Partitions versus sets : a case of duality

Laurent Lyaudet

Université d'Orléans - LIFO,
Rue Léonard de Vinci, B.P. 6759,
F-45067 Orléans Cedex 2, France
Laurent.Lyaudet@gmail.com

Frédéric Mazoit [1]

Université Bordeaux - LaBRI,
351, cours de la Libration,
F-33405 Talence Cedex, France
Frederic.Mazoit@labri.fr

Stéphan Thomassé [1]

Université Montpellier II - CNRS,
LIRMM, 161 rue Ada, 34392 Montpellier Cedex, France
thomasse@lirmm.fr

Abstract

In a recent paper, Amini et al. introduced a general framework to prove duality theorems between tree decompositions and their dual combinatorial object. They unify all known ad-hoc proofs in one duality theorem based on submodular partition functions. This general theorem remains however a bit technical and relies on this particular submodularity property. Instead of partition functions, we propose here a simple combinatorial property of set of partitions which also gives these duality results. Our approach is both simpler, and a little bit more general.

1 Introduction

In the past 30 years, several decompositions of graphs and discrete structures such as tree-decompositions and branch-decompositions of graphs [5,6],

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tree-decompositions of matroids [3] or branch-decomposition of more general structures [4] have been introduced. Most of these decompositions admit some dual combinatorial object (brambles, tangles...), in the sense that a decomposition exists if and only if the dual object does not.

In [1], the authors present a general framework for proving these duality relations. Precisely, a *partitioning tree* on a finite set E is a tree T which leaves are identified to the elements of E in a one-to-one way. Every internal node v of T corresponds to the partition of E which parts are the set of leaves of the subtrees obtained by deleting v. Such a partition is a *node-partition*. A partitioning tree T is *compatible* with a set of partitions $\mathcal{P}$ of E if every node-partition of T belong to $\mathcal{P}$. For some specific sets of partitions $\mathcal{P}$, one can get classical tree decompositions. To illustrate our purpose, let $G = (V, E)$ be a graph (which is not too trivial, i.e. not a union of stars). The *border* of a partition μ of E is the set of vertices incident with edges in at least two parts of μ. For every integer k, let $\mathcal{P}_k$ be the set of partitions of E whose border contain at most $k+1$ vertices. Now, there exists a partitioning tree compatible with $\mathcal{P}_k$ if and only if the tree-width of G is at most k.

The dual objects of partitioning trees are *brambles*. A *$\mathcal{P}$-bramble* is a nonempty set of pairwise intersecting subsets of E which contains a part of every partition in $\mathcal{P}$, and a $\mathcal{P}$-bramble is *principal* if it contains a singleton. A non-principal $\mathcal{P}$-bramble and a partitioning tree compatible with $\mathcal{P}$ cannot both exist at the same time, but there may be none of them.

In [1], the authors propose a sufficient condition for a set of partitions $\mathcal{P}$ to be such that there exists a partitioning tree compatible with $\mathcal{P}$ if and only if no non-principal $\mathcal{P}$-bramble does (duality property). The condition they introduced is expressed by the mean of weight functions on partitions. Precisely, they prove that if a partition function is (*weakly*) *submodular*, the set of partitions with weight bounded by a fixed constant enjoys the duality property. For example, the weight function corresponding to tree-width (the size of the border of a partition) is submodular, therefore, if the tree-width of G is more than k, there is no partitioning tree compatible with $\mathcal{P}_k$, hence a bramble exists. This provides a alternative proof of [1], also presented in [2]. This kind of argument provides duals for some other tree-decompositions.

While [1]'s framework unifies several ad-hoc proof techniques of duality between decompositions and their dual objects, its core theorem mimics a proof of [6]. The argument is quite technical and does not give a real insight of the reason why the duality property holds. Moreover, at least one partition function, the function $\max_f$ which corresponds to branchwidth, is not weakly submodular. Since this function is a limit of weakly submodular functions, Amini et al. also manage to apply their theorem to branchwidth but this is not truly satisfying.

The goal of this paper is twofold. First we give a simpler proof of the duality theorem, then we slightly extend (and simplify) the definition of *weak submodularity* so that the function $\max_f$ becomes weakly submodular.

To do so, we consider *partial partitioning trees*, in which the leaves of a tree T are labelled by the parts of some partition of E, called the *displayed partition* of T. When the displayed partition consists of singletons, we have our previous definition of partitioning trees. The set of displayed partitions of partial partitioning trees compatible with $\mathcal{P}$ (i.e. such that every node-partition belongs to $\mathcal{P}$) is denoted by $\mathcal{P}^{\uparrow}$. Observe that in T, internal nodes of degree two can be simplified, so we can assume that all internal nodes have degree at least three.

We do not make any distinction between principal and non principal $\mathcal{P}$-brambles. Instead we define a *set of small sets* to be a subset of 2^E closed under taking subset, and whose elements are *small*. We say that a set of partitions $\mathcal{P}^{\uparrow}$ is *dualising* if for any set of small sets $\mathcal{S}$, there exists a *big bramble* (i.e. a bramble containing no part in $\mathcal{S}$) if and only if $\mathcal{P}^{\uparrow}$ contains no *small partition* (i.e a partition whose parts all belong to $\mathcal{S}$). Thus the classical duality results are derived when $\mathcal{S}$ consists of the empty set and the singletons. Note that since a $\mathcal{P}$-bramble Br meets all partitions in $\mathcal{P}$, if $\mathcal{P}$ contains a small partition, Br cannot contain only big parts. Hence, a class of partitions cannot both admit a big bramble and a small partition.

In Section 2, we fix some notations and give some basic definitions. In Section 3, we give an equivalent and yet easier notion than the dualising property: the refining property. In Section 4, we give a sufficient condition on $\mathcal{P}$ so that $\mathcal{P}^{\uparrow}$ is refining (and thus dualising). Finally, in Section 5, we extend the definition of *weak submodularity* to match our sufficient condition for duality, and we prove that the partition function $\max_f$ is weakly submodular and thus, that branchwidth fully belongs to the unifying framework.

2 Brambles

Let E be a finite set. We denote by 2^E the set of subsets of E. A *partition* of E is a set of pairwise disjoint subsets of E which cover E and whose parts are non empty. The sets $\mathcal{P}$ and $\mathcal{Q}$ denote sets of partitions of E. Greek letters α, β,... denote sets of nonempty subsets of E, while capital letters A, B,... denote nonempty subsets of E. We write X^c for the complement $E \setminus X$ of X. We denote a finite union $\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_p$ by $(\alpha_1|\alpha_2|\ldots|\alpha_p)$ and also shorten $(\{A\}|\alpha|\{B\})$ into $(A|\alpha|B)$. The *size* of a subset α of 2^E is just the number of sets in α. For any F, $\alpha \setminus F$ denotes the set $\{A \setminus F \ ; \ A \in \alpha\}$, where empty sets have been removed. The *overlap* of α is the set $\mathrm{ov}(\alpha)$ of the elements that belong to at least two parts of α.

Let T and T' be two partial partitioning trees respectively displaying $(\alpha|A)$ and $(A^c|\beta)$ with u a leaf of T labelled A and u' a leaf of T' labelled A^c. Take the disjoint union of T and T'. Link the respective neighbours of u and u' and remove u and u'. What we get is a new partitioning tree which displays $(\alpha|\beta)$. We say that $(\alpha|\beta)$ is the *merged* partition of $(\alpha|A)$ and $(A^c|\beta)$. It is easy to check that the set $\mathcal{P}^\uparrow$ of all displayed partitions of partial partitioning trees is exactly the least superset of $\mathcal{P}$ which is closed under merging of partitions.

Lemma 1 *For any $(\alpha|A) \in \mathcal{P}^\uparrow \backslash \mathcal{P}$, there exists $(\gamma|C) \in \mathcal{P}$ and $(C^c|\mu|A) \in \mathcal{P}^\uparrow$ such that $(\alpha|A) = (\gamma|\mu|A)$, where $(\gamma|C)$ has at least three parts.*

PROOF. Let T be some partial partitioning tree which displays $(\alpha|A)$. Since $(\alpha|A)$ does not belong to $\mathcal{P}$, T has at least two internal nodes.

The partition $(\gamma|C)$ can be any node-partition of an internal node of T which is adjacent to only one internal node and not adjacent to the leaf A. $\square$

We say that such a partition $(\gamma|C)$ *decomposes* $(\alpha|A)$. To extend this notion to $\mathcal{P}^\uparrow$, we also say that $(\alpha|A)$ *decomposes* $(\alpha|A)$, when $(\alpha|A) \in \mathcal{P}$.

Starting with some subset β of 2^E, one can perform two operations:

- (Deletion) Suppress an element in some set of β. Precisely, if $\beta = (B|\gamma)$ and $b \in B$, the result of the deletion operation is $(B \setminus \{b\}|\gamma)$.
- (Partition) Partition some set of β. Precisely, if $\beta = (B|\gamma)$ and δ is a partition of B, the result of the partition operation is $(\delta|\gamma)$.

We say that α is *finer* than β if it can be obtained from β by a sequence of deletions and partitions. Observe that in some cases, the deletion operation can result in an empty set. In these cases, since we do not allow the empty set in our families of sets, we simply delete the set. When we write that $(\alpha_1|\dots|\alpha_p)$ is finer than $(\beta_1|\dots|\beta_q)$, with $p \leq q$, we usually mean that each α_i is finer than β_i. Note that if α is finer than β, then $\mathrm{ov}(\alpha)$ is included in $\mathrm{ov}(\beta)$.

A $\mathcal{P}$-*bramble*, or just *bramble* when no confusion can occur, is a set Br of subsets of E such that

- Br contains a part of every $\mu \in \mathcal{P}$ (Br *meets* every $\mu \in \mathcal{P}$);
- the elements of Br are pairwise intersecting.

If Br is a $\mathcal{P}$-bramble, we say that $\mathcal{P}$ *admits* the bramble Br.

A set $\mathcal{S}$ of *small* sets is just a subset of 2^E which is closed under taking subsets. A set which does not belong to $\mathcal{S}$ is *big*. By extension, a *big bramble* is a bramble consisting exclusively of big sets, while a *small partition* only contains small parts.

If we consider directly $\mathcal{P}^\uparrow$, we have a dummy duality theorem which states that: Either there is a small partition in $\mathcal{P}^\uparrow$, or there is a set containing a big part of every $\mu \in \mathcal{P}^\uparrow$. Thus the pairwise intersection condition is not required. However this condition is necessary to restrict the obstruction to $\mathcal{P}$.

Lemma 2 *A set Br is a $\mathcal{P}$-bramble if and only if it is a $\mathcal{P}^\uparrow$-bramble.*

PROOF. Let Br be a set of subsets of E. Since $\mathcal{P} \subseteq \mathcal{P}^\uparrow$, if Br is a $\mathcal{P}^\uparrow$ bramble, then Br is a $\mathcal{P}$ bramble too. Now suppose that Br is not a $\mathcal{P}^\uparrow$-bramble. If Br contains disjoint elements, it cannot be a $\mathcal{P}$-bramble so let us suppose that Br contains no part of some partition $\mu \in \mathcal{P}^\uparrow$. Take μ with minimum number of parts. If $\mu \in \mathcal{P}$, then Br is not a $\mathcal{P}$-bramble, otherwise $\mu = (\alpha|\beta)$ for some $(\alpha|A)$, $(A^c|\beta) \in \mathcal{P}^\uparrow$ has less parts than μ. Since μ is minimal, Br contains a part of both $(\alpha|A)$ and $(A^c|\beta)$ and no part of $(\alpha|\beta)$. It contains both A and A^c which are disjoint, and thus Br is not a bramble. □

3 Dualising and refining sets of partitions

We will only apply the theorems of this section to sets of partitions of the form $\mathcal{P}^\uparrow$, but since these results are valid in the general case, we express them for any set $\mathcal{Q}$ of partitions of E.

A set of partitions $\mathcal{Q}$ is *dualising* if for any set of small sets $\mathcal{S}$, either there exists a big $\mathcal{Q}$-bramble, or $\mathcal{Q}$ contains a small partition.

A set of partitions $\mathcal{Q}$ is *refining* if for any $(\alpha|A)$, $(B|\beta) \in \mathcal{Q}$ with A disjoint from B, there exists a partition in $\mathcal{Q}$ which is finer than the covering $(\alpha|\beta)$.

Theorem 3 *If $\mathcal{Q}$ is refining, then $\mathcal{Q}$ is dualising.*

PROOF. Suppose that $\mathcal{Q}$ is refining and that $\mathcal{Q}$ contains no small partition for some set of small sets. There exists a set that contains a big part from every partition in $\mathcal{Q}$, and which is closed under taking superset (just consider the set of all big sets). We claim that such a set Br, chosen inclusion-wise minimal, is a big bramble.

If not, there exists two disjoint sets A and B in Br. Choose them inclusion-wise minimal. Since $Br \setminus \{A\}$ is upward closed and Br is minimal, there exists $(\alpha|A) \in \mathcal{Q}$ which contains no part of $Br \setminus \{A\}$. Similarly, there exists $(B|\beta) \in \mathcal{Q}$ which contains no part of $Br \setminus \{B\}$. Hence Br does not meet $(\alpha|\beta)$, but since $\mathcal{Q}$ is refining, it contains a partition which is finer than $(\alpha|\beta)$ and which is not met by Br, a contradiction. □

Conversely,

Theorem 4 *If $\mathcal{Q}$ is dualising, then $\mathcal{Q}$ is refining.*

PROOF. Assume for contradiction that $\mathcal{Q}$ is not refining. Let $(\alpha|A)$ and $(B|\beta) \in \mathcal{Q}$ with A and B disjoint and such that $\mathcal{Q}$ contains no partition which is finer than $(\alpha|\beta)$. Choose, as small sets, all the sets included in some part of $(\alpha|\beta)$.

- Since $\mathcal{Q}$ contains no partition which is finer than $(\alpha|\beta)$, there is no small partition.
- Since a bramble Br cannot both contain A and B, to meet both $(\alpha|A)$ and $(B|\beta)$, it must contain a small set. Thus Br cannot be a big bramble.

This proves that $\mathcal{Q}$ is not dualising. □

We would like to emphasise that in the following, we only use Theorem 3.

4 Pushing sets of partitions

We now introduce a property on $\mathcal{P}$ which implies that $\mathcal{P}^{\uparrow}$ is refining and thus, by Theorem 3, that $\mathcal{P}^{\uparrow}$ is dualising.

A set of partitions $\mathcal{P}$ is *pushing* if for every pair of partitions $(\alpha|A)$ and $(B|\beta)$ in $\mathcal{P}$ with $A^c \cap B^c \neq \emptyset$, there exists a nonempty $F \subseteq A^c \cap B^c$ such that $(\alpha \setminus F|A \cup F) \in \mathcal{P}$ or $(B \cup F|\beta \setminus F) \in \mathcal{P}$.

To prove that if $\mathcal{P}$ is pushing, then $\mathcal{P}^{\uparrow}$ is refining, we have to strengthen the refining property as follows. If a partition α is only obtained from β by deletions, we say that α is *strongly finer* than β, and a set $\mathcal{Q}$ of partition of E is *strongly refining* if for any $(\alpha|A)$, $(B|\beta) \in \mathcal{Q}$ with A disjoint from B, there exists a partition in $\mathcal{Q}$ strongly finer than the covering $(\alpha|\beta)$. Clearly if $\mathcal{Q}$ is strongly refining, then it is refining, the following theorem thus implies that if $\mathcal{P}$ is pushing, then $\mathcal{P}^{\uparrow}$ is refining.

Theorem 5 *If $\mathcal{P}$ is pushing, then $\mathcal{P}^{\uparrow}$ is strongly refining.*

PROOF. Suppose for a contradiction that $\mathcal{P}$ is pushing, that $(\alpha|A)$, $(B|\beta)$ both belong to $\mathcal{P}^{\uparrow}$ with A disjoint from B, and yet $\mathcal{P}^{\uparrow}$ contains no partition strongly finer than $(\alpha|\beta)$. Choose $(\alpha|\beta)$ with minimum number of parts, and then with minimum overlap among counter-examples with minimal size. Let $O = A^c \cap B^c$ be the overlap of $(\alpha|\beta)$. Observe that since $(\alpha|\beta)$ is not a partition of E, O is nonempty.

We claim that there exist no $(\gamma|C)$, $(D|\delta) \in \mathcal{P}^{\uparrow}$ with C disjoint from D, such that $(\gamma|\delta)$ is strongly finer than $(\alpha|\beta)$ and has an overlap which is a strict subset of O. If not, our choice of $(\alpha|A)$, $(B|\beta)$ implies that $\mathcal{P}^{\uparrow}$ contains a partition λ which is strongly finer than $(\gamma|\delta)$ and thus λ is strongly finer than $(\alpha|\beta)$, a contradiction.

By Lemma 1, let $(\gamma|C)$ and $(D|\delta)$ be respectively decomposing $(\alpha|A)$ and $(B|\beta)$. Since $A \subseteq C$ and $B \subseteq D$, we have $C^c \cap D^c \subseteq O$. If $C^c \cap D^c$ is nonempty, since $\mathcal{P}$ is pushing, there exists a nonempty subset F of O such that, say, $(\gamma \setminus F, C \cup F) \in \mathcal{P}$. If C^c and D^c are disjoint, they cannot both contain O. There thus exists a non empty $F \subseteq O$ which is disjoint from, say, C^c, and therefore $(\gamma|C) = (\gamma \setminus F, C \cup F)$. In both cases, $(\gamma \setminus F, C \cup F) \in \mathcal{P}$.

- If $(\gamma|C) = (\alpha|A)$, then $(\gamma \setminus F|\beta)$ is strongly finer than $(\alpha|\beta)$ and its overlap is $O \setminus F$, which is strictly included in O, a contradiction.
- If $(\gamma|C) \neq (\alpha|A)$, we consider $(C^c|\mu|A) \in \mathcal{P}^{\uparrow}$ such that $(\gamma|\mu|A) = (\alpha|A)$. Since $(C^c|\mu|A)$ has less parts than $(\alpha|A)$, there exists $(C'|\mu'|\beta') \in \mathcal{P}^{\uparrow}$ which is strongly finer than $(C^c|\mu|\beta)$. We assume that $C' \subseteq C^c$ is nonempty, since $(\mu'|\beta') \in \mathcal{P}^{\uparrow}$ would be strongly finer than $(\alpha|\beta)$. If $O \not\subseteq C'^c$, then $(\gamma|\mu'|\beta')$ is strongly finer than $(\alpha|\beta)$, with an overlap strictly included in O, a contradiction. If $O \subseteq C'^c$, then C' and $C \cup F$ are disjoint. But then $(\gamma \setminus F|\mu'|\beta')$ is strongly finer than $(\alpha|\beta)$, and its overlap (which is a subset of $O \setminus F$) is a strict subset of O, a contradiction. □

Observe that $\mathcal{P}$ being pushing implies $\mathcal{P}^{\uparrow}$ begin refining, but we could not avoid the strong version of refinement in our proof. For instance, the relaxed statement of theorem5, with refining only, makes the first claim of the proof to fail. We could imagine that there exists $(\gamma|C)$ and $(D|\delta) \in \mathcal{P}^{\uparrow}$ with C and D disjoint, $(\gamma|\delta)$ finer than $(\alpha|\beta)$ with a smaller overlap but with more parts than $(\alpha|\beta)$.

5 Submodular partition functions

A *partition function* is a function from the set of partitions of E into $\mathbb{R}\cup\{+\infty\}$. In [1], the authors define *submodular* partition functions Ψ such that for every partitions $(\alpha|A)$ and $(B|\beta)$, we have:

$$\Psi(\alpha|A) + \Psi(B|\beta) \geq \Psi(\alpha \setminus B^c|A \cup B^c) + \Psi(\beta \setminus A^c|B \cup A^c).$$

It is routine to observe that if Ψ is partition submodular, then for every k, the set $\mathcal{P}_k$ of partitions with Ψ value at most k is pushing, just consider for this $F = A^c \cap B^c$ in the definition of the pushing property. Hence $\mathcal{P}_k^{\uparrow}$ is dualising as soon as Ψ is submodular. From this follows the duality theorems for tree-width of matroids and graphs, as explicited in [1].

However, in order to also obtain duality for branchwidth, the authors introduce *weakly submodular* partition functions as partition functions such that for every partitions $(\alpha|A)$ and $(B|\beta)$, at least one of the following holds:

- there exists $A \subset F \subseteq (B \setminus A)^c$ with $\Psi((\alpha|A)) > \Psi((\alpha \setminus F|A \cup F))$;
- $\Psi((\beta|B)) \geq \Psi((\beta \setminus A^c|B \cup A^c))$.

Since $(\beta|B)$ and $(\beta \setminus A^c|B \cup A^c)$ are equal when $A^c \cap B^c = \emptyset$, this definition is only really interesting when $A^c \cap B^c \neq \emptyset$.

We introduce now a more convenient property, still called *weak submodularity*, in which partition functions satisfy that for every $(\alpha|A)$ and $(B|\beta)$ with $A^c \cap B^c \neq \emptyset$, there exists a nonempty $F \subseteq A^c \cap B^c$ such that at least one of the following holds:

- $\Psi((\alpha|A)) \geq \Psi((\alpha \setminus F|A \cup F))$;
- $\Psi((\beta|B)) \geq \Psi((\beta \setminus F|B \cup F))$.

This definition indeed generalises the previous one.

- Suppose that there exists $A \subset F \subseteq (B \setminus A)^c$ with $\Psi((\alpha|A)) > \Psi((\alpha \setminus F|A \cup F))$. Set $F' := F \cap (A^c \cap B^c)$. Since $F = F' \cup A$, $(\alpha \setminus F|A \cup F) = (\alpha \setminus F'|A \cup F')$. Thus $\Psi((\alpha|A)) > \Psi((\alpha \setminus F'|A \cup F'))$ and F' is certainly nonempty.
- Suppose that $\Psi((\beta|B)) \geq \Psi((\beta \setminus A^c|B \cup A^c))$. Set $F := A^c \cap B^c$. Since $(\beta \setminus A^c|B \cup A^c) = (\beta \setminus F|B \cup F)$, $\Psi((\beta|B)) \geq \Psi((\beta \setminus F|B \cup F))$ and F is nonempty.

Claim 6 *A set of partition $\mathcal{P}$ is pushing if and only if $\mathcal{P} = \{\mu \ ; \ \Psi(\mu) \leq k\}$ for some weakly submodular partition function Ψ and $k \in \mathbb{R} \cup \{+\infty\}$.*

Obviously given a weakly submodular partition function Ψ, the class of partitions $\mathcal{P}_k = \{\alpha \ ; \ \Psi(\alpha) \leq k\}$, for some $k \in \mathbb{R}$, is pushing. Conversely if $\mathcal{P}$ is pushing, then defining Ψ as $\Psi(\alpha) = 0$ if $\alpha \in \mathcal{P}$ and $\Psi(\alpha) = 1$ otherwise, we obtain a weakly submodular partition function.

A *connectivity function* is a function $f : 2^E \mapsto \mathbb{R} \cup \{+\infty\}$ which is *symmetric* (i.e. for any $A \subseteq E$, $f(A) = f(A^c)$) and *submodular* (i.e. for any A, $B \subseteq E$, $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$). For any connectivity function f, we define the partition function $\max_f$ by $\max_f(\alpha) = \max\{f(A) \ ; \ A \in \alpha\}$ (α a partition of E). The weak submodularity of the $\max_f$ function gives the duality theorems concerning branchwidth and rankwidth.

Lemma 7 *The function* $\max_f$ *is a weakly submodular partition function.*

PROOF. Let $(\alpha|A)$ and $(B|\beta)$ be two partitions of E such that $A^c \cap B^c$ is nonempty. Let F with $A \setminus B \subseteq F \subseteq (B \setminus A)^c$ be such that $f(F)$ is minimum. We claim that $\max_f((\alpha|A)) \geq \max_f((\alpha \setminus F|A \cup F))$.

Indeed, we have $f(F \cap A) \geq f(F)$ by definition of F, and by submodularity, since $f(F) + f(A) \geq f(A \cap F) + f(A \cup F)$, we have $f(A) \geq f(A \cup F)$. For every X in α, we have by submodularity of f:

$$f(X) + f(F^c) \geq f(X \cap F^c) + f(X \cup F^c) \tag{1}$$

Since $f(F)$ is minimum, $f(F) \leq f(F \setminus X)$, and thus f being symmetric:

$$f(X \cup F^c) \geq f(F^c) \tag{2}$$

Adding (1) and (2), we obtain $f(X) \geq f(X \cap F^c)$. Thus $\max_f((\alpha|A)) \geq \max_f((\alpha \setminus F, A \cup F))$, as claimed.

Similarly, $\max_f((B|\beta)) \geq \max_f((B \cup F^c|\beta \setminus F^c))$. Now at least one of $F_A := F \cap (A^c \cap B^c)$ and $F_B := F^c \cap (A^c \cap B^c)$, say F_A, is nonempty. Since $(\alpha \setminus F|A \cup F) = (\alpha \setminus F_A|A \cup F_A)$, there exists a nonempty $F_A \subseteq A^c \cap B^c$ with $\max_f((\alpha|A)) \geq \max_f((\alpha \setminus F_A, A \cup F_A))$ which proves that $\max_f$ is weakly submodular. □

Together with Theorems 3 and 5, Lemma 7 gives a new proof of the branchwidth and rankwidth duality theorems.

6 Conclusion

In the present paper, we solve some shortcomings of [1] by changing a bit the original framework and, mainly, by exhibiting a specific property of sets of partitions instead of defining these sets via the use of partition function. Here are some points in which our approach differs significantly:

- In [1], the "interesting" brambles are the non-principal ones. These brambles do not contain elements that appear as leaves of partitioning trees, i.e. singletons. The duality property thus relates partitioning trees and non-principal brambles.

  In the present paper we relax the condition on the leaves of a partitioning tree by only requiring that these are small sets. In this setting, the duality property relates partial partitioning trees displaying a small partition to big brambles.
- By introducing the refinement property, we give an equivalent version of the dualising property. This simplifies the technicalities of the previous proofs, as well as it highlights the fact that this dualising/refinement property is a natural definition in the study of sets of partitions.
- Finally, the previous definition of weak submodularity being not entirely satisfactory (lack of symmetry, problem with branchwidth) we propose a new definition which simplifies and unifies the previous one.

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