Orientations of Simplices Determined by Orderings on the Coordinates of their Vertices

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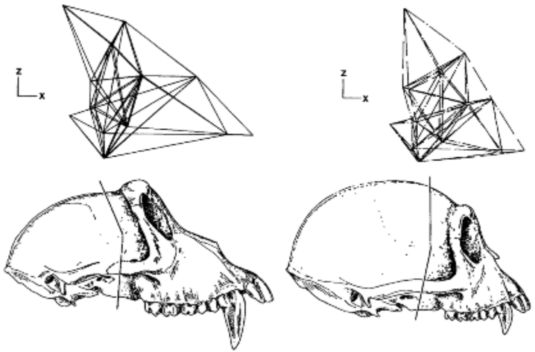
Motivation

Study of the 3D shape of anatomical structures.



Applications:

Anatomy, anthropology, paleontology, medicine

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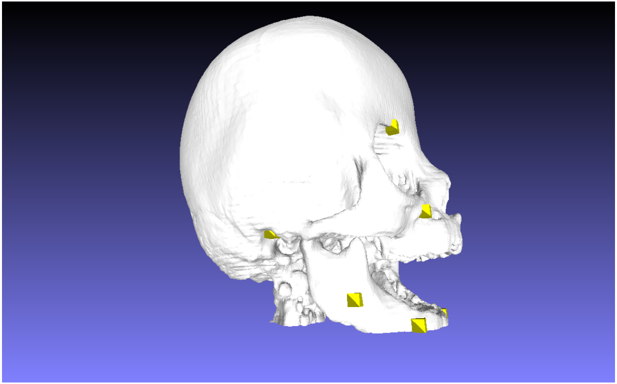

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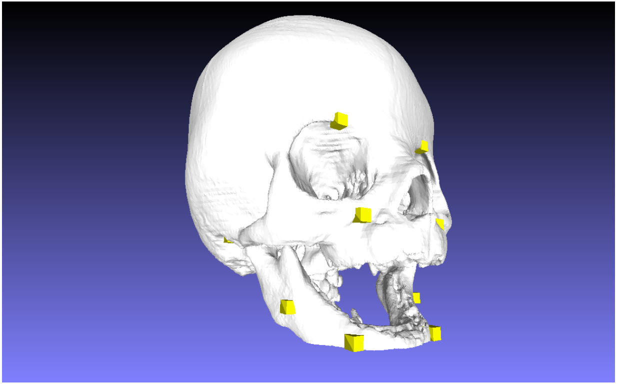

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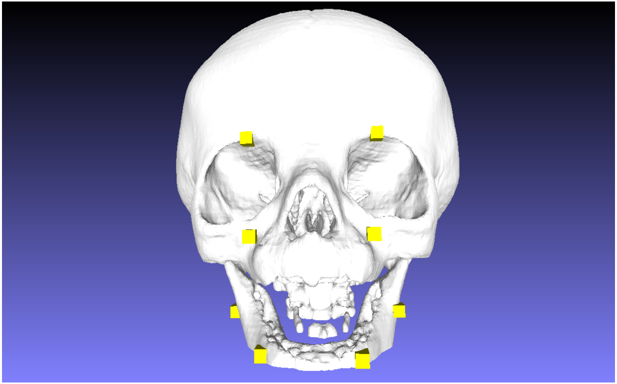

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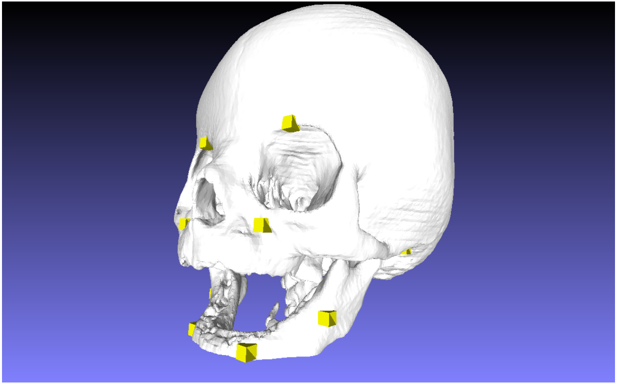

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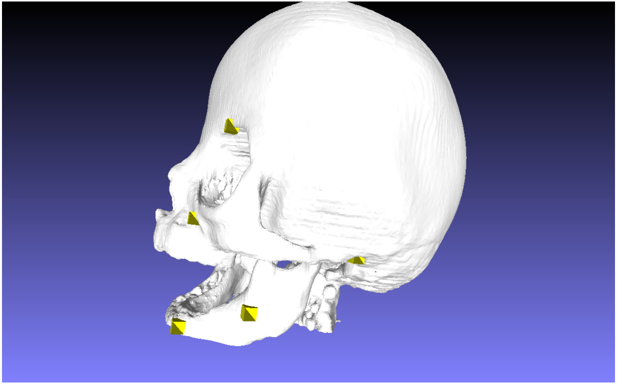

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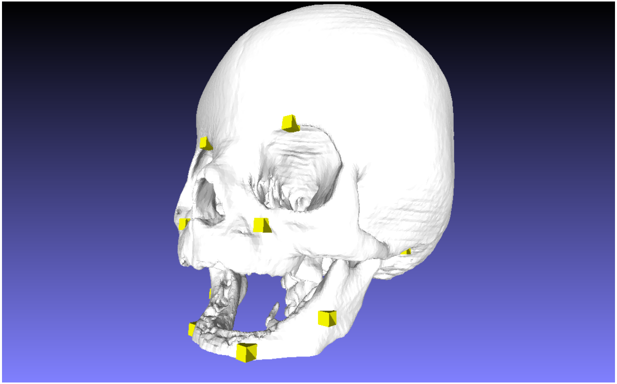

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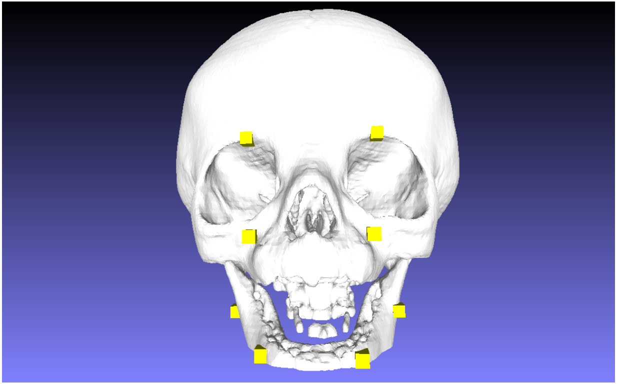

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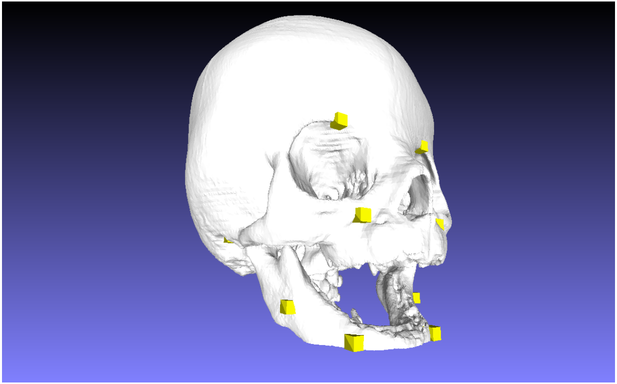

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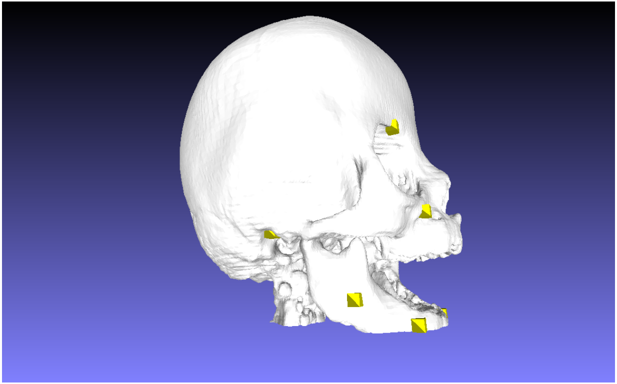

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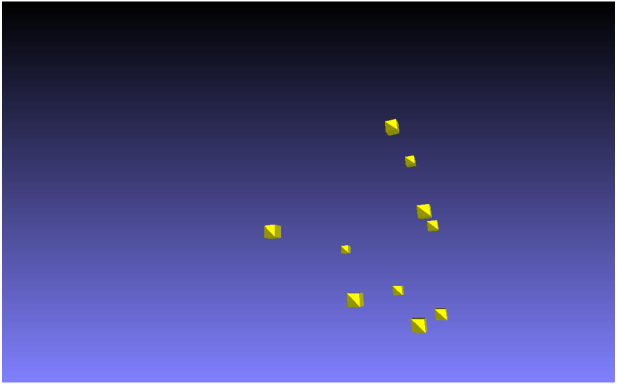

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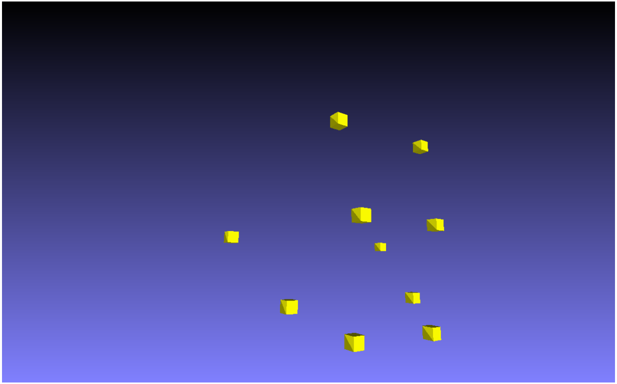

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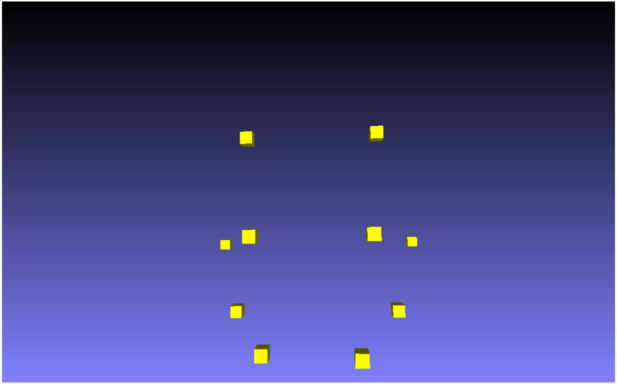

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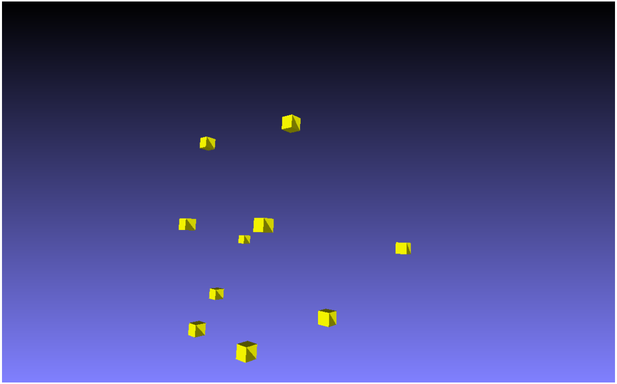

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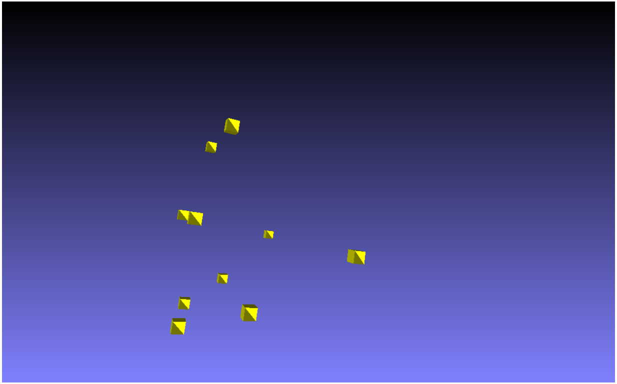

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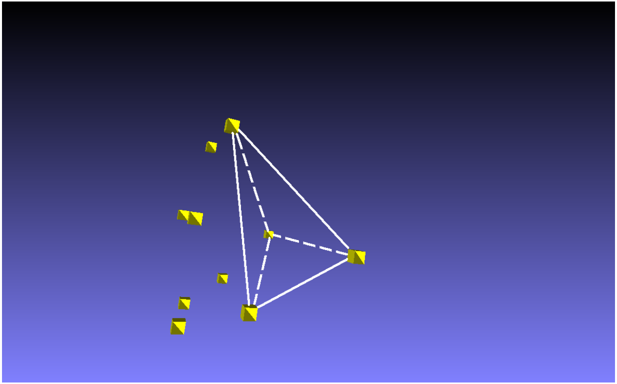

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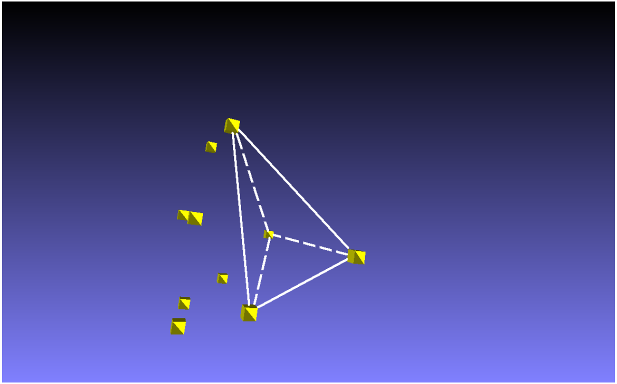
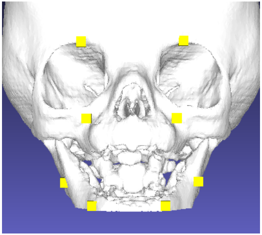


- We encode the shape of the anatomical structure with the orientation of all quadruplets of points.

$\Longrightarrow$ combinatorial study of 3D anatomical structures

The problem

But for an anatomical structure, we can define ordering relations between the coordinates of the points;



View from the front

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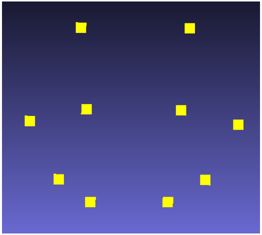

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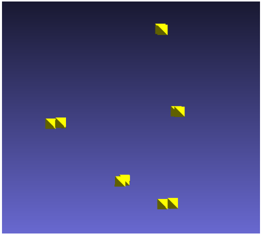

View from the right

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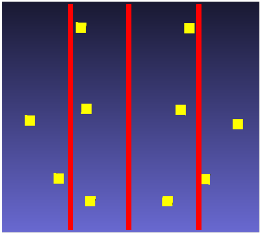
Some points are to the left of others



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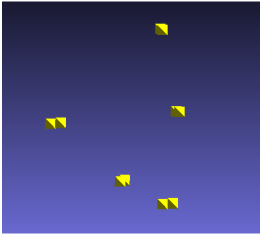

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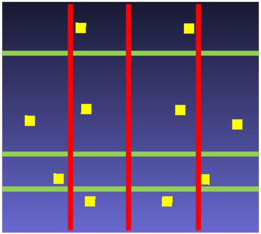
Some points are on top of others



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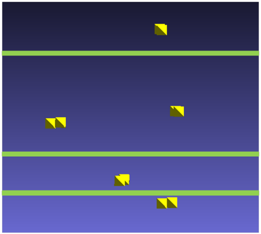

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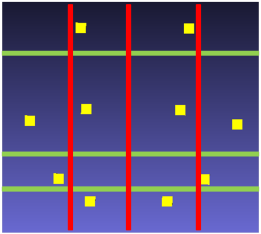

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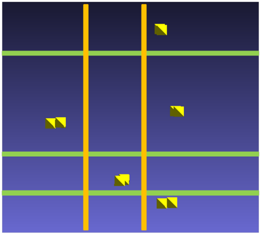

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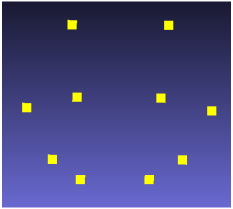
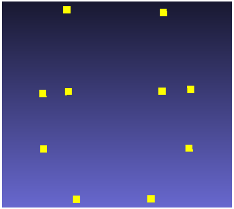
The problem

Landmark point positions change due to morphological variability or differences...



Skull 1

. . .



Skull k

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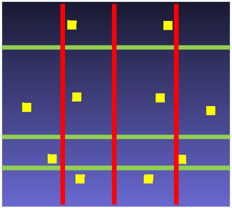

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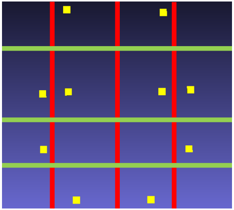

Skull 1 Skull k

but they still respect the orderings.

The problem

Landmark point positions change due to morphological variability or differences...

Skull 1 . . . Skull k

but they still respect the orderings.

Question

Can we determine quadruplets of points whose orientation depend only on the orderings (i.e. independently of the coordinate values)?

Orientation of a simplex

Definition

The orientation of a simplex ($\in \{+, -, 0\}$):

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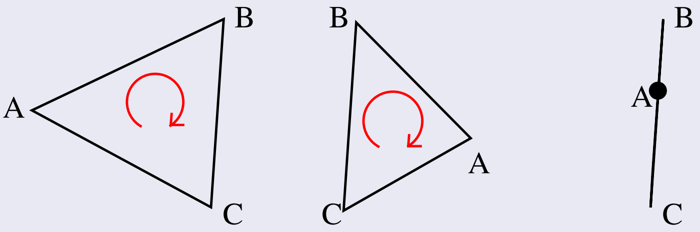
Triplet (A, B, C):



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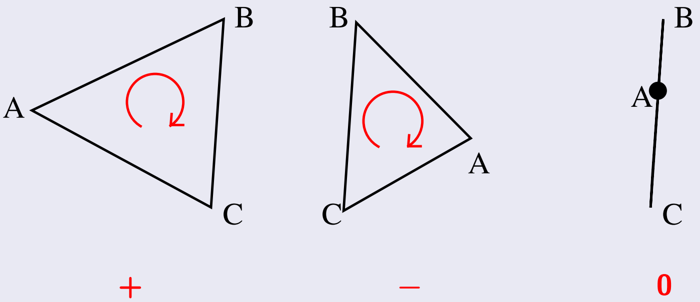

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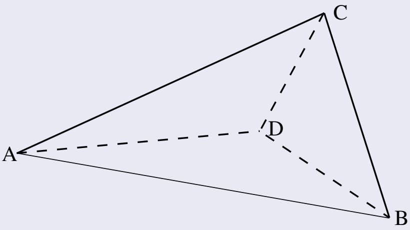

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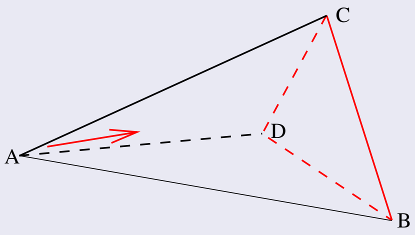

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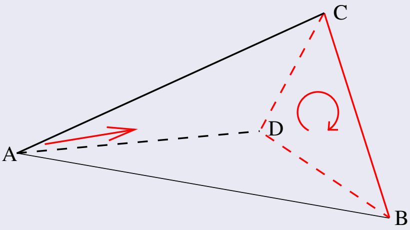

Orientation of the triplet (B, C, D)

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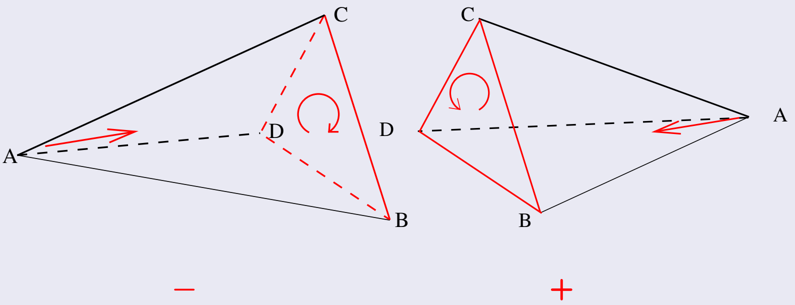

Orientation of a simplex

Definition

The orientation of a simplex ($\in \{+, -, 0\}$):

Orientations of simplices = chirotopes of an oriented matroid

Formalism

Notations

- M: a formal matrix

$$M = \begin{pmatrix} 1 & 1 & \dots & 1 \\ x_{1,1} & x_{2,1} & \dots & x_{n,1} \\ x_{1,2} & x_{2,2} & \dots & x_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1,n-1} & x_{2,n-1} & \dots & x_{n,n-1} \end{pmatrix}$$

where $x_{j,i}$ is a formal variable

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- $\mathcal{P}$: a set of n points P_j in a space of dimension $n-1$

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- M: a formal matrix
- $\mathcal{P}$: a set of n points P_j in a space of dimension $n-1$
- Assigns M with $P_{j,i}$ (i-th coordinate of the point P_j)

$$M_{\mathcal{P}} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ P_{1,1} & P_{2,1} & \dots & P_{n,1} \\ P_{1,2} & P_{2,2} & \dots & P_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ P_{1,n-1} & P_{2,n-1} & \dots & P_{n,n-1} \end{pmatrix}$$

Orientation of $\mathcal{P}$ $=$ sign of $\det(M_{\mathcal{P}})$

Formalism

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- M: a formal matrix
- $\mathcal{P}$: a set of n points P_j in a space of dimension $n-1$
- the real matrix $M_{\mathcal{P}}$

Orientation of $\mathcal{P}$ = sign of $\det(M_{\mathcal{P}})$

Remark

Orientation of $\mathcal{P} = 0 \iff \mathcal{P}$ is contained in an hyperplane

Configuration of orderings

Definition

We call *configuration of* $n-1$ *orderings on* $\mathcal{E}$, a set $\mathcal{C}$ of $n-1$ orderings on a set $\mathcal{E}$ of size n.

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Example: a configuration $\mathcal{C}$ of 3 orderings in $\{A, B, C, D\}$

$$A <_x B <_x C <_x D$$
$$B <_y D <_y C \quad \text{and} \quad B <_y A <_y C$$
$$D <_z A$$

$\mathcal{P}$ *satisfies* $\mathcal{C}$

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A set of points $\mathcal{P}$ *satisfies* $\mathcal{C}$ if
$\forall i \in \{1, \ldots, n-1\}, \forall e, f \in \mathcal{E}, \qquad e <_i f \Longrightarrow x_{e,i} < x_{f,i}$

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$\mathcal{P} = \{P_1\,(0,4,3);\ P_2\,(2,2,3);\ P_3\,(3,5,0);\ P_4\,(5,3,1)\}$ satisfies $\mathcal{C}$:

$$x(P_1) < x(P_2) < x(P_3) < x(P_4)$$
$$y(P_2) < y(P_4) < y(P_3) \quad \text{and} \quad y(P_2) < y(P_1) < y(P_3)$$
$$z(P_4) < z(P_1)$$

Fixed configuration

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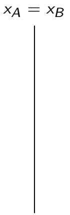

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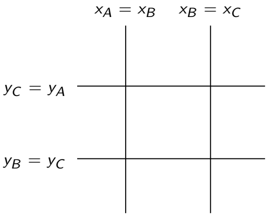

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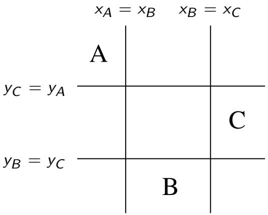

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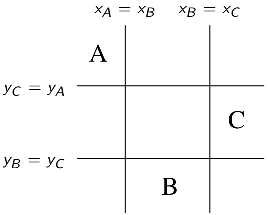

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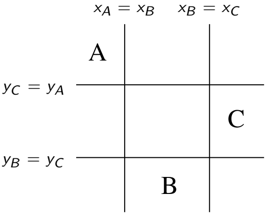

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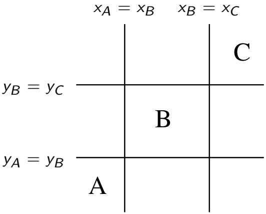

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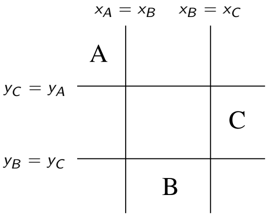

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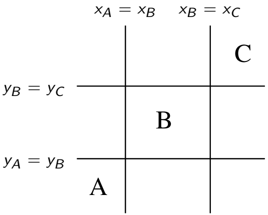

non-fixed configuration

Equivalence

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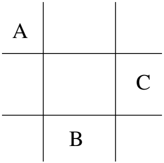

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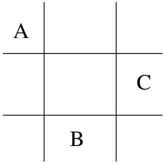
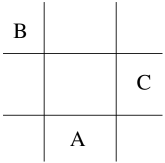
a relabelling of $\mathcal{E}$

$B <_x A <_x C$
$A <_y C <_y B$





Equivalence

Definition

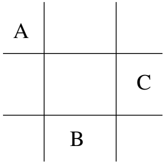
Two configurations of $n - 1$ orderings are *equivalent* if they are equal up to a relabelling of $\mathcal{E}$, a permutation of orderings, and reversion(s) of orderings.

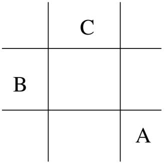
Example in 2D:

$A <_x B <_x C$
$B <_y C <_y A$

a permutation of orderings

$B <_x C <_x A$
$A <_y B <_y C$





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a reversion of an ordering

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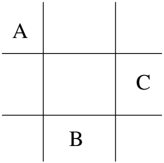

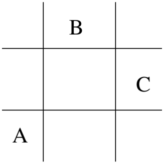

Sign of $det(M)$: $\sigma_{\mathcal{C}}(det(M))$

Definition

The *sign of* $det(M)$ *w.r.t.* $\mathcal{C}$, denoted $\sigma_{\mathcal{C}}(det(M))$, belongs to $\{\boxed{+}, \boxed{-}, \boxed{\pm}\}$:

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- if $\mathcal{C}$ is non-fixed: there exist $\mathcal{P}_1$ and $\mathcal{P}_2$ satisfying $\mathcal{C}$ such that $det(M_{\mathcal{P}_1}) < 0$ and $det(M_{\mathcal{P}_2}) > 0$.

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  $\Longrightarrow \sigma_{\mathcal{C}}(det(M)) = \boxed{\pm}$.

The problem (rewording)

Question (reminder)

Can we determine quadruplets of points whose orientation depend only on the orderings (i.e. independently of the coordinate values)?

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Determine the fixity of the configurations (determine if they are fixed or non-fixed).

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Determine the fixity of the configurations (determine if they are fixed or non-fixed).

The problem (rewording 2)

Does there exist $\mathcal{P}$ satisfying $\mathcal{C}$ such that $\det(M_{\mathcal{P}}) = 0$?

Linear extensions

Definition

A *linear extension of a configuration* $\mathcal{C}$ is a configuration where each ordering of $\mathcal{C}$ is replaced by one of its linear extensions.

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Example:

$$\mathcal{C}$$

$$A <_x B <_x C <_x D$$

$$B <_y D <_y C \quad \text{and} \quad B <_y A <_y C$$

$$D <_z A$$

a linear extension of $\mathcal{C}$

$$A <_x B <_x C <_x D$$

$$B <_y D <_y A <_y C$$

$$D <_z A <_z C <_z B$$

Linear extensions

Proposition 1

$\mathcal{C}$ is non-fixed $\iff \exists$ a non-fixed linear extension of $\mathcal{C}$.

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$\mathcal{C}$ is fixed $\iff$ all linear extension of $\mathcal{C}$ are fixed.

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Proposition 1 (rewording)

$\mathcal{C}$ is fixed $\iff$ all linear extension of $\mathcal{C}$ are fixed.

$\implies$ We will concentrate only on linear configurations.

Computing $\sigma_{\mathcal{C}}(det(M))$

Definition

When $det(M)$ can be written as

$$\text{“}det(M) = \sum \prod (x_{e,i} - x_{f,i})\text{”}$$

it is called *an expression of* $det(M)$

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$$\det(M) = \det \begin{pmatrix} 1 & 1 & 1 \\ x_A & x_B & x_C \\ y_A & y_B & y_C \end{pmatrix}$$

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&= \det \begin{pmatrix} 1 & 0 & 0 \\ x_A & x_B - x_A & x_C - y_A \\ y_A & y_B - y_A & y_C - y_A \end{pmatrix}
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$$det(M) \;=\; (x_B - x_A)(y_C - y_A) - (y_B - y_A)(x_C - x_A)$$

Definition

The *sign* $x_{e,i} - x_{f,i}$ *w.r.t.* $\mathcal{C}$, denoted $\sigma_{\mathcal{C}}(x_{e,i} - x_{f,i})$, belongs to $\{\boxed{+}, \boxed{-}\}$ such that:

$$\sigma_{\mathcal{C}}(x_{e,i} - x_{f,i}) = \boxed{+} \text{ if } f <_i e \text{ in } \mathcal{C};$$
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Computing $\sigma_{\mathcal{C}}(det(M))$

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Definition

The *sign of an expression of* $det(M)$ *w.r.t.* $\mathcal{C}$ is

- $\boxed{+}$ or $\boxed{-}$ if it can be calculated
- $\boxed{?}$ if not

Computing $\sigma_{\mathcal{C}}(det(M))$

Observation 1

If det(M) has such an expression whose sign is $\boxed{+}$ or $\boxed{-}$, then $\mathcal{C}$ is fixed.

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$$\boxed{+}$$

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$\boxed{+}$ $\boxed{+}$ $\boxed{-}$ $\boxed{+}$

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$$\color{red}{A <_x B} <_x C$$
$$B <_y A <_y C$$

$$\det(M) = (x_B - x_A)(y_C - y_A) - (y_B - y_A)(x_C - x_A)$$

$$\boxed{+}\ \boxed{+}\ \boxed{+} \quad - \quad \boxed{-}\ \boxed{+}\ \boxed{-} \quad = \quad \boxed{+}$$

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$\Longrightarrow \mathcal{C}$ is fixed

Computing $\sigma_{\mathcal{C}}(det(M))$

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Example (2):

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$$\boxed{+} \quad \boxed{+} \qquad \boxed{-} \quad \boxed{-}$$

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$$\boxed{+}\,\boxed{+}\,\boxed{+} \quad - \quad \boxed{-}\,\boxed{-}\,\boxed{+} \quad = \quad \boxed{?}$$

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$$\boxed{+}\quad\boxed{+}\qquad\qquad\boxed{-}\quad\boxed{-}$$
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we can not directly conclude

Key theorem / conjecture 1

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If $\det(M)$ has such an expression whose sign is $\boxed{+}$ or $\boxed{-}$, then $\mathcal{C}$ is fixed.

Theorem / Conjecture 1

$\mathcal{C}$ is fixed if and only if $\det(M)$ has an expression whose sign is $\boxed{+}$ or $\boxed{-}$.

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proved in dimension 2 and 3 ($n = 3$ and 4)

conjecture in higher dimensions

Characterization in dimension 2

Theorems 1 and 2

Up to equivalence, there are exactly two configurations of 2 orderings

$$A <_x B <_x C \qquad\qquad A <_x B <_x C$$
$$A <_y C <_y B \qquad\qquad A <_y B <_y C$$

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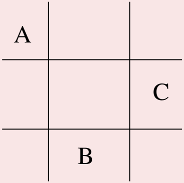
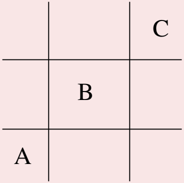


fixed configuration



non-fixed configuration

Characterization of the fixed configurations in 3D

Theorem 3: fixed configurations

The following are equivalent:

- $\mathcal{C}$ is fixed
- the sign of an expression of $(det(M)) \in \{\boxed{+}, \boxed{-}\}$
- 1. up to equivalence, $\mathcal{C}$ satisfies

$$B <_x C <_x A$$
$$C <_y A <_y B$$
$$A <_z B <_z C$$

  and

  2. $\exists\, X \in \{A, B, C\}$ such that we have either
     - $X < D$ in all the orderings

     or
     - $X > D$ in all the orderings

Fixed configurations in 3D

Up to equivalence, there are exactly 4 fixed configurations:

$$B <_x C <_x A <_x D$$
$$C <_y A <_y B <_y D$$
$$A <_z B <_z C <_z D$$

$$B <_x C <_x D <_x A$$
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An other characterization in 3D

$\mathcal{C}$

$$B <_x D <_x C <_x A$$
$$C <_y A <_y B <_y D$$
$$A <_z B <_z D <_z C$$

a configuration induced by $\mathcal{C}$
w.r.t. the ordering $<_y$

$$D <_x C <_x A$$
$$A <_z D <_z C$$

An other characterization in 3D

$\mathcal{C}$	a configuration induced by $\mathcal{C}$ w.r.t. the ordering $<_y$
$B <_x D <_x C <_x A$ $C <_y A <_y B <_y D$ $A <_z B <_z D <_z C$	$D <_x C <_x A$ $A <_z D <_z C$

Theorem 4: non-fixed configurations

Let $\mathcal{C}'$ be a configuration induced by $\mathcal{C}$ on $\mathcal{E}'$ w.r.t. $<_i$. Let $P \in \mathcal{E} \backslash \mathcal{E}'$.

$\mathcal{C}$ is non-fixed if and only if

- $\mathcal{C}'$ is non-fixed and
- P is extreme in the ordering $<_i$ of $\mathcal{C}$,

An other characterization in 3D

Theorem 4: non-fixed configurations

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Example:

$C <_x D <_x A <_x B$

$A <_y C <_y B <_y D$

$A <_z B <_z C <_z D$

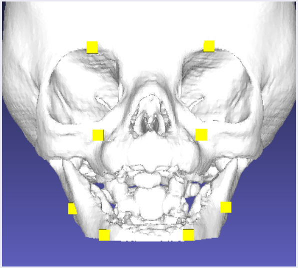
non-fixed configuration induced by $\mathcal{C}$

extreme point

Experimentation

In this example

Set of skulls for morphometrical analysis of craniofacial morphology (dental classes)

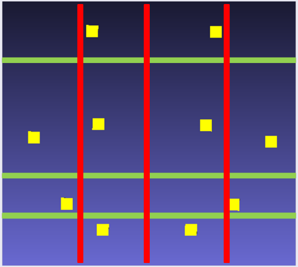


- 10 points 3D

Experimentation

In this example

Set of skulls for morphometrical analysis of craniofacial morphology (dental classes)



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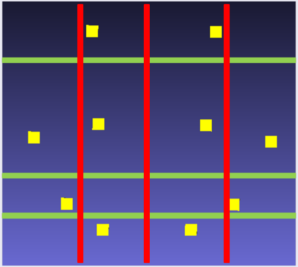
- 210 configurations
- 8,112 linear extensions

Experimentation

In this example

Set of skulls for morphometrical analysis of craniofacial morphology (dental classes)



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- $\Longrightarrow$ 20 fixed configurations

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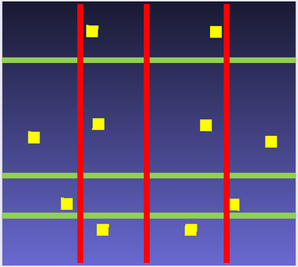

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Goal

Find the quadruplets of points which characterize significantly the morphological differences.

Conjecture

Theorem / Conjecture 1

$\mathcal{C}$ is fixed if and only if $\det(M)$ has an expression whose sign is $\boxed{+}$ or $\boxed{-}$.

proved in dimension 2 and 3 ($n = 3$ and 4)

conjecture in higher dimensions

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Thanks!