropriate impulse response. However, in many real life situations, this impulse response cannot be precisely specified. The filtered value can thus be considered as biased by this arbitrary choice of one impulse response among all possible impulse responses considered in this specific context. In this paper, we propose a new approach to perform filtering that aims at computing an interval valued signal containing all outputs of filtering processes involving a coherent family of conventional linear filters. This approach is based on a very straightforward extension of the expectation operator involving appropriate concave capacities.

1. Introduction

In the last 10 years, there has been increasing interest in interval valued data [17]. Replacing a precise value by an interval value generally reflects the variability or uncertainty that underlies the observation process. Different interpretations are possible. For example, an interval value can be seen as: a range in which one could have a certain level of confidence of finding the true value of the observed variable [33], a range of values that the real data can have when the measurement process involves quantization and/or sampling [17,18], a representation of the known detection limits, sensitivity or resolution of a sensor [21], etc.

In signal processing, it is often very difficult to gain access to a process with truly reliable information about the properties of the expected variations and errors that contaminate the observations. In the automatic control context, some very well established state estimation processes, e.g. Kalman filtering, can lead to algorithms based on some very intricate frameworks involving sets of parameters that are very difficult to adjust even for a process control expert. To overcome this difficulty, some interval observation based algorithms have been proposed for state estimation (see, e.g. [31,4]). The same problem arises in the sensor fusion setting. In [33], an interval estimation fusion based on sensor interval estimates is considered that leads to fused values that have advantages with respect to their precise counterparts in terms of robustness, specificity and ease of extension to higher dimensions. In [1], Brito considered replacing precise valued by interval valued data to perform a robust data analysis. In the same way, Denoeux et al. propose different extensions of statistical tools to deal with interval valued data [8,7].

However, within any interval-based signal processing application, there is still a strong need for a reliable representation of the variability domain of each involved observation. An important issue concerns the meaning of the interval and the consistency of this meaning with respect to the tools used for further analysis or processing, as recommended by the *Guide to the*

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doi:10.1016/j.ijar.2010.06.003

Expression of Uncertainty in Measurement [12]. In many practical applications, a confidence interval with a probability of one is too wide to obtain useful values. Therefore, one challenge of practical interest is to be able to obtain an interval valued observation with a particular guaranteed property but which is also as specific as possible.

Single or double bootstrapping has been proposed as a nice way for recovering the confidence interval of an estimated value [25]. However, the computing resources required by such methods may be prohibitive. In [19], a Monte-Carlo method based on a Cauchy distribution is used to provide the interval valued expected error of a function whose inputs are intervals. Quantile estimates can also be used to obtain specific confidence intervals [3].

Random set theory could be required to use these interval data (see, e.g. [15] and references therein). Kreinovich et al. [21] propose to extend the expectation or variance of interval valued data. In [14], the authors propose to design an interval observer of a time varying process that provides online estimation of upper and lower bounds of unmeasured states of observations. This estimation is said to contain, with certainty, the true observation based on supposed pre-knowledge of the input signal bounds and reasonable knowledge on the equation that applies to the observed process. An interesting property is that generally the width of the observed interval directly depends on the uncertainty range of the unknown variables.

This property can be very useful for signal processing applications. In fact, it would be necessary to account for known pre-calibrated observation variability, but also for a lack of knowledge on the proper model to be used. In this context, we here propose a new interpretation of an interval valued piece of information.

Consider a linear process involving an expectation-like operation for which you have partial information about the suitable probability density function to be used. Thus, a way to account for this lack of knowledge is to consider every possible (suitable) function. Then the exact output value of the process cannot be precisely computed. It seems natural to replace the notion of precise expected value by an imprecise expected value that represents all possible precise expected values given by all possibly appropriate models. An interval can be a compact and useful representation of this imprecise expected value.

The problem of computing an expected value based on an ill-known probability has been proposed in the past in the decision making context (see [32] and references therein). The more general framework involves sets of probabilities called credal sets [26], an approach which often leads to computationally complicated methods involving at least linear constrained programming. A simpler way, which has been successfully used in the past (see references in [19]), to account for a partial lack of knowledge on the suitable probability to be used, is the p-box approach [30], where an interval-based representation of the cumulative distribution function (cdf) is considered. Some useful generalizations have also been proposed [9,10]. However, as shown by Kreinovich and Ferson in [20], this representation can sometimes lead to NP-hard computations or to estimates that are not specific enough, in contradiction with the underlying confidence representation. In [32], Yager and Kreinovich propose a precise expectation operator based on an interval valued probability density function. The output value accounts for the imprecise knowledge on the probability density function (pdf) but it does not provide the set of all outputs obtained by considering all the pdfs contained in the credal set induced by the imprecise pdf. A more specific and very useful way of representing a family of probability density functions is to consider a class of non-additive confidence measures called concave Choquet capacities [2].

In this paper, we propose to represent the notion of partial lack of probabilistic information by replacing the pdf by a concave capacity representing the credal set called the core of this capacity. We thus propose a very simple extension of the expectation operator whose output is an interval instead of a single value. We show that this interval is the smallest to include every precise expected value based on a probability that belongs to the core of the involved capacity. This extension makes use of asymmetric Choquet integrals [6,24]. Then, by considering conventional linear filtering as a linear combination of at most two linear expectations, we make use of our new interval valued expectation operator to design interval valued output filtering processes that contain all outputs of filtering processes involving a particular family of conventional linear filters. The methods we propose do not directly deal with the measurement uncertainty associated to the acquired samples, but rather with the uncertainty (or lack of experience) of the operator or with the intrinsic imprecision associated handling with continuous signals with digital operations. However, some links between our approach and measurement random variations exist that have been studied in [23].

This article is organized as follows: Section 2 presents the framework and notations. Section 3 deals with the construction of our imprecise expectation operator. Section 4 explains how to use our imprecise operator for performing imprecise filtering. Section 5 presents some illustrative experiments and comparisons of this new approach with the conventional approach.

2. Framework and notations

Basic definitions of finite impulse response filters, Choquet integrals and generalized real intervals are described in this section. See [5,13,6] for details on Choquet integrals and [17] concerning generalized real intervals.

2.1. Finite impulse response digital filters

In signal processing, filtering consists of modifying a real input signal by blocking pre-specified particular components (usually frequency components). By contrast with an analog filter, which directly operates on a continuous signal, a digital filter operates on digital samples and performs a mathematical manipulation that results in output samples having theoretically pre-required properties. Since the digital samples to be processed are usually obtained by sampling a continuous signal

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and since the output samples are generally converted in a continuous output signal via a D/A converter, most digital filters are designed to mimic analog filters. More precisely, the pre-required properties of the output digital signal are specified in the continuous domain.

In digital filtering, finite impulse response (FIR) filters are the most popular type of linear filters. They are usually defined by their responses to the individual frequency components that constitute the input signal. In this context, the mathematical manipulation consists of convolving the input samples with a particular digital signal called the impulse response of the filter. This impulse response is simply the response of the digital filter to a Kronecker impulse input.

Let $X = (X_n)_{n=1,\dots,N}$ be a sequence of N digital samples of a signal. Let $\rho = (\rho_i)_{i\in\mathbb{Z}}$ be the finite impulse response of the considered filter. The computation of Y_k, the kth component of the filter output, is given by $Y_k = \sum_{n=1}^{N}\rho_{k-n}X_n$. When the impulse response is positive and has a unitary gain ($\forall i \in \mathbb{Z},\ \rho_i \geqslant 0$ and $\sum_{i\in\mathbb{Z}}\rho_i = 1$), it can be considered as a probability mass function inducing a probability measure P on each subset A of $\mathbb{Z}$ by $P(A) = \sum_{i\in A}\rho_i$. These special types of impulse responses are often called *summative kernels* [22], or simply *kernels*, when used to ensure interplay between continuous and discrete domains. Thus, computing Y_k is equivalent to computing a discrete expectation operator involving a probability measure P_k induced by $(\rho_{k-n})_{n\in\mathbb{Z}}$, the probability distribution obtained by translating the probability distribution ρ over $k : Y_k = \sum_{n=1}^{N}\rho_{k-n}X_n = E_{P_k}(X)$. According to this interpretation, the probability P_k defines a probabilistic neighborhood of the kth sample. As the impulse response is finite, it has a bounded support, i.e. $\exists N \in \mathbb{N},\ i \notin [-N, N] \Rightarrow \rho_i = 0$. This bounded support will be referred to throughout this paper as the radius of the summative kernel.

2.2. Choquet integrals

A first straightforward idea to account for ill-known probability weights is to replace the precise probability in the expression of the discrete expectation operator by an interval valued probability $\left[\underline{\rho}_n, \bar{\rho}_n\right]$. However, as shown in [32], when this aggregation has to be consistent with the usual precise probability based aggregation, this approach leads to a very simple but precise expected value, which does not comply with the desired properties. Our aim is to obtain the set of all values that would have been obtained by considering every possible probability weight. Hence, in this paper, we propose a new method based on capacities and Choquet integrals.

Let Ω be a finite set of indices corresponding to samples of the signal and $\mathcal{P}(\Omega)$ be the power set of Ω. A capacity, or a confidence measure, can be defined on this reference set.

Definition 1. A capacity ν is a set function $\nu : \mathcal{P}(\Omega) \to [0, 1]$ such that $\nu(\emptyset) = 0$, $\nu(\Omega) = 1$, and $\nu(A) \leqslant \nu(B)$ for all $A \subseteq B$.

Given a capacity ν, its conjuguate ν^c, is defined as: $\nu^c(A) = 1 - \nu(A^c)$, for any subset A of Ω, with A^cbeing the complementary set of A in Ω. A capacity ν such that for all A, B in $\mathcal{P}(\Omega)$, $\nu(A \cup B) + \nu(A \cap B) \leqslant \nu(A) + \nu(B)$ is said to be concave (or submodular or 2-alternating). In this paper, we only consider concave capacities. The core of a capacity ν, denoted core(ν), is the set of probabilities P on $\mathcal{P}(\Omega)$ such that $\nu(A) \geqslant P(A)$ for all subsets A of Ω.

Definition 2. Let ν be a capacity on $\mathcal{P}(\Omega)$, and $X \in (\mathbb{R}^+)^N$ be a finite positive real function, then the Choquet integral of X with respect to ν is defined by:

$$C_\nu(X) = \sum_{n=1}^{N} X_{(n)}\left(\nu(A_{(n)}) - \nu(A_{(n+1)})\right),$$

where $X_{(n)}$ indicates that the indices have been permuted so that $X_{(1)} \leqslant \dots \leqslant X_{(N)}$, $A_{(i)} = \{(i), \dots, (N)\}$, and $A_{(N+1)} = \emptyset$.

In this paper, we need to use the Choquet integral of signals X which take on a negative value. Given $X \in \mathbb{R}^N$ a signal, X^- and X^+ are positive vectors defined as: $X_n^- = \max(-X_n, 0)$ and $X_n^+ = \max(X_n, 0)$, for all n in $\{1, \dots, N\}$, so $X = X^+ - X^-$.

Definition 3. Let $X \in \mathbb{R}^N$ be a signal and ν be a capacity on $\mathcal{P}(\Omega)$, then the asymmetric Choquet integral of X with respect to ν is defined by:

$$\check{C}_\nu(X) = C_\nu(X^+) - C_{\nu^c}(X^-).$$

In this paper, we use the following result proved by Denneberg [5]:

Theorem 4. *If ν is a concave capacity on $\mathcal{P}(\Omega)$, then for all $X \in \mathbb{R}^N$, $\check{C}_{\nu^c}(X) = \inf_{P\in\text{core}(\nu)} E_P(X)$ and $\check{C}_\nu(X) = \sup_{P\in\text{core}(\nu)} E_P(X)$, where E_P is the expectation operator corresponding to the probability P.*

So, if ν is a concave capacity, for all X in $\mathbb{R}^N$, we have $\check{C}_{\nu^c}(X) \leqslant \check{C}_\nu(X)$. Note that, if for all P in core(ν), $E_P(X)$ is a constant value, then $\check{C}_{\nu^c}(X) = \check{C}_\nu(X)$. And if X is a constant function equal to c, then $\check{C}_{\nu^c}(X) = \check{C}_\nu(X) = c$.

2.3. Generalized real intervals

A real interval is denoted $[a] = [\underline{a}, \bar{a}]$ with $\underline{a}, \bar{a}$ reals such that $\underline{a} \leqslant \bar{a}$ and the set of the real intervals is denoted $\mathbb{IR}$.

Definition 5. Let $[a]$ and $[b]$ be two real intervals.

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The Minkowski addition of $[a]$ and $[b]$ is defined by:

$$[a] \oplus [b] = [\underline{a} + \underline{b}, \bar{a} + \bar{b}].$$

The dual Minkowski addition of $[a]$ and $[b]$ is defined by:

$$[a] \boxplus [b] = [\min(\underline{a} + \bar{b}, \bar{a} + \underline{b}), \max(\underline{a} + \bar{b}, \bar{a} + \underline{b})].$$

If $0 \in [b]$, then the Minkowski addition can be interpreted as a dilation, and the dual Minkowski addition can be interpreted as an erosion [27]. We also define two subtraction operators:

Definition 6. Let $[a]$ and $[b]$ be two real intervals.

The Minkowski subtraction of $[a]$ and $[b]$ is defined by:

$$[a] \ominus [b] = [\underline{a}, \bar{a}] \oplus [-\bar{b}, -\underline{b}] = [\underline{a} - \bar{b}, \bar{a} - \underline{b}].$$

The dual subtraction of $[a]$ and $[b]$ is defined by:

$$[a] \boxminus [b] = [\underline{a}, \bar{a}] \boxplus [-\bar{b}, -\underline{b}] = [\min(\underline{a} - \underline{b}, \bar{a} - \bar{b}), \max(\underline{a} - \underline{b}, \bar{a} - \bar{b})].$$

The $\boxminus$ operator is identical to the difference operator defined by Hukahara in [16] when $[a] = [b] \oplus [x]$ has a solution for $[x]$.

In this paper the following properties are used.

Proposition 7. *Let $[a]$ and $[b]$ be two real intervals. For all x in $[a]$, y in $[b]$ and α, β two positive real numbers we have $\alpha x - \beta y \in \alpha[a] \ominus \beta[b]$.*

Proposition 8. *Let $[a]$ be a real interval, $[a] \ominus [a]$ equals $\{0\}$ if and only if $[a]$ is a singleton.*

3. Imprecise expectation operators corresponding to a concave capacity

Definition 9. The imprecise expectation operator corresponding to a concave capacity v is defined by:

$$\underline{\overline{E}}_v : \mathbb{R}^N \to \mathbb{IR} : X \mapsto [\check{C}_{v^c}(X), \check{C}_v(X)].$$

Using Definition 6, we can easily prove the following decomposition.

Proposition 10.

$$\forall X \in \mathbb{R}^N, \quad \underline{\overline{E}}_v(X) = \underline{\overline{E}}_v(X^+) \ominus \underline{\overline{E}}_v(X^-).$$

4. Imprecise expectation for imprecise filtering

As mentioned in Section 2.1, linear filtering consists of convolving an input signal with a particular function called the impulse response of the filter. Since precise knowledge of the filter's impulse response cannot usually be justified, it would be useful to be able to account for this lack of knowledge by replacing a single impulse response by a coherent family of impulse responses. What we mean by *coherent* is that this family should reflect, in a sense, the practitioner's difficulty in choosing the most appropriate filter in a specific context. For example, when considering a low-pass filter, a practitioner can be unsure about the order, or the type of filter to use (Butterworth, Bessel, Chebyshev, Elliptic, etc.). It can also be hard to define the appropriate cutoff frequency. In this last example, the coherent family of filters can be the set of all third order Butterworth filters having a cutoff frequency belonging to an interval $[f_{\min}, f_{\max}]$.

In this section, we propose to use the new operator presented in the previous section to provide new filtering tools. The output of such filters is interval valued. Each interval represents the convex hull of a set of all the single valued output obtained by each filter belonging to the considered family. We do not consider the impulse response as being a random function but rather as an imprecisely defined function. Our approach is based on the fact that any FIR filter can be rewritten as a weighted sum of at most two usual expectation operations. A coherent family of filters is modeled by using two concave capacities derived from the knowledge a practitioner has about the impulse response of one or more filters that should be appropriate for the considered application. We propose two approaches to derive an imprecise filter from a precise filter. The first approach consists of constructing the most specific possibilistic capacities that encompass the probabilities involved in the filtering. It leads to a family of filters having nested radii. The second approach simply consists of designing capacities that account for the imprecision induced by sampling of an ideal continuous filter.

4.1. FIR filtering and expectation operators

We consider only discrete linear filters having a finite impulse response. A positive impulse response of a filter having a unitary gain is called a summative kernel [22]. More precisely, $\rho = (\rho_i)_{i\in\mathbb{Z}}$ is a summative kernel if and only if $\forall i \in \mathbb{Z}, \rho_i \geqslant 0$ and $\sum_{i\in\mathbb{Z}}\rho_i = 1$.

Proposition 11. *Any real finite impulse response of a discrete filter can be expressed as a linear combination of, at most, two summative kernels.*

Proof. Let $\varphi = (\varphi_i)_{i\in\mathbb{Z}}$ be the real finite impulse response of a discrete filter such that $\sum_{i\in\mathbb{Z}}\varphi_i < \infty$. Let $\varphi_i^+ = \max(0, \varphi_i)$ and $\varphi_i^- = \max(0, -\varphi_i)$. Let $A^+ = \sum_{i\in\mathbb{Z}}\varphi_i^+$ and $A^- = \sum_{i\in\mathbb{Z}}\varphi_i^-$. We have $\varphi_i = \varphi_i^+ - \varphi_i^-$. Let $\rho_i^+ = \frac{\varphi_i^+}{A^+}$ and $\rho_i^- = \frac{\varphi_i^-}{A^-}$. By construction, ρ_i^+ and ρ_i^- are summative kernels. Now $\varphi_i^+ = \rho_i^+ A^+$ and $\varphi_i^- = \rho_i^- A^-$, therefore $\varphi_i = \rho_i^+ A^+ - \rho_i^- A^-$. □

Corollary 12. *Any discrete linear filtering operation can be considered as a weighted sum of, at most, two expectation operations.*

Corollary 13. *Let P_k^+ (resp. P_k^-) be the probability measure based on the summative kernel ρ_{k-i}^+ (resp. ρ_{k-i}^-), X an input signal and Y the corresponding output signal, then $Y_k = A^+ E_{P_k^+}(X) - A^- E_{P_k^-}(X)$.*

Proof. Let $X = (X_n)_{n=1\ldots N}$ be the discrete signal to be filtered and $\varphi = (\varphi_i)_{i\in\mathbb{Z}}$ the impulse response of the filter. Y_k, the kth component of the filter output, is computed by the discrete convolution of the impulse response of the filter and the imput signal: $Y_k = \sum_{n=1}^N \varphi_{k-n} X_n$. Since $\varphi_i = \varphi_i^+ - \varphi_i^-$, $Y_k = \sum_{n=1}^N (\varphi_{k-n}^+ - \varphi_{k-n}^-) X_n = \sum_{n=1}^N \varphi_{k-n}^+ X_n - \sum_{n=1}^N \varphi_{k-n}^- X_n$. Now $\varphi_i^+ = \rho_i^+ A^+$ and $\varphi_i^- = \rho_i^- A^-$, therefore $Y_k = A^+ \sum_{n=1}^N \rho_{k-n}^+ X_n - A^- \sum_{n=1}^N \rho_{k-n}^- X_n$. By construction, $\sum_{i\in\mathbb{Z}}\rho_i^+ = 1$ and $\sum_{i\in\mathbb{Z}}\rho_i^- = 1$, thus $Y_k = A^+ E_{P_k^+}(X) - A^- E_{P_k^-}(X)$. □

We call the decomposition of the impluse response of the filter φ into $\rho^+ = (\rho_i^+)_{i\in\mathbb{Z}}$, $\rho^- = (\rho_i^-)_{i\in\mathbb{Z}}$, A^+, and A^- the *canonical* decomposition of φ.

4.2. How can an imprecise filtering operator be derived from a conventional filtering operator?

Since no tool is currently available to derive a suitable capacity from objective criteria, we propose, in a signal processing context, to derive an imprecise filtering operator from one or several conventional filters that have been validated for this particular application.

Let us consider a discrete filter whose finite impulse response is $\varphi = (\varphi_i)_{i\in\mathbb{Z}}$. Let us also consider the canonical decomposition of φ into ρ^+, ρ^-, A^+ and A^-. Let P^+ (resp. P^-) be the probability measure induced by ρ^+ (resp. ρ^-). The method we propose consists of finding two capacities υ^+ and υ^- such that $P^+ \in \text{core}(\upsilon^+)$ and $P^- \in \text{core}(\upsilon^-)$. By translating the confidence measures, we also define υ_k^+ and υ_k^- such that $P_k^+ \in \text{core}(\upsilon_k^+)$ and $P_k^- \in \text{core}(\upsilon_k^-)$.

Let $X = (X_n)_{n=1\ldots N}$ be the discrete signal to be filtered. Due to Theorem 4 and as we look for a capacity υ^+ such that P^+ belongs to core(υ^+), we have $E_{P^+}(X) \in \underline{\overline{E}}_{\upsilon^+}(X)$. Similarly, we have $E_{P^-}(X) \in \underline{\overline{E}}_{\upsilon^-}(X)$. According to Corollary 13, Y_k, the kth component of the output of the precise filter, is given by $Y_k = A^+ E_{P_k^+}(X) - A^- E_{P_k^-}(X)$. As A^+ and A^- are positive reals, Proposition 7 concerning the Minkowski subtraction implies that:

$$Y_k = A^+ E_{P_k^+}(X) - A^- E_{P_k^-}(X) \in [Y_k] = A^+ \underline{\overline{E}}_{\upsilon_k^+}(X) \ominus A^- \underline{\overline{E}}_{\upsilon_k^-}(X), \tag{1}$$

with $[Y_k]$ being the kth component of the output of the imprecise filter.

Note that, in expression (1), the Minkowski subtraction $\ominus$ can be replaced by the dual Minkowski subtraction $\boxminus$. This replacement would lead to a filtering operator having different behaviors and properties. Replacing the Minkowski operator could account for a strong link between the two dominated summative kernels ρ^+ and ρ^-, as will be shown in Section 5.2.

4.3. Imprecise filtering based on a possiblity distribution

A possibility measure is a particular concave capacity. In fact, a possibility measure is the only measure (except for the probability measure) that can be defined by a single precise discrete mass function on the considered space (here $\mathbb{Z}$). $\pi = (\pi_i)_{i\in\mathbb{Z}}$ is a possibility distribution if and only if: $\pi_i \geqslant 0$, $\forall i \in \mathbb{Z}$ and $\max_{i\in\mathbb{Z}}\{\pi_i\} = 1$. A possibility distribution induces, on each finite subset A of $\mathbb{Z}$, a possibility measure denoted $\Pi(A)$ that is computed as follows: $\Pi(A) = \max_{i\in A}\{\pi_i\}$.

As shown in [22], the most suitable way to associate a discrete possibility distribution with a discrete probability distribution $\rho = (\rho_i)_{i\in\mathbb{Z}}$, in an objective context,[1] consists of computing the most specific transformation proposed by Dubois [11]. This transformation leads to defining a possibility distribution $\pi = (\pi_i)_{i\in\mathbb{Z}} : \pi_i = \sum_{k\in\mathbb{Z}}\rho_k \chi(\rho_k \leqslant \rho_i)$, with $\chi(\rho_k \leqslant \rho_i) = 1$ if $\rho_k \leqslant \rho_i$ and 0 otherwise. The possibility measure Π, induced by π is then the most specific one that dominates the probability measure P induced by ρ. This possibility distribution can also be thought of as a kind of possibilistic neighborhood called a maxitive kernel [22].

[1] A confidence measure is said to be objective when it refers to properties of the physical world, e.g. measurement error modeling.

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Another transformation can also be used, which is usually preferred in a subjective context,[2] called the subjective transformation and leading to a possibility distribution $\tilde{\pi} = (\tilde{\pi}_i)_{i\in\mathbb{Z}}$, such that: $\tilde{\pi}_i = \sum_{k\in\mathbb{Z}} \min(\rho_k, \rho_i)$. This subjective transformation always defines a possibility distribution that is less specific than the previous one, i.e. $\forall i \in \mathbb{Z}, \tilde{\pi}_i \geqslant \pi_i$.

Finally, since a broad range of summative kernels used for signal processing are monomodal and symmetric, a triangular maxitive kernel is a good candidate since it defines a possibility measure that dominates any probability measure defined by a summative kernel having the same mode and any smaller radius [22]. Note that, when a summative kernel κ is dominated by a maxitive kernel π, then any summative kernel having the same shape, but a smaller radius, is also dominated by π.

4.4. Imprecise filtering based on accounting-for-sampling

Another way to derive an imprecise filtering operator from a precise filtering operator is to account for the imprecision due to sampling. Indeed, most discrete filters used in digital signal processing are designed to be a sampled version of a continuous filter. Let $\rho = (\rho_i)_{i\in\mathbb{Z}}$ be the discrete probability distribution associated with the considered filter. Let us suppose that this distribution is bounded, i.e. there is a finite subset Ω of $\mathbb{Z}$ such that $\forall i \notin \Omega, \rho_i = 0$. The fact that this discrete filter is derived from a continous filter means that each value ρ_i associated with the ith sampled time (iT), with T being the sample time, is obtained by integrating a continuous function $\rho(t)$ in a neighborhood of iT defined by a continuous sampling kernel η:

$$\rho_i = \int_{-\infty}^{\infty} \rho(t\eta(t - iT))dt. \tag{2}$$

The neighborhood sampling kernel (e.g. splines [29]) is usually a monomodal positive function with a support in the interval $[-T,T]$, such that $\int_{-\infty}^{\infty} \eta(t)dt = 1$. A way to estimate the original continuous function $\rho(t)$ from the discrete kernel ρ is to use an interpolating kernel. Most of these kernels lead to functions $\rho(t)$ whose value, when $t \in [iT,(i+1)T]$, belongs to the interval $[\rho_i, \rho_{i+1}]$. Thus a simple way to account for this imprecision is to define a capacity v by: $\forall A \in \mathcal{P}(\Omega),\ v(A) = \frac{\sum_{k\in\Omega}\phi_A(k)}{\sum_{k\in\Omega}\phi_\Omega(k)}$ with

$$\phi_A(k) = \begin{cases} \max(\rho_k, \rho_{k+1}), & \text{if } k \in A \text{ and } (k+1) \in A, \\ \rho_k, & \text{if } k \in A \text{ and } (k+1) \notin A, \\ \rho_{k+1}, & \text{if } k \notin A \text{ and } (k+1) \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 14. *The above capacity is concave.*

Proof. Let $A, B \in \mathcal{P}(\Omega)$, v is concave if $v(A) + v(B) - v(A\cup B) - v(A\cap B) \geqslant 0$. Let $\alpha = \sum_{k\in\Omega}\phi_\Omega(k)$, thus v is concave if $\alpha v(A) + \alpha v(B) - \alpha v(A\cup B) - \alpha v(A\cap B) \geqslant 0$, i.e. if $\sum_{k\in\Omega}(\phi_A(k) + \phi_B(k) - \phi_{A\cup B}(k) - \phi_{A\cap B}(k)) \geqslant 0$, because $\alpha > 0$.

To prove the inequality, we consider all possible situations in the following table:

$k \in A$	$k+1 \in A$	$k \in B$	$k+1 \in B$	$\phi_A(k) + \phi_B(k) - \phi_{A\cup B}(k) - \phi_{A\cap B}(k)$
yes	yes	yes	yes	0
yes	yes	yes	no	0
yes	yes	no	yes	0
yes	yes	no	no	0
yes	no	yes	yes	0
yes	no	yes	no	0
yes	no	no	yes	$\rho_k + \rho_{k+1} - \max(\rho_k, \rho_{k+1}) \geqslant 0$
yes	no	no	no	0

So $\forall k \in \Omega$, $\phi_A(k) + \phi_B(k) - \phi_{A\cup B}(k) - \phi_{A\cap B}(k) \geqslant 0$ which proves that v is a concave capacity. □

5. Experiments

5.1. Signal filtering experiments

This first experiment is based on a signal issued from the STS[3] pressure sensor of the AUV[4] Taipan300 acquired during sea-trial experiments at Salagou Lake in France [28]. The measurement range is between 0 and 10 m, with an accuracy of 0.5% of its full scale, which corresponds to 5 cm in absolute terms.

[2] A confidence measure is said to be subjective when it refers to interpreting probabilities as betting rates, in an exchangeable bet setting.
[3] Sensor Technik Sirnach AG.
[4] AUV stands for autonomous underwater vehicle.

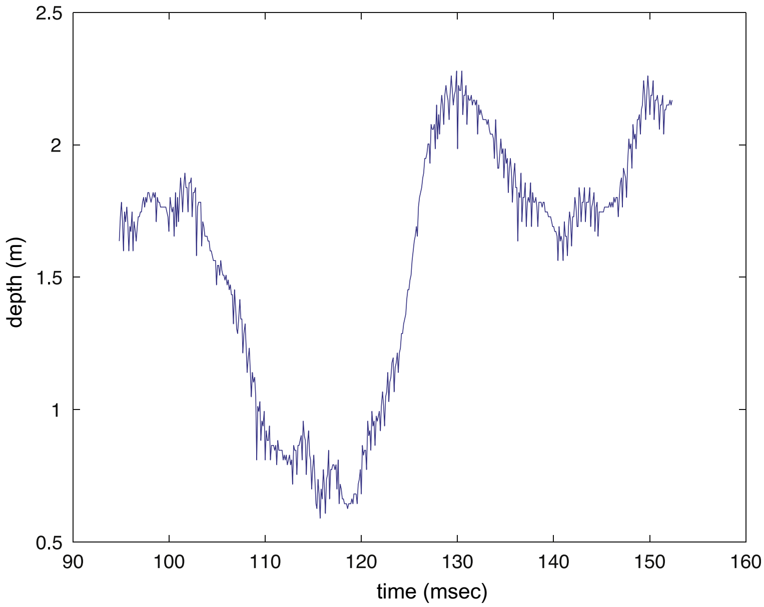

Fig. 1. Raw depth signal.

During this experiment, the depth of the AUV was controlled by a sliding-mode [28] at a subsurface depth of around 2 m. The signal is partly reproduced in Fig. 1. It is composed of a signal induced by the depth of the vehicle corrupted by variations caused by waves, errors in trajectory control and measurement errors. The wave height (about 0.5 m) is not negligible at this depth. Moreover, the main goal of these sea-trial experiments was to tune the different coefficients of the command law used and the pitch was not yet properly controlled.

Since the raw signal depicted in Fig. 1 is very noisy, it is usually post-processed by using a low-pass filter. For this experiment, we used an order-3 Butterworth low-pass filter with a cutoff frequency equal to 0.15 times the sample rate frequency. The filter shape is depicted in Fig. 2, its canonical decomposition is depicted in Fig. 3.

In the first part of the experiment, we consider dominating the probability measure by a possibility measure. In the second part of the experiment, we consider dominating the probability measure by a capacity that accounts for sampling.

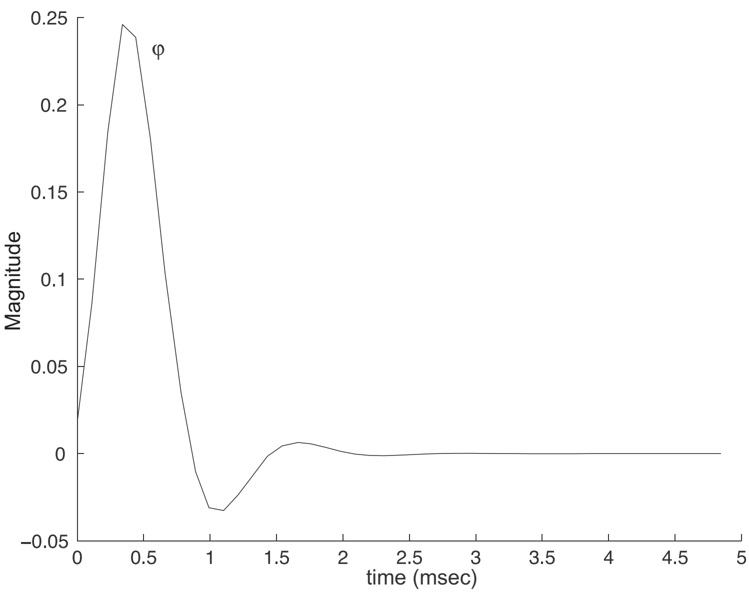

Fig. 2. Butterworth filter impulse response φ.

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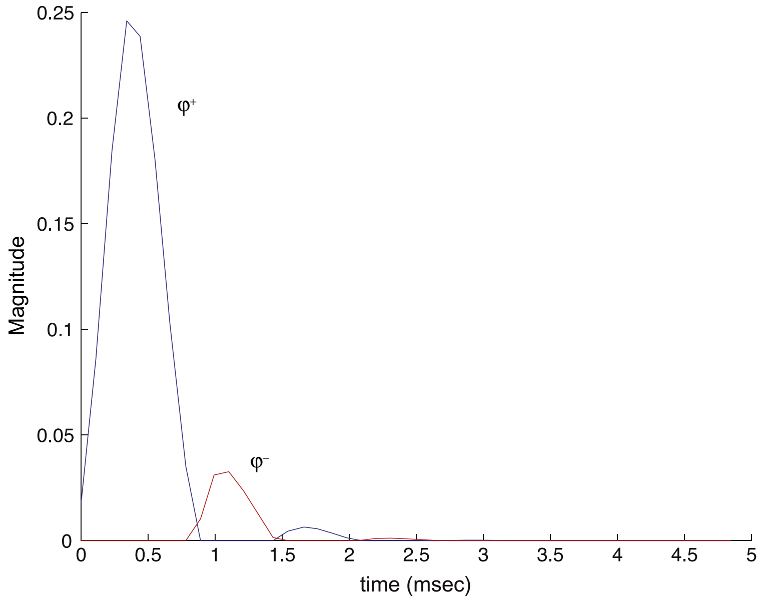

Fig. 3. Decomposition into two positive impulse responses φ^+ and φ^-.

The first part of the experiment is plotted in Fig. 4, the second part of the experiment is plotted in Fig. 5. In all figures, the precise Butterworth filtered depth value is plotted in black and superimposed on the original signal (in cyan). The imprecise filtered value is plotted in red (lower value) and blue (upper value). Moreover, for readers with a B/W printout, the upper, lower and precise values are indicated by arrows, while the original signal usually appears in light gray. When considering Figs. 4 and 5, it can easily be seen that the precise filtered value belongs to the imprecise filtered value. The imprecise filtered value provided by the accounting-for-sampling approach is more specific than the possibilistic based approach. Moreover, when considering the possibilistic domination, the imprecision of the imprecise filtered value reflects the overall distance between the filtered signal and the original signal, providing a good measure of the noise level. Note that, with the Butterworth filter being

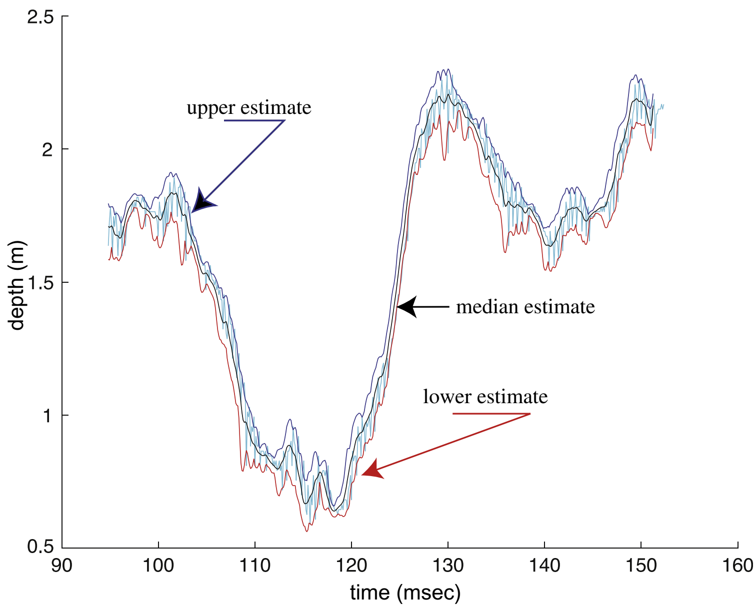

Fig. 4. Depth signal filtered by the precise Butterworth filter (black) and the imprecise possibility-based Butterworth filter (blue-upper, red-lower) superimposed with the original signal (cyan) (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.).

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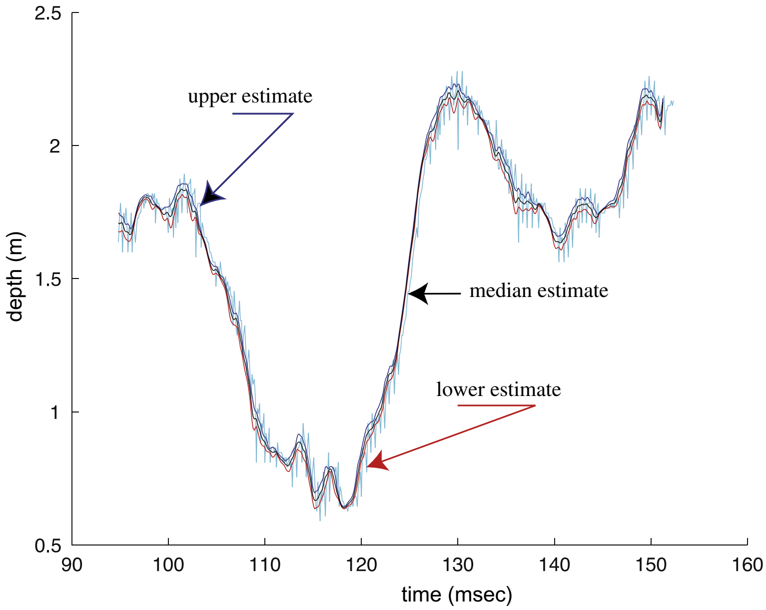

Fig. 5. Depth signal filtered by the precise Butterworth filter (black) and the imprecise accounting-for-sampling capacity based Butterworth filter (blue-upper, red-lower) superimposed with the original signal (cyan) (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.).

a real-time filter (i.e. its impulse response is causal), the filtered signal is slightly delayed compared to the original signal. The same applies for the imprecise filtered signal.

5.2. Experiments on deriving a signal

Signal derivation has been a great topic of interest in the image processing community since it is very useful for performing edge detection and image segmentation. In general, derivation consists of convolving a signal with a discrete kernel obtained by sampling the derivative of an interpolative continuous kernel. In the image community, the Canny–Deriche filter is said to be optimal for edge detection since it respects Canny's detection criteria. The Canny–Deriche derivative filter φ is given by: $\varphi_i = -\frac{(1-\beta)^2}{\beta} i \beta^{|i|}$, where $\beta \in [0,1]$ is a parameter that defines the filter radius. The shape of the filter is depicted in Fig. 6 for $\beta \simeq 0, 4$. This kernel can be considered as bounded by assuming that $\varphi_i = 0$ if $|\varphi_i| \leqslant \epsilon$, where ϵ represents the precision of the computer.

The second experiment we propose consists of deriving a signal by using a Canny–Deriche filter and its imprecise version. This experiment aims at highlighting the usefulness of the dual Minkowski difference in such a setting.

Since most kernels used for derivation are positive and symmetric, the derivation kernel is composed of two identical symmetric functions. The Canny–Deriche filter acts in the same way. Therefore, the canonical decomposition of the filter leads to $\varphi_i = A\rho_i - A\rho_{-i}$, where ρ is a causal filter, i.e. such that $\rho_i = 0$ if $i < 0$. Thus, let $X = (X_n)_{n=1\ldots N}$ be the discrete signal to be filtered, then the value of $\dot{X}_k$, the kth estimate of the temporal derivative of X is given by:

$$\dot{X}_k = A(E_{P_k^+}(X) - E_{P_k^-}(X)), \tag{3}$$

where P_k^+ (resp. P_k^-) is the probability measure based on ρ_i (resp. ρ_{-i}) at the kth sample. $E_{P_k^+}(X)$ (resp. $E_{P_k^-}(X)$) can be seen as a right (resp. left) mean value of signal X around the kth time sample. If these two mean values are equal, then the derivative equals 0.

Now, let us consider the family of all symmetric filters φ whose canonical decomposition is $\varphi_i = A\rho_i - A\rho_{-i}$ with P^+ (resp. P^-) based on ρ_i (resp. ρ_{-i}) dominated by ν^+ (resp. ν^-). In this case, ν^+ and ν^- are mirror functions on each singleton of $\mathbb{Z}$. A situation can arise that, for any of these filters $E_{P_k^+}(X) = E_{P_k^-}(X)$ and therefore $\dot{X}_k = 0$. Thus $\overline{E}_{\nu_k^+}(X) = \underline{E}_{\nu_k^-}(X)$, but due to Property 8 we have $[\dot{X}_k] = A\left(\underline{E}_{\nu_k^+}(X) \ominus \overline{E}_{\nu_k^-}(X)\right) \neq 0$. That is to say that $[\dot{X}_k]$ is bigger than the set of all derivatives obtained with this family of filters. This faulty conclusion originates from the filter decomposition that does not account for the strong link between the two capacities ν^+ and ν^-. This link can be accounted for by replacing the Minkowski difference by the dual Minkowski difference in the previous equation. By considering the dual Minkowski difference, when a situation occurs such that $\overline{E}_{\nu_+}(X) = \underline{E}_{\nu^-}(X)$, it is interpreted as a consensus of all filters based on the fact that the estimated derivative equals 0. This replacement leads to a more specific (but also more risky) interval, as will be illustrated in the experiment.

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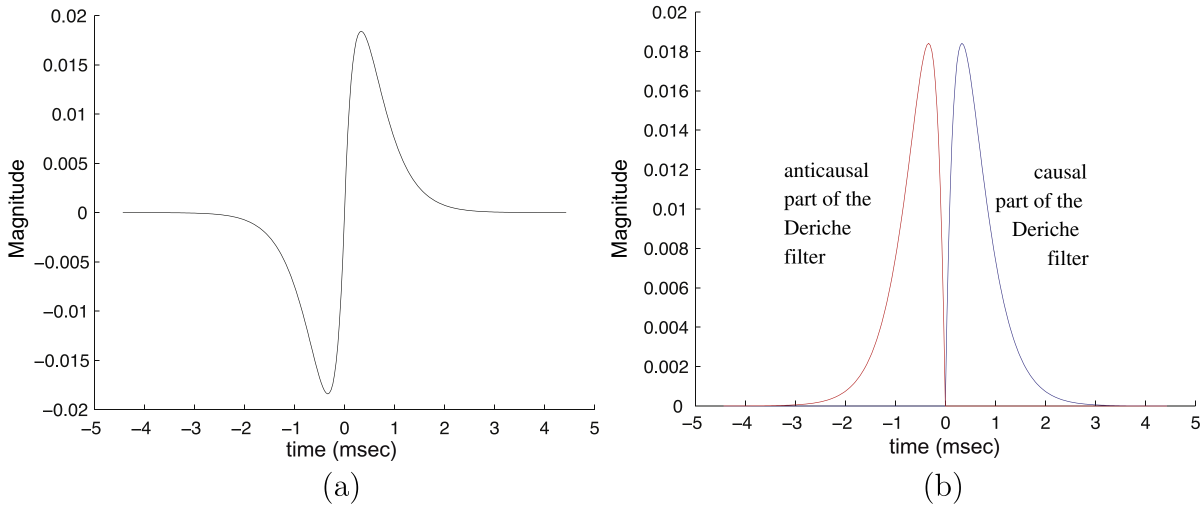

Fig. 6. (a) Canny–Deriche FIR and (b) Canny–Deriche FIR canonical decomposition.

Let us consider the synthetic signal of the form $x(t) = \lambda t \cos(wt)$ depicted in Fig. 7(a) and the same signal degraded by additive gaussian noise whose standard deviation increases with time (i.e. the noise is not stationary), as depicted in Fig. 7(b).

Since the signal is known, its derivative is also known, which enables us to illustrate the behavior of the derivation algorithm. Figs. 8–11 present the derivative of both original (a) and noisy signals (b). In each figure, the real derivative is plotted in magenta, the precise estimated derivative is plotted in black while the imprecise derivative estimate is plotted in blue for upper and red for lower estimates. Figs. 8 and 9 present an imprecise estimate that uses the Minkowski difference. Figs. 10 and 11 present an imprecise estimate based on the dual Minkowski difference. In each figure, we have expanded a specific detail of the signal (around time = 14 ms). Naturally, since the derivative is estimated via a smoothing filter, the amplitude of the estimated derivative is always slightly lower than the real derivative.

When comparing Figs. 8 and 9 to Figs. 10 and 11 it can easily be seen that the imprecise estimation of the derivative is always more precise when using the dual Minkowski difference. This increased precision is an advantage when the signal is not noisy, i.e. the only noise is due to sampling (Figs. 7 and 8): the dual Minkowski difference based imprecise estimate, the precise estimate and the real derivative are almost identical. However, when the signal is noisy, the fact that the two intervals $\underline{\overline{E}}_{\nu^+}(X)$ and $\underline{\overline{E}}_{\nu^-}(X)$ are equal cannot be considered as a sign that, for any $P^+ \in \text{core}(\nu^+)$ and $P^- \in \text{core}(\nu^-)$, $E_{P_k^+}(X) = E_{P_k^-}(X)$. Therefore, the imprecise estimate based on the dual Minkowski difference can be seen as too risky an interval valued estimate since it does not always include the real derivative or the precise estimated derivative (Fig. 11). On the other hand, when the filter is properly designed, the Minkowski difference approach always includes the real possibly noisy signal (Figs. 8 and 9). Moreover, the precise expected value is always included in the imprecise expected value due to Theorem 4.

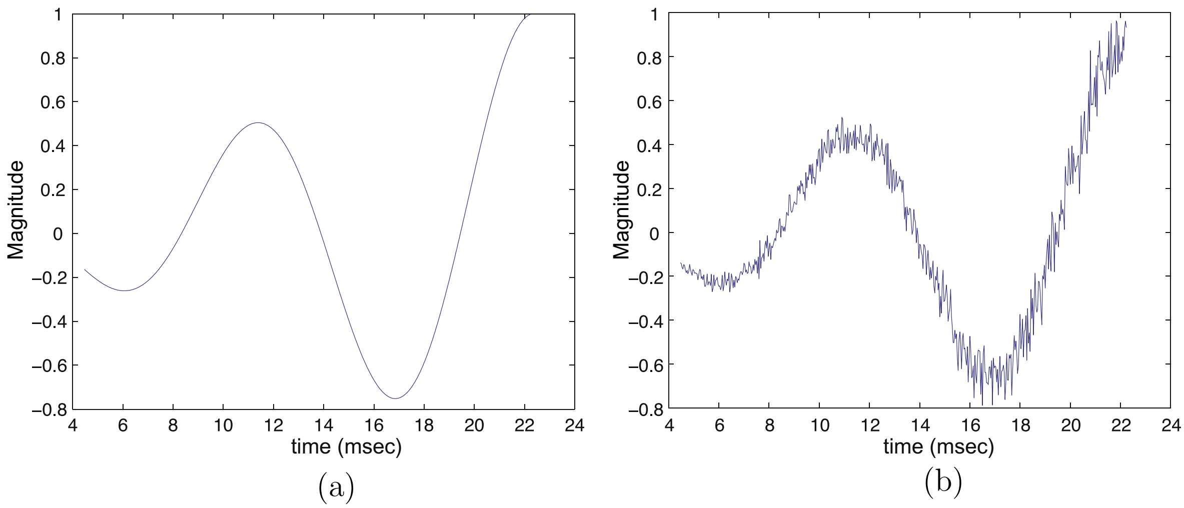

Fig. 7. Original signal without noise (a) and with noise (b).

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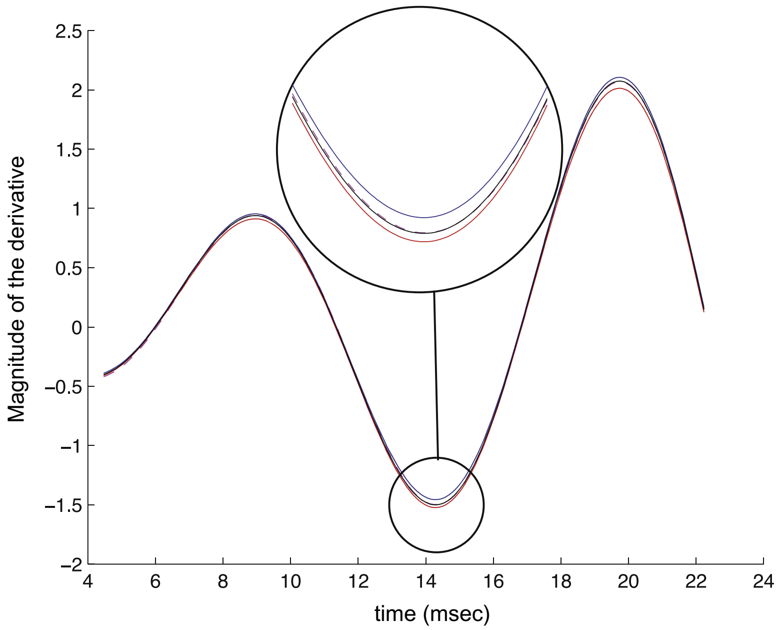

Fig. 8. Derivative estimate of the original signal based on the Minkowski difference.

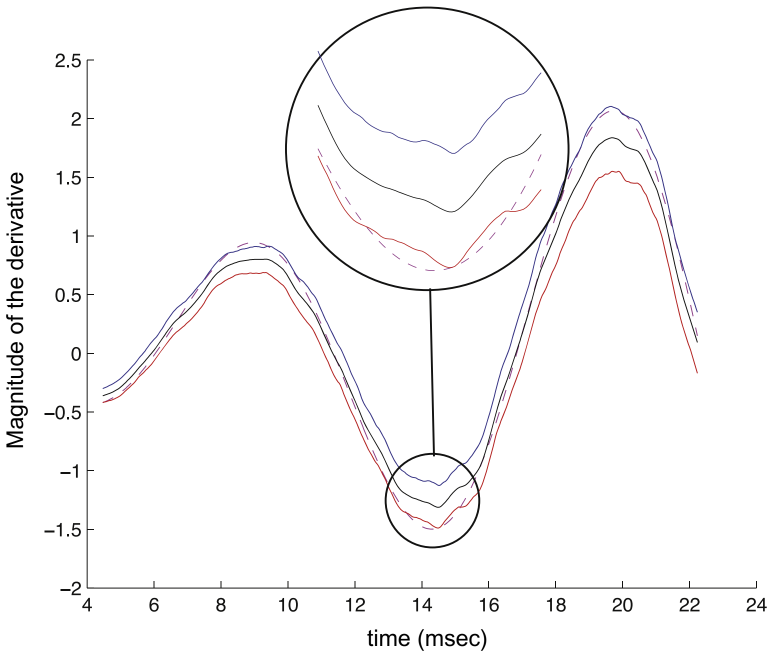

Fig. 9. Derivative estimate of the noisy signal based on the Minkowski difference.

5.3. Comparison with the classical precise valued approach

It would be useful to compare the filtering ability of the new proposed approach with some traditional approach. However, this comparison is not straightforward since, according to our knowledge, no other imprecise filtering approach exists. We thus need to compare two methods whose outputs are different by nature.

To achieve this comparison, we need to compute a distance that should reflect the ability of each method to reject the noise that corrupts the signal to be estimated. By noise we mean any perturbation in the signal value, including random noise, sampling, quantification or other signals. Our approach will be based on the $L1$ distance and three different interval

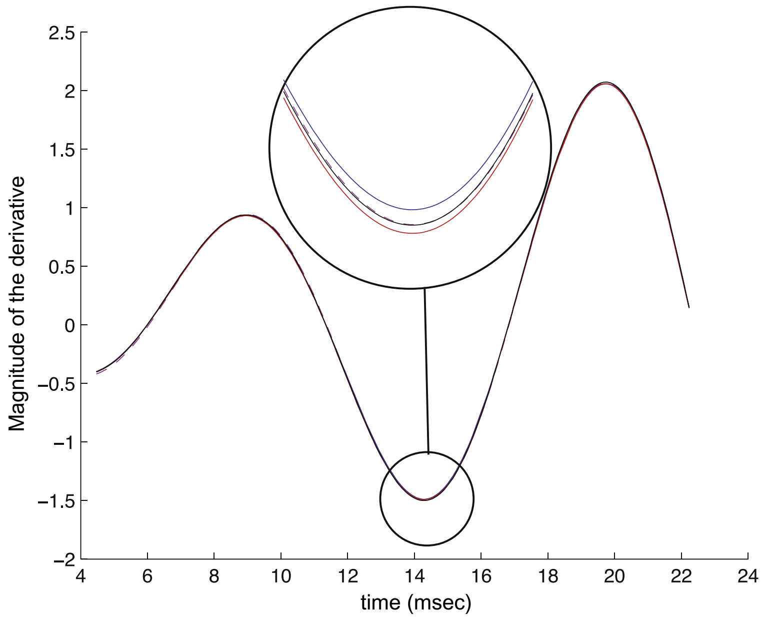

Fig. 10. Derivative estimate of the original signal based on the dual Minkowski difference.

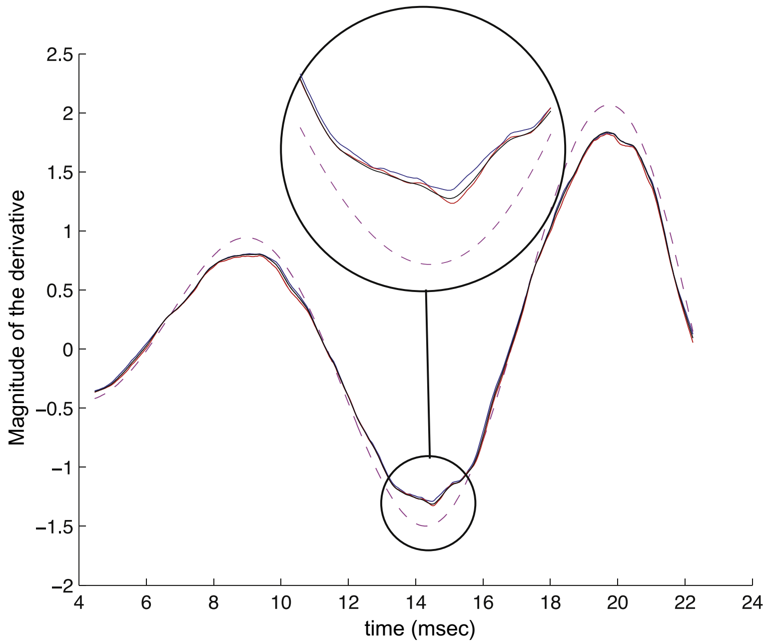

Fig. 11. Derivative estimate of the noisy signal based on the dual Minkowski difference.

valued extensions. The $L1$ distance between a noise-free signal $X = (X_n)_{n=1...N}$ and its precise estimate $Y = (Y_n)_{n=1...N}$ is defined by: $d_1(X, Y) = \sum_{n=1}^{N}(|X_n - Y_n|)$. The three extensions we propose to compare a noise-free signal $X = (X_n)_{n=1...N}$ and its imprecise estimate $[Y] = ([Y_n])_{n=1...N} = ([\underline{Y}_n, \overline{Y}_n])_{n=1...N}$ are:

- The Hausdorff distance defined by $d_{H1}(X, [Y]) = \sum_{n=1}^{N} \max_{y \in [Y_n]} |X_n - y| = \sum_{n=1}^{N} \max(|X_n - \underline{Y}_n|, |X_n - \overline{Y}_n|)$. This distance is, by nature, not fair for our approach. A Hausdorff distance between an imprecise-valued quantity and a precise-valued quantity increases with the imprecision of the imprecise-valued quantity and actually, the imprecision of the interval valued

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estimate $[Y]$ increases with the noise level. This property is due to the fact that all traditional filters included in the set of filters represented by the concave capacities are likely to disagree about the value of the filtered signal in the presence of noise [23]. By construction, the least value of $d_{H1}(X,[Y])$ equals $\frac{1}{2}d_1(\underline{Y},\overline{Y})$.

- The shortest distance extension defined by $d_{S1}(X,[Y]) = \sum_{n=1}^{N}\min_{y\in[Y_n]}|X_n - y|$ reflects the ability of the interval valued output $[Y]$ to include the real precise signal value X. This distance would be too fair for the imprecise filtering approach since the more imprecise the output of the filter is, the more likely it will include the noise-free signal. A too high output imprecision is not a desired property.
- The $L1$ distance between X and the median signal $\widetilde{Y} = (\widetilde{Y}_n)_{n=1\dots N}$ defined by: $\forall n \in [1\dots N]$, $\widetilde{Y}_n = \frac{1}{2}(\underline{Y}_n + \overline{Y}_n)$. This distance can be considered as good trade off between the two previous approaches, since $\widetilde{Y}$ is the closest precise signal, in d_{H1} distance, to the interval valued signal $[Y]$. This latter distance, denoted d_{M1}, is computed by: $d_{M1}(X,[Y]) = \sum_{n=1}^{N}(|X_n - \widetilde{Y}_n|)$.

Note that any of the proposed extensions coincides with the d_1 distance if $[Y]$ is precise.

In this section, the noise rejection ability of both approaches will be compared on the two most frequently used filtering applications that are low-pass filtering and random noise filtering. For both experiments, we will use a set of low frequency synthetic signals, with each signal being composed of a weighted sum of five sine waves. Frequencies are chosen in the range $[0, 0.01f_s]$, with f_s being the sampling frequency. Both weights and frequencies are randomly chosen. The results presented here average the results of 200 different experiments. For extending the classical filtering approach, we use the possibility domination approach presented in Section 4.3. The results obtained by accounting for the sampling approach are similar.

For the low-pass filtering experiment, the noise will consist of a weighted sum of five high frequency signals. Frequencies are chosen in the range $[0.2f_s, 0.3f_s]$. Weights and frequencies are randomly chosen. We use an order five Butterworth filter with a cutoff frequency $f_c = 0.02f_s$.

Fig. 12 shows the four distances d_1, d_{H1}, d_{S1} and d_{M1} between the noise-free signal X and its imprecise estimate $[Y]$ for different noise level values. As expected, $d_{H1}(X,[Y]) \geqslant d_{M1}(X,[Y]) \geqslant d_{S1}(X,[Y])$ whatever the noise-to-signal ratio. The d_{S1} measure decreases when the noise level increases because the specificity of $[Y]$ decreases. However, this lack of specificity (that is reflected by the fact that d_{H1} decreases with the noise level) is offset by the fact that $d_{M1}(X,[Y])$ remains almost stable. The two distances $d_1(X,Y)$ and $d_{M1}(X,[Y])$ are so close that they are superimposed on Fig. 12. Thus, the median of the interval valued output of the imprecise filter has a behavior that is comparable to the precise valued output of the conventional filtering approach.

For the random noise filtering experiment, the considered noise is composed of a centered normal random variable in the time domain. This noise is added to the previous low frequency signal (see Fig. 14). The signal is filtered with a centered kernel having a Gaussian shape with a standard deviation equal to 10 times the sample time. Convolving this kernel with the signal does not make a great change in the noise-free signal since the signal has a highest frequency that is much lower than the cutoff frequency of the considered filter. Fig. 13 shows the three distances d_{H1}, d_{M1} and d_{S1} between X and $[Y]$ superimposed on the distance d_{H1} between X and Y for different noise level values.

Any remark about the previous experiment is also valid for this experiment. For random noise, an interesting result is reported in [23]: the imprecision of $[Y]$ can be sought as a signature of the noise level. The most relevant explanation of this property is the following. The set of filters represented by the concave capacity have cutoff frequencies that are much higher than the highest frequency of the original signal. Thus, when the signal is noise-free, all of those filters provide the same output, and therefore the imprecise output is very specific. If the signal is noisy, then the output signals will generally be different, depending on the shape of the impulse response of each filter. Then the output of the imprecise-valued filter, which is the convex envelope of all those precise responses, is a marker of the variability induced by the random noise (see [23] for a more complete explanation).

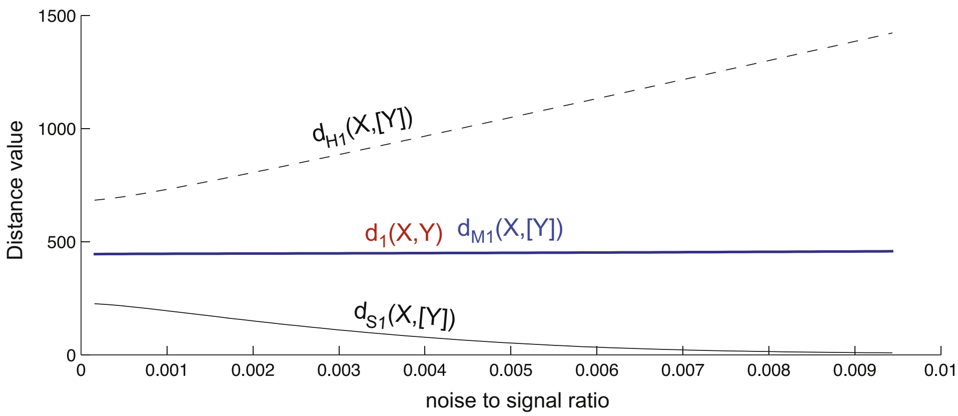

Fig. 12. Low-pass filtering: d_{H1}, d_{M1} and d_{S1} distances between the noise-free signal and interval valued estimated signal for different noise-to-signal ratio values.

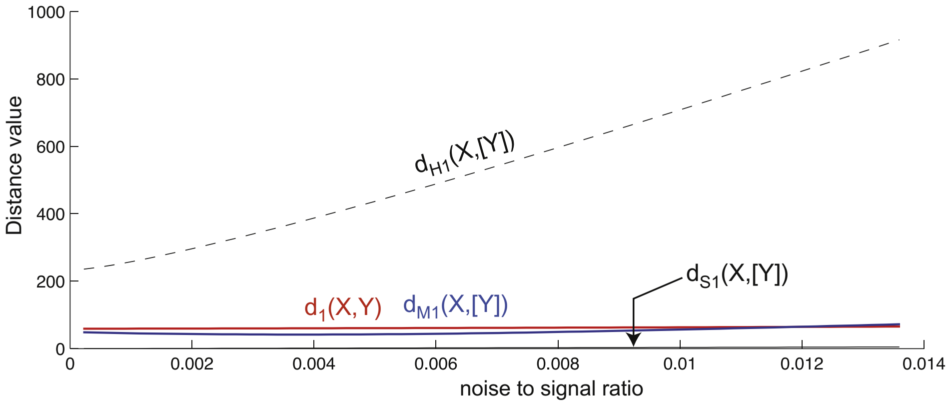

Fig. 13. Comparisons of distances for different noise-to-signal ratio values.

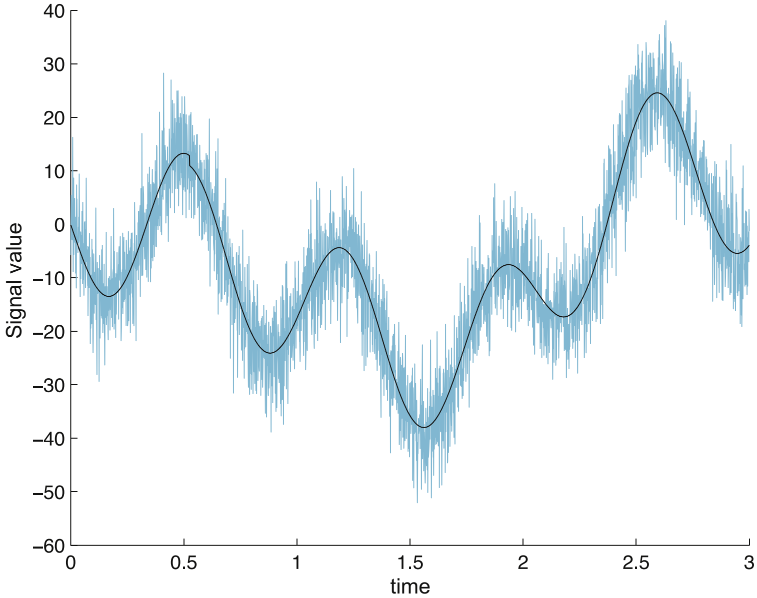

Fig. 14. A sample of the noise-free signal (black) superimposed on the noisy signal (cyan) for a signal-to-noise ratio of 0.012 (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.).

6. Concluding remarks

In this paper, we have proposed a new interpretation and a new way of computing the imprecision associated with an observed value. According to this interpretation, the imprecision of an observation can be due to the observation process but also to poor knowledge on the proper post-processing to be used to filter the raw measured signal. This paper focuses on this last flaw, which is barely considered.

This new method is an extension of the conventional signal filtering approach that enables us to handle imperfect knowledge about the impulse response of the filter to be used. The single precise impulse response is replaced by a set of impulse responses that is consistent with the user's expert knowledge. More precisely, linear filtering is considered as a linear combination of two expectation operations which are extended to concave capacities. Methods to design these capacities are presented.

We have proposed two extensions based on the Minkowski and the dual Minkowski sum. Using the extension based on the Minkowski sum leads to computing an imprecise-valued filtered signal which is a convex hull of all precise valued filtered signals that belong to the considered set of impulse responses. However, as mentioned in the article, this output can be too imprecise, i.e. containing output signals that are not outputs of the set of filters envisaged by the user. On the other hand, the approach based on the dual Minkowski sum can lead to a too specific filtered output. The study of this imprecise-valued filtered signal specificity of is a natural follow-up of this work. Our actual approach only considers a precise signal input. It thus would be useful extend our work to an imprecise signal input, whose imprecision could come from a previous imprecise filtering or be due to pre-calibration of the expected signal error. This could be a way to deal with the measurement uncertainty that is only indirectly taken into account within our approach.

Acknowledgement

The authors are indebted to Olivier Parodi for the depth data used in this article.

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