Cyclic Orderings and Cyclic Arboricity of Matroids

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Abstract

We derive a general result concerning cyclic orderings of the elements of a matroid. As corollaries we obtain two further results. The first corollary proves a conjecture of Gonçalves [7], stating that the circular arboricity of a matroid is equal to its fractional arboricity. This generalises a well-known result from Nash-Williams on covering graphs by spanning trees, and a result from Edmonds on covering matroids by bases. The second corollary is the proof of a special case of a conjecture of Kajitani *et al* [8] about the possibility of ordering the elements of a matroid of rank r to have a cyclic ordering in which every r consecutive elements form a base of the matroid.

1 Introduction

1.1 Cyclic Orderings of Matroids

Matroids are combinatorial structures which model the independence property in linear spaces. The crucial axiom on the bases of a matroid is the *exchange axiom*. There are actually several forms of this (see [11]), the important one for us is: if B and B' are bases and $b' \in B' \setminus B$, then there exists $b \in B \setminus B'$ such that $(B \setminus \{b\}) \cup \{b'\}$ is a base. If a non-empty set of subsets $\mathcal{B}$ on a ground set E satisfies this axiom, it is the base set of a matroid $\mathcal{M}$. Observe that it follows directly that all bases have the same cardinality. A set $F \subseteq E$ is an *independent set* if it is contained in a base. A *rank* function is then associated to $\mathcal{M}$ by letting the rank $r(F)$ of any subset F of E to be equal to the maximum cardinality of an independent subset included in F.

For those readers more familiar with graphs than with matroids, a classical class of matroids is formed by taking the edge set of a connected graph as the ground set, and as bases

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the edge sets of spanning trees. In such a matroid, independent sets are sets of edges without a cycle, and the rank of a set of edges is the number of edges of a maximum forest contained in that set.

Similarly, those with a background in linear algebra can take a set of vectors from their favourite vector space as the ground set. In that case, the normal concept of bases and independence can be used. And the rank of a set corresponds to the dimension of the set (more formally, the dimension of the subspace spanned by the vectors in the set).

For (much) more about matroids, we refer to the textbook of Oxley [11]. One difference with Oxley's book is that instead of the closure $\mathrm{cl}(F)$ of a set $F \subseteq E$, we will use *span* $\mathrm{sp}(F)$. (See the definition in Section 2.) We will always use E to denote the ground set of a matroid. All our matroids are assumed to be finite and contain no loops (i.e, there is no element $e \in E$ such that $\{e\}$ is a dependent set).

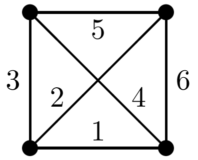
The base exchange axiom immediately implies that given two bases B and B', there exists a sequence of exchanges which transforms B into B', i.e., there exists a sequence of bases $B = B_0, \ldots, B_i, \ldots, B_r = B'$ for which every pair of consecutive bases differ on one element. Assuming now that B and B' are disjoint (hence r is equal to the rank), a specific way to get such a sequence is as follows : Let $B' = \{b'_1, b'_2, \ldots, b'_r\}$. Take $b_1 \in B$ such that $(B \setminus \{b_1\}) \cup \{b'_1\}$ is a base, then take $b_2 \in B$ such that $(B \setminus \{b_1, b_2\}) \cup \{b'_1, b'_2\}$ is a base, etc. We then obtain a sequence $(b_1, b_2, \ldots, b_r, b'_1, b'_2, \ldots, b'_r)$ of elements of E such that every set of r consecutive terms forms a base (indeed, these can be taken as the sets B_i).

A natural question is if this sequence provides a *cyclic ordering* in which each r cyclically consecutive elements form a base. I.e., is it the case that the sequences $(b'_2, \ldots, b'_r, b_1)$, $(b'_3, \ldots, b'_r, b_1, b_2)$, etc., form bases as well ? It is easy to come up with examples that show this is false in general. Take for instance the sequence of vectors $\binom{1}{0}, \binom{0}{1}, \binom{1}{1}, \binom{2}{0}$ in $\mathbb{R}^2$. But note that a simple reordering to the sequence $\binom{1}{0}, \binom{0}{1}, \binom{2}{0}, \binom{1}{1}$ does indeed give a cyclic ordering in which each two cyclically consecutive elements form a base.

It is an intriguing conjecture that such a cyclic ordering is in fact always possible.

Conjecture 1.1
Given two bases $B = b_1, \ldots, b_r$ and $B' = b'_1, \ldots, b'_r$ of a matroid, there exists a cyclic ordering of the sequence $(b_1, b_2, \ldots, b_r, b'_1, b'_2, \ldots, b'_r)$ such that every r cyclically consecutive terms in the cyclic ordering form a base.

To illustrate this conjecture further, we can give two examples based on the edge set of the complete graph K_4, sketched below. The graphical matroid $\mathcal{M}(K_4)$ has all sets of three edges



that do not form a cycle as its bases. So the sequence $(3, 1, 5, 2, 4, 6)$ is a cyclic ordering in which each three cyclically consecutive terms form a base.

Another matroid $\mathcal{M}_2$ on the edge set of K_4 can be found by taking as bases all sets of three edges incident with the same vertex and their complements. So $\{1,2,3\}$ and $\{4,5,6\}$ are bases, but $\{1,2,5\}$ is not. A cyclic ordering according to Conjecture 1.1 for this matroid is, for instance, $(1,2,3,5,4,6)$.

A conjecture requiring a more specific cyclic ordering of the elements of the sequence $(b_1,b_2,\dots,b_r,b'_1,b'_2,\dots,b'_r)$ was in fact already asked by Gabow [6], and later again by Wiedemann [14] and Cordovil and Moreira [2].

Conjecture 1.2
Let $B=b_1,\dots,b_r$ and $B'=b'_1,\dots,b'_r$ be two bases of a matroid $\mathcal{M}$. Then it is possible to permute the elements of B to a sequence $(b_{\pi(1)},\dots,b_{\pi(r)})$ and permute the elements of B' to a sequence $(b'_{\pi'(1)},\dots,b'_{\pi'(r)})$ such that the combined sequence $(b_{\pi(1)},\dots,b_{\pi(r)},b'_{\pi'(1)},\dots,b'_{\pi'(r)})$ is a cyclic ordering in which every r cyclically consecutive elements form a base.

The answer to Conjecture 1.2 (and hence to Conjecture 1.1) is "yes" if $\mathcal{M}$ is a graphical matroid [2, 8, 14], but as far as we are aware the answer of both is unknown even if $\mathcal{M}$ is a vector space over some field.

Kajitani *et al* [8] also formulated a much more general conjecture. Let us introduce for this the key-definition of this paper.

For a matroid $\mathcal{M}$, we define the *maximal density* $\gamma(\mathcal{M})$ as $\gamma(\mathcal{M})=\max\limits_{\varnothing\neq A\subseteq E}\dfrac{|A|}{r(A)}$. (To avoid division by zero, here we must assume that the matroid has no loops.) Following Catlin *et al* [1], we call $\mathcal{M}$ *uniformly dense* if $\gamma(\mathcal{M})=\dfrac{|E|}{r(E)}$. Observe that every matroid admitting a partition of its ground set into bases is uniformly dense. A matroid $\mathcal{M}$ of rank r is *cyclically orderable* if there exists a cyclic ordering $(e_1,\dots,e_m)$ of E such that every r cyclically consecutive terms form a base of $\mathcal{M}$.

Conjecture 1.3
A matroid is cyclically orderable if and only if it is uniformly dense.

This conjecture would imply that if $\mathcal{M}$ is a uniform dense matroid with $\gamma(\mathcal{M})=\dfrac{P}{Q}$, then there exist P bases so that each element of E appears in exactly Q of them. This weaker result was proved by Fraisse and Hell [5].

As one of the main results of this paper, we prove Conjecture 1.3 for a special class of matroids.

Theorem 1.4
Let $\mathcal{M}$ be a matroid such that $|E|$ and $r(E)$ are coprime (i.e, $\gcd(|E|,r(E))=1$*). Then $\mathcal{M}$ is cyclically orderable if and only if it is uniformly dense.*

The proof of this theorem can be found in Section 3.

1.2 Variants of Arboricity of Matroids

We next move towards the second main result of this paper. The *arboricity* $\Upsilon(\mathcal{M})$ of a matroid $\mathcal{M}$ is the minimum number of bases needed to cover all elements of the matroid. Since every base can contain at most $r(A)$ elements for any subset A of the matroid, the arboricity of a matroid is always at least its maximal density. A classical result of Edmonds [3], extending the result for graphs by Nash-Williams [10], guarantees that this lower bound gives in fact the right answer.

Theorem 1.5 (Edmonds [3])
For a matroid $\mathcal{M}$ *we have* $\Upsilon(\mathcal{M}) = \lceil \gamma(\mathcal{M}) \rceil$.

The *fractional arboricity* $\Upsilon_f(\mathcal{M})$ of a matroid $\mathcal{M}$ is defined as follows: To each base B of $\mathcal{M}$ assign a non-negative real value $x(B)$, such that for each element e the sum $\sum\limits_{B \ni e} x(B)$ is at least one. Then $\Upsilon_f(\mathcal{M})$ is the minimum of $\sum\limits_{B \in \mathcal{B}(\mathcal{M})} x(B)$ we can obtain under these conditions. Again we have that $\Upsilon_f(\mathcal{M})$ is at least the maximal density $\gamma(\mathcal{M})$, but as has been observed by several authors (see, e.g., Catlin *et al* [1] and Scheinerman and Ullman [12, Section 5.4]), the fact that we have equality follows easily from Edmonds' Theorem mentioned above.

Proposition 1.6
For a matroid $\mathcal{M}$ *we have* $\Upsilon_f(\mathcal{M}) = \gamma(\mathcal{M})$.

Proof Suppose $\gamma(\mathcal{M}) = P/Q$ for some positive integers P, Q. Let $\mathcal{M}^Q$ be the matroid obtained from $\mathcal{M}$ by replacing each element by Q parallel elements. Then we have $\gamma(\mathcal{M}^Q) = P$, and hence by Edmonds' Theorem we can cover $\mathcal{M}^Q$ by P bases. Interpreting these bases as bases of $\mathcal{M}$ and setting $x(B) = 1/Q$ for each of them, gives the required equality. $\square$

We now define a third kind of arboricity, *circular arboricity* $\Upsilon_c(\mathcal{M})$. Let S_d be the topological cycle with circumference d. When speaking about a (left-closed, right-open) *cyclic interval* $[x, y)$ of S_d, we implicitly assume that we start at x and follow S_d in the clockwise order until reaching y. Given a matroid $\mathcal{M}$, we want to map the elements of E to S_d such that for every cyclic unit interval $[x, x+1)$, the elements mapped into that cyclic interval form an independent set. Then we define $\Upsilon_c(\mathcal{M})$ as the infimum over all values of d for which such a mapping is possible. Since we assume the matroid to be finite and loopless, it is easy to see that this infimum is actually attained and is in fact a rational number.[1]

Proposition 1.7
For a matroid $\mathcal{M}$ *we have* $\Upsilon_f(\mathcal{M}) \leq \Upsilon_c(\mathcal{M}) \leq \Upsilon(\mathcal{M})$.

Proof Set $d = \Upsilon(\mathcal{M})$, and take bases $B_1, \ldots, B_d$ covering all elements of $\mathcal{M}$. By removing elements from certain bases if they appear in more than one base, we find d independent

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[1] We would have the same definition if cyclic intervals were open on both sides or left-open, right-closed. With closed cyclic unit intervals we get the same value for the circular arboricity, but it would be a real infimum in that instance.

sets $I_1, \dots, I_d$ such that each element appears in exactly one independent set. Now for each element e, if $e \in I_i$, then map e to the point i on the cycle S_d. Then every cyclic unit interval contains exactly one of the independent sets I_i, which proves $\Upsilon_c(\mathcal{M}) \leq d$.

Next set $d = \Upsilon_c(\mathcal{M})$. Suppose $\Phi : E \longrightarrow S_d$ is a mapping satisfying the requirements for the circular arboricity. By going around the cycle, this mapping gives an ordering $(e_1, \dots, e_m)$ of E (by taking $(e_1, \dots, e_m)$ such that $0 \leq \Phi(e_1) \leq \Phi(e_2) \leq \cdots \leq \Phi(e_m) < d$). With each element e_i we associate an independent set I_i by taking the elements that get mapped into $[\,\Phi(e_i), \Phi(e_i) + 1\,)$. If necessary, we can add arbitrarily chosen extra elements to extend I_i to a base B_i. For $i = 1, \dots, m$, let $x(B_i)$ be equal to the difference between $\Phi(e_{i+1})$ and $\Phi(e_i)$ (measuring by going from $\Phi(e_i)$ to $\Phi(e_{i+1})$ in clockwise direction), where we take $e_{m+1} = e_1$. Note that $x(B_i)$ is zero if $\Phi(e_i) = \Phi(e_{i+1})$. For all other bases B of $\mathcal{M}$, set $x(B) = 0$. It is easy to check that for all elements e in E we have $\sum\limits_{B \ni e} x(B) \geq 1$, and that $\sum\limits_{B \in \mathcal{B}(\mathcal{M})} x(B) = d$, proving that $\Upsilon_f(\mathcal{M}) \leq d$. □

Our second main result is that we in fact always have equality between fractional and circular arboricity.

Theorem 1.8
For a matroid $\mathcal{M}$ we have $\Upsilon_c(\mathcal{M}) = \Upsilon_f(\mathcal{M})$.

This result was conjectured for graphical matroids by Gonçalves [7, Section 3.8]. Its proof will be given in Section 4.

1.3 Weighted Orderings of Matroids

We finally describe our most general result from which we can derive Theorems 1.4 and 1.8. A *weighted matroid* is a matroid $\mathcal{M}$ together with a weight function ω from E into the non-negative rational numbers. The weight $\omega(A)$ of a subset $A \subseteq E$ is the sum of the weights of its elements. Similar to the unweighted case, we define the maximal density $\gamma(\mathcal{M}, \omega)$ as $\max\limits_{\varnothing \neq A \subseteq E} \dfrac{\omega(A)}{r(A)}$. Again, a weighted matroid is *uniformly dense* if E achieves the maximal density.

Let Φ be a mapping from E into S_d. The *cyclic interval* associated with an element e of E is the cyclic interval $J(e) = [\,\Phi(e), \Phi(e) + \omega(e)\,)$ of S_d. Thus, to every point x of S_d, one can associate the set of elements $E_\Phi(x) \subseteq E$ such that $e \in E_\Phi(x)$ whenever $x \in J(e)$. The mapping Φ is a *fractional cyclic independence order* of $(\mathcal{M}, \omega)$ if $E_\Phi(x)$ is independent for every point x of S_d.

If d is chosen large enough, there exists certainly a fractional cyclic independence order of $(\mathcal{M}, \omega)$ into S_d. But we can in fact characterise the values of d for which such a mapping exists.

Theorem 1.9
Let $(\mathcal{M}, \omega)$ be a weighted matroid, and d a positive rational number. Then there exists a fractional cyclic independence order of $(\mathcal{M}, \omega)$ into S_d if and only if d is greater than or equal to the maximal density $\gamma(\mathcal{M}, \omega)$.

The remainder of this paper is organised as follows. In the next section we will prove Theorem 1.9. We will also state the result that can be obtained from it using matroid duality. In the two following sections we will show how Theorems 1.4 and 1.8 follow from Theorem 1.9, and discuss some corollaries that can be obtained from those theorems in turn.

2 Proof of Theorem 1.9

We use the notations and conventions from the first section. Since the weights $\omega(e)$ and the number d in the hypothesis of Theorem 1.9 are assumed to be rational, it is obvious that the theorem is equivalent to Theorem 2.1 below. Here a *D-gon*, for some positive integer D, can be thought of as the sequence $(1, 2, \ldots, D)$ in cyclic order (in clockwise direction), or as the integer elements of the topological cycle S_D. This means we also assume that for a mapping Φ of E into the D-gon, each image $\Phi(e)$ is a point from the D-gon, and the cyclic interval $[\,\Phi(e), \Phi(e)+\omega(e)\,)$ can be thought of as just the sequence $(\Phi(e), \Phi(e)+1, \ldots, \Phi(e)+\omega(e)-1)$ taken modulo D. Since we are now working with integers only, we drop the word "fractional".

Theorem 2.1
Let $(\mathcal{M}, \omega)$ be a weighted matroid with all weights non-negative integers, and D a positive integer. Then the following statements are equivalent :

(a) *There exists a cyclic independence order Φ of $(\mathcal{M}, \omega)$ into the D-gon.*

(b) *For all $A \subseteq E$ we have $\omega(A) \leq D \cdot r(A)$.*

We use a few more concepts from matroid theory in our proof. A set is *dependent* if it is not independent. A minimal dependent set (minimality with respect to subset inclusion) is called a *circuit*. The *span* $\mathrm{sp}(A)$ of a set $A \subseteq E$ is the maximal set containing A whose rank is equal to the rank of A. It is easy to show that an element $e \in E$ is in $\mathrm{sp}(A)$ if and only if $e \in A$ or if there is a circuit in $A \cup \{e\}$ containing e.

Again, the reader more familiar with graph theory should be able to follow the proof below by translating everything into graph terms, where the ground set is the set of edges of some graph. In that context, an independent set is an acyclic set (i.e., the edge set of a spanning forest), a circuit is a cycle, the rank of a set A is the maximal number of edges of an acyclic subset of A, and the span of A is the maximal collection of edges in a subgraph that has the same connected components as A.

For convenience, in the remainder we often use $A - e$ to denote $A \setminus \{e\}$, and $A + e$ for $A \cup \{e\}$.

We need the following well-known properties of the span operator in matroids.

Property 2.2
Let $\mathcal{M}$ be a matroid with ground set E.

(i) *If for $A, A' \subseteq E$ we have $\mathrm{sp}(A) = \mathrm{sp}(A')$, then for all $B \subseteq E$ we have $\mathrm{sp}(A \cup B) = \mathrm{sp}(A' \cup B)$.*

(ii) *If $e \in A \subseteq E$, then $e \in \mathrm{sp}(A - e)$ if and only if e is contained in a circuit of A.*

(iii) *Let $A \subseteq E$, $e \in A$ and $f \notin A$, and suppose that there is a circuit C in $A + f$ such that $f \in C$ and $e \notin C$. Then $\mathrm{sp}(A - e) = \mathrm{sp}(A + f - e))$.*

Proof of Theorem 2.1 Observe first that if there is a subset $A \subseteq E$ such that $\omega(A) > D \cdot r(A)$, then no matter how we map E into the D-gon, for one of the points x of the D-gon we get that $E_\Phi(x) \cap A$ has more than $r(A)$ elements, and hence $E_\Phi(x)$ cannot be an independent set. In particular there is no cyclic independence order.

So we only have to prove (b) $\Rightarrow$ (a). For this, let D satisfy the condition in (b). We prove (a) by induction on $|E|$. (It is trivially true if $|E| = 1$.) If there is an element e such that $\omega(e) = 0$, then we can remove e from the matroid and are done by induction on the smaller matroid. So from now on we assume $\omega(e) > 0$ for all $e \in E$.

Given two mappings Φ, Φ' of E into the D-gon, we say that Φ *is better than* Φ' if for every point x of the D-gon, the span of $E_\Phi(x)$ contains the span of $E_{\Phi'}(x)$. We also say that Φ *is strictly better than* Φ' if the inclusion is strict for some point x, and that Φ is *best possible* if no other mapping is strictly better.

Since there are only finitely many mappings into the D-gon, we can choose a best possible mapping Φ. Our goal is to prove that Φ is a cyclic independence order. Assume that this is not the case. Then there exists some point x of the D-gon for which $E_\Phi(x)$ is not independent, i.e., $E_\Phi(x)$ contains a circuit. Since $\omega(E) \leq D \cdot r(E)$, that means that at the same time there must be a point x' of the D-gon for which $E_\Phi(x')$ does not span E.

If $e \in E$ belongs to a circuit contained in $E_\Phi(\Phi(e))$, then a *push* of e simply consists of replacing $\Phi(e)$ by $\Phi(e)+1$ (although it might be better intuitively to think of it as replacing the cyclic interval $[\,\Phi(e), \Phi(e)+\omega(e)\,)$ by $[\,\Phi(e)+1, \Phi(e)+\omega(e)+1\,)$). Observe that performing one push always results in a better mapping since the span of $E_\Phi(\Phi(e))$ does not decrease. Moreover, a push will give a strictly better mapping if adding e to $E_\Phi(\Phi(e)+\omega(e)+1)$ does increase the span of that set, i.e., if e does not belong to a circuit of $E_\Phi(\Phi(e)+\omega(e)+1) \cup \{e\}$. However, since Φ is assumed to be best possible, no sequence of pushes can result in a strictly better mapping.

On the other hand, there always exists an element that we can push. This follows from our earlier observation that for some point x, there exists a circuit C in $E_\Phi(x)$. Going counterclockwise back along the D-gon, starting from x, let y be the last point of the D-gon for which C is contained in $E_\Phi(y)$. By definition of y, there exists $e \in C$ such that $\Phi(e) = y$. In particular, we can push e.

From the arguments in the previous paragraphs, there exists an infinite sequence of pushes. To make the sequence of pushes deterministic, we do the following: Start with some initial ranking $e_1, \ldots, e_m$ of the elements. Every time we push an element, we rearrange this ranking by moving that element to the back of the sequence. (So elements towards the end of the ranking have been pushed "more recently" than those towards the beginning.) Whenever we have a choice between pushing elements, we always push the first pushable element according to the ranking at that moment.

Considering the deterministic sequence of pushes thus obtained, let us say that an element e of E is *bounded* if it is pushed a finite number of times; otherwise it is *unbounded*. Starting with Φ this means that after a finite number of pushes we obtain a mapping for which all bounded elements have reached their final position. Continuing with the sequence, the sequence of mappings eventually must become a periodic sequence. Let $\Phi_1, \ldots, \Phi_T$ be the mappings occurring in this periodic sequence. We will analyse the properties of this sequence of these mappings in some detail.

We first remark that by the definition of pushing, each of the mappings Φ_i is also best possible, which guarantees the following.

Claim 1 *For all i, j and all points x, $\mathrm{sp}\big(E_{\Phi_i}(x)\big) = \mathrm{sp}\big(E_{\Phi_j}(x)\big)$.*

Let E^U be the set of unbounded elements, and E^B the bounded ones. As some $E_{\Phi_1}(x)$ do not span E, the set E^B of bounded elements is non-empty. Also, by our supposition that we are dealing with an infinite sequence of pushes, the set E^U is not empty.

Let e be an element in E^B. Set $x_e = \Phi_1(e)$. Since e has reached its final position by the time we consider the mappings $\Phi_1, \ldots, \Phi_T$, we have $\Phi_i(e) = \Phi_1(e)$ for all i. And so in fact $x_e = \Phi_i(e)$ for all i.

Claim 2 *For all i, e does not belong to a circuit of $E_{\Phi_i}(x_e)$.*

Indeed, suppose this is false for some i. Then e is pushable in Φ_i. Since this holds in each of the (infinitely many) later appearances of Φ_i, eventually e will become the first among the pushable elements in Φ_i. So e will eventually be pushed, a contradiction.

By this claim, all the pushes of elements from $E_{\Phi_i}(x_e)$, for any i, involve circuits that do not contain e. So, using Claim 1 and Property 2.2 (iii), we have that $\mathrm{sp}\big(E_{\Phi_i}(x_e) - e\big) = \mathrm{sp}\big(E_{\Phi_j}(x_e) - e\big)$ for all i, j. By Property 2.2 (ii) we also have that $e \notin \mathrm{sp}\big(E_{\Phi_i}(x_e) - e\big)$ for all i.

Now, additionally, let f be an element in E^U. Since f cycles infinitely around the D-gon, there is a j such that $f \in E_{\Phi_j}(x_e)$. But that means trivially that $\mathrm{sp}\big(E_{\Phi_j}(x_e) + f - e\big) = \mathrm{sp}\big(E_{\Phi_j}(x_e) - e\big)$. Using Claim 1, Property 2.2 (i), and the relations above, this gives that for all i, j :

$$\mathrm{sp}\big(E_{\Phi_i}(x_e) + f - e\big) \;=\; \mathrm{sp}\big(E_{\Phi_j}(x_e) + f - e\big) \;=\; \mathrm{sp}\big(E_{\Phi_j}(x_e) - e\big) \;=\; \mathrm{sp}\big(E_{\Phi_i}(x_e) - e\big).$$

Since for all i, $e \notin \mathrm{sp}\big(E_{\Phi_i}(x_e) - e\big)$, this leads to $e \notin \mathrm{sp}\big(E_{\Phi_i}(x_e) + f - e\big)$

We can repeat these arguments for every $f \in E^U$ to obtain the following.

Claim 3 *For all i and $e \in E^B$ we have $e \notin \mathrm{sp}\big((E_{\Phi_i}(x_e) \cup E^U) - e\big)$.*

Since $E^U \subseteq (E_{\Phi_i}(x_e) \cup E^U) - e$, this immediately gives that $e \notin \mathrm{sp}(E^U)$. But this holds for all $e \in E^B$, and so $E^B \cap \mathrm{sp}(E^U) = \varnothing$. We have proved our final, crucial, claim :

Claim 4 *$\mathrm{sp}(E^U) = E^U$.*

This means that the contracted matroid $\mathcal{M}/E^U$ has exactly E^B as its elements. Moreover, $\Phi_1|_{E^B}$ is a cyclic independence order of the contracted matroid $(\mathcal{M}/E^U, \omega|_{E^B})$ into the D-gon.

Now consider $\mathcal{M}\backslash E^B$, the submatroid of $\mathcal{M}$ restricted to E^U, and let r' be the rank function of this restricted matroid. Since $r'(A) = r(A)$ for all $A \subseteq E^U$, we have that $\omega(A) \leq D{\cdot}r'(A)$ for all $A \subseteq E^U$. Hence by the induction hypothesis, there exists a cyclic independence order Φ' from $(\mathcal{M}\backslash E^B, \omega|_{E^U})$ into the D-gon.

Finally, combine the mappings $\Phi_1|_{E^B}$ and Φ' to a mapping Ψ of E into the D-gon:

$$\Psi(e) \;=\; \begin{cases} \Phi_1|_{E^B}(e), & \text{if } e \in E^B; \\ \Phi'(e), & \text{if } e \in E^U. \end{cases}$$

For each point x, the set $E_\Psi(x)$ obtained from this mapping has the property that $E_\Psi(x) \cap E^B$ is independent in $\mathcal{M}/E^U$ and $E_\Psi(x) \cap E^U$ is independent in $\mathcal{M} \backslash E^B$. That means the whole set $E_\Psi(x)$ is independent in $\mathcal{M}$, proving that Ψ is a cyclic independence order from $(\mathcal{M}, \omega)$ into the D-gon. □

We next describe in some detail the corollary of Theorem 1.9 by considering the dual of a matroid. A mapping Φ of the elements E of a weighted matroid $(\mathcal{M}, \omega)$ in S_d is called a *fractional cyclic full-rank order* if for every point x of S_d, we have that $r(E_\Phi(x)) = r(E)$. (Recall that $E_\Phi(x)$ is the set of elements e of E for which the cyclic interval $[\,\Phi(e), \Phi(e)+\omega(e)\,)$ contains x.)

We first give the integer version, similar to Theorem 2.1.

Corollary 2.3
Let $(\mathcal{M}, \omega)$ be a weighted matroid with all weights non-negative integers, and D a positive integer. Then the following statements are equivalent:

(a) *There exists a cyclic full-rank order Φ of $(\mathcal{M}, \omega)$ into the D-gon.*

(b) *For all $A \subseteq E$ we have $\omega(A) \geq D \cdot (r(E) - r(E \setminus A))$.*

Proof Let $\mathcal{M}^*$ be the dual matroid of $\mathcal{M}$, with rank function r^*, and define a new integer weight ω^* by setting, for all $e \in E$, $\omega^*(e) = D - \omega(e)$. Then for all $A \subseteq E$ we have

$$\begin{aligned} \omega(A) \geq D \cdot (r(E) - r(E \setminus A)) &\iff D \cdot |A| - \omega^*(A) \geq D \cdot (r(E) - r(E \setminus A)) \\ &\iff \omega^*(A) \leq D \cdot (|A| - r(E) + r(E \setminus A)) \\ &\iff \omega^*(A) \leq D \cdot r^*(A). \end{aligned}$$

By Theorem 2.1, this means (b) is equivalent to the existence of a cyclic independence order Φ^* of $(\mathcal{M}^*, \omega^*)$ into the D-gon, i.e., a mapping of E into cyclic intervals $[\,\Phi^*(e), \Phi^*(e) + \omega^*(e)\,)$ such that for all points x we have that $E_{\Phi^*}(x)$ is independent in $\mathcal{M}^*$. But $E_{\Phi^*}(x)$ is independent in $\mathcal{M}^*$ if and only if $E \setminus E_{\Phi^*}(x)$ has full rank in the original matroid $\mathcal{M}$. So if we form a new mapping $\Phi(e) = \Phi^*(e) + \omega^*(e)$, then the cyclic intervals $[\,\Phi^*(e), \Phi^*(e) + \omega^*(e)\,)$ and $[\,\Phi(e), \Phi(e) + \omega(e)\,)$ are exactly each other's complement on the D-gon. Hence $E_\Phi(x) = E \setminus E_{\Phi^*}(x)$ for all points x, and so $r(E_\Phi(x)) = r(E)$, proving that Φ is a cyclic full-rank order. □

From this we can easily form the dual version of Theorem 1.9.

Corollary 2.4 *Let $(\mathcal{M}, \omega)$ be a weighted matroid, and d a positive rational number. Then there exists a fractional cyclic full-rank order of $(\mathcal{M}, \omega)$ into S_d if and only if d is smaller than or equal to* $\displaystyle\min_{\varnothing \neq A \subseteq E} \frac{\omega(A)}{r(E) - r(E \setminus A)}$ *(here we assume that division by zero gives a quotient $+\infty$, hence never the minimum).*

3 Cyclic Orderings of Matroids

The main goal of this section is to prove Theorem 1.4. We also give one corollary.

The following is an equivalent formulation of the theorem.

Theorem 3.1
Let $\mathcal{M}$ be a matroid of rank r and m elements, such that $\gcd(r,m)=1$. Then the following statements are equivalent:

(a) *There exists a cyclic ordering $(e_1,\ldots,e_m)$ of the elements of $\mathcal{M}$ such that every cyclic interval $e_i,\ldots,e_{i+r-1}$ of length r is a base of $\mathcal{M}$.*

(b) *For all $A\subseteq E$, we have $r\cdot|A|\le m\cdot r(A)$.*

Proof Again, if there is a subset $A\subseteq E$ such that $r\cdot|A|>m\cdot r(A)$, then no matter how we order E, there will be a cyclic interval $e_i,\ldots,e_{i+r-1}$ that contains more than $r(A)$ elements from A. That means that those elements can not be independent, and hence certainly not a base.

Next assume that $r\cdot|A|\le m\cdot r(A)$, for all $A\subseteq E$. Setting $\omega(e)=r$ for all $e\in E$, and $D=m$, we can apply Theorem 2.1 to conclude that there exists a cyclic independence order Φ of $(\mathcal{M},\omega)$ into the m-gon. We will prove that under the condition that $\gcd(r,m)=1$, this mapping is in fact a cyclic ordering such that every cyclic interval forms a base.

Recall from the paragraph preceding Theorem 2.1 that with every element e we associate a sequence $J(e)=(\Phi(e),\Phi(e)+1,\ldots,\Phi(e)+r-1)$, taken modulo D, on the m-gon. And that then for every point x on the m-gon, the set $E_\Phi(x)$ of all $e\in E$ with $x\in J(e)$ is an independent set. Since all sequences $J(e)$ have the same length r, from this it follows immediately that for each cyclic interval $I_r(x)=(x,x+1,\ldots,x+r-1)$ of length r on the m-gon, the set of elements mapped into $I(x)$ forms an independent set.

First notice that since $r\,|E|=m\cdot r(E)$, the number of elements $e\in E$ that are mapped into a cyclic interval $I_r(x)$ on the m-gon is exactly r: Since these elements form an independent set, there cannot be more than r. And if there would be fewer than r mapped into $I_r(x)$, then some other cyclic interval would have more than r elements, which is also impossible.

Next suppose there exists a point x on the m-gon so that there is no element $e\in E$ with $\Phi(e)=x$. Since $\gcd(r,m)=1$, there exist integers a,b so that $a\cdot m+b\cdot r=1$. Since then also $(a+i\,r)\,m+(b-i\,m)\,r=1$ for all integers i, we can choose positive integers α,β so that $\alpha\,m=\beta\,r+1$. Now consider the sequence of β cyclic intervals:

$$I_r(x+1),\ I_r(x+r+1),\ I_r(x+2\,r+1),\ \ldots,\ I_r(x+(\beta-1)\,r+1),$$

taken modulo m. Since consecutive intervals from this sequence are consecutive on the m-gon (in the sense that the last element of one interval is immediately followed by the first element of the next interval), their union can be viewed as one long interval $I^*=(x+1,x+2,\ldots,x+\beta\,r)$ of length $\beta\,r$, that wraps itself several times around the m-gon. Since $\beta\,r=\alpha\,m-1$, every point on the m-gon appears α times in I^*, except x, which only appears $\alpha-1$ times.

Since each of the intervals in the sequence gets r elements from E mapped into it, the total number of elements mapped into the long sequence I^* is $\beta\,r$. On the other hand, since there is no element from E mapped into x, and I^* contains each point from the m-gon, except x, α times, the total number of elements mapped into I^* should be $\alpha\,|E| = \alpha\, m$. Since $\alpha\, m \neq \beta\, r$, this gives a contradiction.

We can conclude that for every point x on the m-gon, there is at least one $e \in E$ with $\Phi(e) = x$. Since $|E| = m$, that means that for every point x there is exactly one $e \in E$ with $\Phi(e) = x$. This gives an ordering of E along the m-gon which is exactly the cyclic ordering we were looking for. □

An easy corollary is the following, also conjectured, without the condition $\gcd(w, m) = 1$, by Kajitani *et al* [8].

Corollary 3.2
Let $\mathcal{M}$ be a matroid with m elements. Suppose w is a positive integer such that $\gcd(w, m) = 1$, and such that for all $A \subseteq E$ we have $w \cdot |A| \leq m \cdot r(A)$. Then there exists a cyclic ordering $(e_1, \ldots, e_m)$ of E such that every cyclic interval $e_i, \ldots, e_{i+w-1}$ of w elements is independent.

Proof Let $\mathcal{M}_w$ be the matroid whose independent sets are the independent sets of $\mathcal{M}$ with at most w elements. Then $\mathcal{M}_w$ is a uniformly dense matroid of rank w, and hence the corollary follows immediately from Theorem 3.1. □

4 Circular Arboricity of Matroids and Related Results

In this final section we prove Theorem 1.8, settling a conjecture made for graphs in the PhD thesis of Gonçalves [7, page 140].

Theorem 4.1
For a matroid $\mathcal{M}$ we have $\gamma(\mathcal{M}) = \Upsilon_f(\mathcal{M}) = \Upsilon_c(\mathcal{M})$.

Proof By Propositions 1.6 and 1.7 it is enough to prove that $\Upsilon_c(\mathcal{M}) \leq \gamma(\mathcal{M})$. Take positive integers P, Q such that $\gamma(\mathcal{M}) = \dfrac{P}{Q}$. Give all elements $e \in E$ the weight $\omega(e) = Q$. Then we get for the weighted density $\gamma(\mathcal{M}, \omega) = P$, hence $\omega(A) \leq P \cdot r(A)$ for all $A \subseteq E$. By Theorem 2.1 there exists a cyclic independence order Φ of $(\mathcal{M}, \omega)$ into the P-gon. This cyclic independence order has the property that for every point x of the P-gon, the set $E_\Phi(x)$ of elements e such that $x \in \{\, \Phi(e),\, \Phi(e) + 1,\, \ldots,\, \Phi(e) + Q - 1 \,\}$ is independent. But that is equivalent to saying that for every point x of the P-gon the set $F_\Phi(x)$ of elements e with $e \in \{\, x,\, x + 1,\, \ldots,\, x + Q - 1 \,\}$ is independent. So if we define the mapping Ψ of E into the topological cycle $S_{P/Q}$ by setting $\Phi(e) = \Phi(e)/Q$, then this mapping has the property that for every point y of $S_{P/Q}$, the elements of E mapped into $[y, y + 1)$ form an independent set. This shows that $\Upsilon_c(\mathcal{M}) \leq \dfrac{P}{Q} = \gamma(\mathcal{M})$, and completes the proof. □

We can use exactly the same idea as in the proof above, but using Corollary 2.3 instead of Theorem 2.1, to obtain the following result.

Theorem 4.2
Let $\mathcal{M}$ be a matroid and d a real number such that for all $A \subseteq E$ with $r(E \setminus A) < r(e)$ we have $d \leq \dfrac{|A|}{r(E) - r(E \setminus A)}$. Then there exists a mapping Φ of E into the topological cycle S_d such that for every point x, the elements $E_\Phi(x)$ mapped into the cyclic interval $[x, x+1)$ form a set of full rank: $r(E_\Phi(x)) = r(E)$.

Because Gonçalves formulated his original conjecture in terms of graphs, we give the corollaries of the last two theorems for the case of graphical matroids. For a graph G, let $V(G)$ denote the set of vertices, $E(G)$ the set of edges, and $c(G)$ the number of components of G.

Corollary 4.3
Let G be a graph and d a real number such that for every subgraph H of G with more than two vertices we have $d \geq \dfrac{|E(H)|}{|V(H)| - 1}$. Then there exists a mapping Φ of the edge set $E(G)$ into the topological cycle S_d such that for every point x, the edges $E_\Phi(x)$ mapped into the cyclic interval $[x, x+1)$ form an acyclic subgraph of G.

Corollary 4.4
Let G be a connected graph and d a positive real number such that for every set of edges A that form a cutset of G we have $d \leq \dfrac{|A|}{c(G - A)}$. Then there exists a mapping Φ of the edge set $E(G)$ into the topological cycle S_d such that for every point x, the edges $E_\Phi(x)$ mapped into the cyclic interval $[x, x+1)$ form a spanning connected subgraph of G.

Theorem 4.1 generalises a result of Edmonds [3], while Theorem 4.2 generalises another result of Edmonds [4]. Their graphical versions, Corollaries 4.3 and 4.4, generalise classical results of Nash-Williams [10], and of Nash-Williams [9] and Tutte [13], respectively.

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