Use of the domination property for interval valued digital signal processing

Olivier Strauss

LIRMM Université Montpellier II, 161 rue Ada, 34392 Montpellier cedex 5, France,
Olivier.Strauss@lirmm.fr

Abstract. Imprecise probability framework is usually dedicated to decision processes. In recent work, we have shown that this framework can also be used to compute an interval-valued signal containing all outputs of processes involving a coherent family of conventional linear filters. This approach is based on a very straightforward extension of the expectation operator involving appropriate concave capacities.

Keywords: Convex capacities, real intervals, linear filtering

1 Introduction

Digital signal processing (DSP) is a significant issue in many applications (automatic control, image processing, speech recognition, monitoring, radar signal analyze, etc.). DSP is mainly dedicated to filtering, analyzing, compressing, storing or transmitting real world analog signals or sampled measurements. When used to mimic real world signal processing, converting the input signal is required from an analog to a digital form, i.e. a sequence of numbers. This conversion is achieved in two steps: sampling and quantization. Sampling consists of estimating the analog signal value associated with discrete values of the reference space (time, spatial localization). Quantization means associating an integer value to a class of real values. For example, in digital image processing, the reference space is a box in $\mathbb{R}^2$, the sampled space is an interval $[1, n] \times [1, m]$ of $\mathbb{N}^2$, the signal is the projected illumination (or any other activity e.g. radioactivity) and the integer range value is the interval $[0, 255]$ if the grey level is coded on 8 bits.

Within the classical approach, the digital signal to be processed is assumed to be composed of precise real valued quantities associated with precisely known values of the reference domain. Naturally, this is not true. Converting an analog signal into a digital signal transforms the information contained in the signal to be processed. Therefore, the classical approach consisting of mimicking analog signal processing by arithmetical operations leads to unquantified computation errors.

One of the most natural ways to represent the loss of information due to quantization is replacing the real precise valued number associated with each sample by an interval valued number. This representation not only solve the problem of representation of real numbers on a digital scale but is also a suitable way for representing the expected fluctuations in the sampled value due to noise or error in measurement. This approach leads to bounded error estimation [7] when using interval generalization of the involved arithmetic operations (see also [1], [2], [5]). Different interpretations are possible for interval valued data, e.g. a range in which one could have a certain level of confidence of finding the true value of the observed variable [14], a range of values that the real data can take when the measurement process involves quantization and/or sampling [7], [8], or a representation of the known detection limits, sensitivity or resolution of a sensor [4], etc. Within any interval-based signal processing application, there is a strong need for a reliable representation of the variability domain of each involved observation. An important issue is the meaning of the interval and the consistency of this meaning with respect to the tools used for further analysis or processing.

There are still three weaknesses in digital signal processing that are difficult to account for by using classical tools which are: a lack of knowledge of the sampling process, the use of linear digital equations to approximately model non-linear continuous processes, an imprecise knowledge of a filtering process.

In recent papers (see e.g. [9], [10], [6], [13]), we have proposed to use the ability of imprecise probability framework to define family of functions to cope with these weaknesses. Consider a process based on a function for which you have partial information. A way to account for this lack of knowledge is to replace this imprecisely known function by a set of functions that is coherent with your knowledge on the suitable function to be used. Moreover, such a model leads to a new interpretation of interval valued data.

2 Linear signal processing and impulse response

In signal processing, filtering consists of modifying a real input signal by blocking pre-specified particular components (usually frequency components). Finite impulse response (FIR) filters are the most popular type of filters. They are usually defined by their responses to the individual frequency components that constitute the input signal. In this context, the mathematical manipulation consists of convolving the input samples with a particular digital signal called the impulse response of the filter. This impulse response is simply the response of the digital filter to a Kroenecker impulse input.

Let $X = (X_n)_{n=1,...,N}$ be a set of N digital samples of a signal. Let $\rho = (\rho_i)_{i\in\mathbb{Z}}$ be the impulse response of the considered filter. The computation of Y_k, the k^{th} component of Y the filter output, is given by $Y_k = \sum_{n=1}^{N} \rho_{k-n} X_n$.

When the impulse response is positive and has a unitary gain ($\forall i \in \mathbb{Z}$, $\rho_i \geq 0$ and $\sum_{i\in\mathbb{Z}} \rho_i = 1$), it can be considered as a probability distribution inducing a probability measure P on each subset A of $\mathbb{Z}$ by $P(A) = \sum_{i\in A} \rho_i$. This special case of impulse response is often called *summative kernels* [10], or simply *kernels*, when used to ensure interplay between continuous and discrete domains. Thus, computing Y_k is equivalent to computing a discrete expectation operator involving a probability measure P_k induced by $(\rho_{k-n})_{n\in\mathbb{Z}}$, the probability distribution obtained by translating the probability distribution ρ in k: $Y_k = \sum_{n=1}^{N} \rho_{k-n} X_n = E_{P_k}(X)$.

When the impulse response is not positive or has not a unitary gain then it can be expressed as a linear combination of, at most, two summative kernels in the following way. Let $\varphi = (\varphi_i)_{i\in\mathbb{Z}}$ be the real finite impulse response of a discrete filter such that $\sum_{i\in\mathbb{Z}} \varphi_i < \infty$. Let $\varphi_i^+ = \max(0, \varphi_i)$ and $\varphi_i^- = \max(0, -\varphi_i)$. Let $A^+ = \sum_{i\in\mathbb{Z}} \varphi_i^+$ and $A^- = \sum_{i\in\mathbb{Z}} \varphi_i^-$. Let $\rho_i^+ = \frac{\varphi_i^+}{A^+}$ and $\rho_i^- = \frac{\varphi_i^-}{A^-}$. By construction, ρ_i^+ and ρ_i^- are summative kernels and $\varphi_i = \rho_i^+ A^+ - \rho_i^- A^-$.

Thus, any discrete linear filtering operation can be considered as a weighted sum of, at most, two expectation operations. Let P_k^+ (rsp.P_k^-) be the probability measure based on the summative kernel ρ_{k-i}^+ (rsp.ρ_{k-i}^-), X an input signal and Y the corresponding output signal, then $Y_k = A^+ E_{P_k^+}(X) - A^- E_{P_k^-}(X)$.

The decomposition of φ into ρ^+ , ρ^-, A^+ and A^- is called its canonical decomposition and is denoted as $\{A^-, A^+, \rho^-, \rho^+\}$.

3 Extension of linear filtering via a pair of two conjugate capacities

Let us consider a pair of capacities ν^+ and ν^- such that $P^+ \in \text{core}(\nu^+)$ and $P^- \in \text{core}(\nu^-)$. By translating the confidence measures, we also define ν_k^+ and ν_k^- such that $P_k^+ \in \text{core}(\nu_k^+)$ and $P_k^- \in \text{core}(\nu_k^-)$. It is thus easy to extend linear filtering to a convex set of impulse responses defined by ν^+ and ν^- by: $[Y_k] = A^+ \overline{\underline{E}}_{\nu_k^+}(X) \ominus A^- \overline{\underline{E}}_{\nu_k^-}(X)$, with $\ominus$ being the Minkowski sum [11], $\overline{\underline{E}}_\nu$ the extension of expectation to concave capacities [12] and $[Y_k]$ the k^{th} component of the output of the imprecise filter. Due to the domination properties [3] it verifies: $Y_k = A^+ E_{P_k^+}(X) - A^- E_{P_k^-}(X) \in [Y_k]$.

It can also be extended by considering two real intervals $[I^+]$ and $[I^-]$ and using $\otimes$, the extension of the multiplication to interval valued quantities:

$$[Y_k] = \left([I^+] \otimes \overline{\underline{E}}_{\nu_k^+}(X)\right) \ominus \left([I^-] \otimes \overline{\underline{E}}_{\nu_k^-}(X)\right). \tag{1}$$

Thus every filter with an impulse response φ whose canonical decomposition $\{A^-, A^+, \rho^-, \rho^+\}$ is such that $P^+ \in \text{core}(\nu^+)$ and $P^- \in \text{core}(\nu^-)$ and $A^+ \in [I^+]$ and $A^- \in [I^-]$ has an output that belongs to $[Y]$ the output of the interval-valued filter defined by Equation 1.

4 Discussion and conclusion

The new method we propose is an extension of the conventional signal filtering approach that enables us to handle imperfect knowledge about the impulse response of the filter to be used. It mostly consist in replacing the usual single precise impulse response by a set of impulse responses that is consistent with the user's expert knowledge. It can be perceived as a surprising way of using the imprecise probability framework. It allows a new interpretation and a new way of computing the imprecision associated with an observed value. According to this interpretation, the imprecision of an observation can be due to the observation process but also to poor knowledge on the proper post-processing to be used to filter the raw measured signal. Defining the pair of convex capacities also defines the convex set of impulse responses. In our recent papers, we have shown different approaches to define pair of capacities that able to handle with a lack of knowledge of the sampling process and an imprecise knowledge of a filtering process (see e.g. [12]). It also ables the propagation of input random noise level to the output filtered value [9] and thus use this information for automatically define thresholds in image analysis processes [6].

Our actual approach only considers a precise signal input. It thus would be useful extend our work to an imprecise signal input, whose imprecision could come from a previous imprecise filtering or be due to pre-calibration of the expected signal error. This could be a way to deal with the measurement uncertainty that is only indirectly taken into account within our approach.

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