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The chromatic number of 2-edge-colored and signed graphs of bounded maximum degree ☆

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ABSTRACT

A 2-edge-colored graph or a signed graph is a simple graph with two types of edges. A homomorphism from a 2-edge-colored graph G to a 2-edge-colored graph H is a mapping $\varphi : V(G) \to V(H)$ that maps every edge in G to an edge of the same type in H. Switching a vertex v of a 2-edge-colored or signed graph corresponds to changing the type of each edge incident to v. There is a homomorphism from the signed graph G to the signed graph H if after switching some subset of the vertices of G there is a 2-edge-colored homomorphism from G to H.

The chromatic number of a 2-edge-colored (resp. signed) graph G is the order of a smallest 2-edge-colored (resp. signed) graph H such that there is a homomorphism from G to H. The chromatic number of a class of graphs is the maximum of the chromatic numbers of the graphs in the class.

We study the chromatic numbers of 2-edge-colored and signed graphs (connected and not necessarily connected) of a given bounded maximum degree. More precisely, we provide exact bounds for graphs with a maximum degree 2. We then propose specific lower and upper bounds for graphs with a maximum degree 3, 4, and 5. We finally propose general bounds for graphs of maximum degree k, for every k.

1. Introduction

This section is devoted to introduce the concepts of 2-edge-colored graphs, signed graphs, homomorphisms, and target graphs. The state of the art and new results are presented in Section 2.

1.1. Signed and 2-edge-colored graphs

A *2-edge-colored graph* or a *signed graph* $G = (V, E, s)$ is a simple graph (V, E) with two kinds of edges: positive and negative edges. We do not allow parallel edges nor loops. The signature $s : E(G) \to \{-1, +1\}$ assigns to each edge its sign. For

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the concepts discussed in this paper, 2-edge-colored graphs and signed graphs only differ on the notion of homomorphism. Note that 2-edge-colored graphs are sometimes referred to as *signified* graphs by some authors.

A *positive neighbor* (resp. *negative neighbor*) of a vertex v is a vertex that is connected to v with a positive (resp. negative) edge. The set of positive (resp. negative) neighbors of a vertex v is the *positive* (resp. *negative*) *neighborhood* of v, denoted by $N^+(v)$ (resp. $N^-(v)$). A 2-edge-colored or signed graph $G = (V, E, s)$ such that $s(e) = +1$ (resp. $s(e) = -1$) for all $e \in E$ is called an *all positive graph* (resp. *all negative graph*).

Switching a vertex v of a 2-edge-colored or signed graph corresponds to reversing the signs of all the edges that are incident to v. Two 2-edge-colored or signed graphs G and G' are *switching equivalent* if it is possible to turn G into G' after any number of switches.

Given a 2-edge-colored or signed graph $G = (V, E, s)$, the *underlying graph* of G is the simple graph (V, E).

A cycle of a 2-edge-colored or signed graph is said to be *balanced* (resp. *unbalanced*) if it has an even (resp. odd) number of negative edges. The notion of balanced cycles allows us to define switching equivalence as follows.

Theorem 1 *(Zaslavsky [12]). Two 2-edge-colored or signed graphs are switching equivalent if and only if they have the same underlying graph and the same set of balanced cycles.*

1.2. Homomorphisms

Given two 2-edge-colored graphs G and H, the mapping $\varphi : V(G) \to V(H)$ is a *homomorphism* if φ maps every edge of G to an edge of H with the same sign. This can be seen as coloring the vertices of G by using the vertices of H as colors. The target graph H gives us the rules that this coloring must follow. If vertices 1 and 2 of H are adjacent with a positive (resp. negative) edge, then every pair of adjacent vertices in G colored with 1 and 2 must be adjacent with a positive (resp. negative) edge.

If G admits a homomorphism to H, then we say that G is *H-colorable* or that H can color G. If G admits a homomorphism to a graph on n vertices, we say that G is *n-colorable*.

The *chromatic number* $\chi_2(G)$ of a 2-edge-colored graph G is the order (the number of vertices) of a smallest 2-edge-colored graph H such that G is H-colorable. The chromatic number $\chi_2(\mathcal{C})$ of a class $\mathcal{C}$ of 2-edge-colored graphs is the maximum of the chromatic numbers of the graphs in the class.

A *2-edge-colored clique* is a 2-edge-colored graph that has the same order and chromatic number.

Lemma 2 *([3]). A 2-edge-colored graph is a 2-edge-colored clique if and only if each pair of non-adjacent vertices is connected by a path of length 2 made of one positive and one negative edge.*

Given two signed graphs G and H, the mapping $\varphi : V(G) \to V(H)$ is a *homomorphism* if there exist two 2-edge-colored graphs G' and H' respectively switching equivalent to G and H such that G' is H'-colorable. However, switching in H is unnecessary (as explained in Section 3.3 of [8]).

The *chromatic number* $\chi_s(G)$ of a signed graph G is the order of a smallest signed graph H such that G admits a homomorphism to H. Note that $\chi_s(G) \le \chi_2(G)$ for any 2-edge-colored graph / signed graph G. The chromatic number $\chi_s(\mathcal{C})$ of a class $\mathcal{C}$ of signed graphs is the maximum of the chromatic numbers of the graphs in the class.

A *signed clique* is a signed graph that has the same order and chromatic number.

Lemma 3 *([8]). A signed graph is a signed clique if and only if every pair of non-adjacent vertices is part of an unbalanced cycle of length 4.*

A class of graphs $\mathcal{C}$ is *colorable* if there exists a finite target graph that can color every graph in the class. Such a target graph is called *universal* target graph for $\mathcal{C}$. A class of graphs is *complete* if, for every two graphs G_1 and G_2 in the class, there is a graph G^* in the class such that G_1 and G_2 are subgraphs of G^*.

A class $\mathcal{C}$ of 2-edge-colored (resp. signed) graphs is *optimally colorable* if there exists a universal target 2-edge-colored (resp. signed) graph T on $\chi_2(\mathcal{C})$ (resp. $\chi_s(\mathcal{C})$) vertices.

Lemma 4. *Every class $\mathcal{C}$ of graphs which is colorable and complete is optimally colorable.*

Proof. Following the proof in [10]: Suppose that $\mathcal{C}$ is colorable and complete but not optimally colorable. There exists a finite set S of graphs in $\mathcal{C}$ which cannot be colored with a single target graph on $\chi(\mathcal{C})$ vertices (such a set can be finite since there exists a finite number of target graphs having at most $\chi(\mathcal{C})$ vertices). Since $\mathcal{C}$ is complete, there exists a graph G in $\mathcal{C}$ that contains every graph in S as subgraphs. Graph G admits a homomorphism to a target graph T on $\chi(\mathcal{C})$ vertices. Therefore, every graph in S can be colored with T, a contradiction. $\square$

The 2-edge-colored graphs are, in some sense, similar to oriented graphs since a pair of vertices can be adjacent in two different ways in both kinds of graphs: with a positive or a negative edge in the case of 2-edge-colored graphs, with a toward or a backward arc in the oriented case.

Table 1
Properties $P_{n,k}$ of the first SP_q graphs.

SP_5	SP_9	SP_{13}	SP_{17}	SP_{25}	SP_{29}	SP_{37}	SP_{41}	SP_{49}	SP_{53}	SP_{61}
$P_{1,2}$	$P_{1,4}$	$P_{1,6}$	$P_{1,8}$	$P_{1,12}$	$P_{1,14}$	$P_{1,18}$	$P_{1,20}$	$P_{1,24}$	$P_{1,26}$	$P_{1,30}$
	$P_{2,1}$	$P_{2,2}$	$P_{2,3}$	$P_{2,5}$	$P_{2,6}$	$P_{2,8}$	$P_{2,9}$	$P_{2,11}$	$P_{2,12}$	$P_{2,14}$
					$P_{3,1}$	$P_{3,1}$	$P_{3,1}$	$P_{3,4}$	$P_{3,3}$	$P_{3,3}$

SP_{73}	SP_{81}	SP_{89}	SP_{97}	SP_{101}	SP_{109}	SP_{113}	SP_{121}	SP_{125}	SP_{137}	SP_{149}
$P_{1,36}$	$P_{1,40}$	$P_{1,44}$	$P_{1,48}$	$P_{1,50}$	$P_{1,54}$	$P_{1,56}$	$P_{1,60}$	$P_{1,62}$	$P_{1,68}$	$P_{1,74}$
$P_{2,17}$	$P_{2,19}$	$P_{2,21}$	$P_{2,23}$	$P_{2,24}$	$P_{2,26}$	$P_{2,27}$	$P_{2,29}$	$P_{2,30}$	$P_{2,33}$	$P_{2,36}$
$P_{3,5}$	$P_{3,6}$	$P_{3,6}$	$P_{3,6}$	$P_{3,8}$	$P_{3,8}$	$P_{3,9}$	$P_{3,12}$	$P_{3,12}$	$P_{3,12}$	$P_{3,13}$
						$P_{4,1}$	$P_{4,2}$	$P_{4,1}$	$P_{4,2}$	$P_{4,2}$

SP_{157}	SP_{169}	SP_{173}	SP_{181}	SP_{193}	SP_{197}	SP_{229}	SP_{233}	SP_{241}	…	SP_{677}
$P_{1,78}$	$P_{1,84}$	$P_{1,86}$	$P_{1,90}$	$P_{1,96}$	$P_{1,98}$	$P_{1,114}$	$P_{1,116}$	$P_{1,120}$	…	$P_{1,338}$
$P_{2,38}$	$P_{2,41}$	$P_{2,42}$	$P_{2,44}$	$P_{2,47}$	$P_{2,48}$	$P_{2,56}$	$P_{2,57}$	$P_{2,59}$	…	$P_{2,168}$
$P_{3,14}$	$P_{3,16}$	$P_{3,17}$	$P_{3,16}$	$P_{3,18}$	$P_{3,19}$	$P_{3,22}$	$P_{3,23}$	$P_{3,23}$	…	$P_{3,76}$
$P_{4,3}$	$P_{4,4}$	$P_{4,3}$	$P_{4,1}$	$P_{4,4}$	$P_{4,4}$	$P_{4,5}$	$P_{4,5}$	$P_{4,6}$	…	$P_{4,28}$
									…	$P_{5,5}$

The notion of homomorphism of oriented graphs has been introduced by Courcelle [5] in 1994 and has been widely studied since then. Due to the similarity above-mentioned, we try to adapt techniques used to study the homomorphisms of oriented graphs of bounded degree to 2-edge-colored graphs of bounded degree.

1.3. Target graphs

A 2-edge-colored graph (V, E, s) is said to be *antiautomorphic* if it is isomorphic to $(V, E, -s)$.

A 2-edge-colored graph $G = (V, E, s)$ is said to be K_n*-transitive* if for every pair of complete subgraphs $\{u_1, u_2, \dots, u_n\}$ and $\{v_1, v_2, \dots, v_n\}$ in G such that $s(u_iu_j) = s(v_iv_j)$ for all $i \neq j$, there exists an automorphism that maps u_i to v_i for all i. For $n = 1$, 2, or 3, we say that the graph is *vertex-transitive*, *edge-transitive*, or *triangle-transitive*, respectively.

Given a 2-edge-colored graph G, let $S_n(G)$ be the set of all complete subgraphs of size n of G. We say that G has *Property* $P_{n,k}$ if $S_n(G)$ is non-empty and, for every complete subgraph $(v_1, v_2, \dots, v_n)$ of $S_n(G)$ and for every sign vector $A = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \{-1, +1\}^n$ there exist at least k A-successors of $(v_1, v_2, \dots, v_n)$, that is k distinct vertices $\{u_1, u_2, \dots, u_k\}$ such that $s(v_iu_j) = \alpha_i$ for $1 \leq i \leq n$ and $1 \leq j \leq k$. Note that if a graph G has Property $P_{n,k}$, then G has Property $P_{n',k'}$ for any $n' \leq n$ and $k' \leq k$.

Given two 2-edge-colored graphs G and H, *Properties* $P_{n,k}$ are useful to extend a partial H-coloring of G. A classical case of application will be the following one: Suppose that H has Property $P_{n,k}$ and G has a vertex v of degree n. If $G - v$ is H-colorable (with some additional conditions), then there exist at least k distinct H-colorings of G with k distinct colors for v.

Let us now introduce another type of property. We say that a graph G has *Property* $C_{n,k}$ if $S_n(G)$ is non-empty and, for every complete subgraph $(v_1, v_2, \dots, v_n)$ of G, we have $|\bigcup_{1 \leq i \leq n} N^+(v_i)| \geq k$ and $|\bigcup_{1 \leq i \leq n} N^-(v_i)| \geq k$. Note that given two integers n and k, a graph having Property $C_{n,k}$ has Property $C_{n',k'}$ for any n' and k' such that $n' \geq n$ and $k' \leq k$.

A target graph having *good* Properties $P_{n,k}$ and/or $C_{n,k}$ will be a good candidate to color some graph families.

Let q be a prime power with $q \equiv 1 \mod 4$. Let $\mathbb{F}_q$ be the finite field of order q.

The *2-edge-colored Paley graph* SP_q has vertex set $V(SP_q) = \mathbb{F}_q$. Two distinct vertices u and v of SP_q are connected with a positive edge if $u - v$ is a square in $\mathbb{F}_q$ and with a negative edge otherwise. This definition is consistent since $q \equiv 1 \pmod 4$ so -1 is always a square in $\mathbb{F}_q$ and, if $u - v$ is a square, then $v - u$ is also a square.

Properties $P_{n,k}$ of the 2-edge-colored Paley graph SP_q can be easily computed for $n = 1$ and $n = 2$ based on its construction:

Lemma 5 *([9]). Graph SP_q is vertex-transitive, edge-transitive, antiautomorphic and has Properties $P_{1,\frac{q-1}{2}}$ and $P_{2,\frac{q-5}{4}}$.*

However, for $n \geq 3$, it is no longer possible to determine the Properties $P_{n,k}$ from the construction rules and we therefore use a brute force algorithm that computes them, and so does it for Properties $C_{n,k}$.

Table 1 gives the properties $P_{n,k}$ of the first SP_q graphs.

Given a 2-edge-colored graph G with signature s_G, we create the *antitwinned graph* of G denoted by $\rho(G)$ as follows. Let G^{+1}, G^{-1} be two copies of G. The vertex corresponding to $v \in V(G)$ in G^i is denoted by v^i. The vertex set, edge set, and signature of $\rho(G)$ are defined as follows:

- $V(\rho(G)) = V(G^{+1}) \cup V(G^{-1})$
- $E(\rho(G)) = \{u^iv^j : uv \in E(G),\ i, j \in \{-1, +1\}\}$
- $s_{\rho(G)}(u^iv^j) = i \times j \times s_G(uv)$

Table 2
Properties $C_{n,k}$ of the first $\rho(SP_q^+)$ graphs.

$\rho(SP_5^+)$	$\rho(SP_9^+)$	$\rho(SP_{13}^+)$	$\rho(SP_{17}^+)$	$\rho(SP_{25}^+)$	$\rho(SP_{29}^+)$	$\rho(SP_{37}^+)$	$\rho(SP_{41}^+)$
$C_{1,5}$	$C_{1,9}$	$C_{1,13}$	$C_{1,17}$	$C_{1,25}$	$C_{1,29}$	$C_{1,37}$	$C_{1,41}$
$C_{2,7}$	$C_{2,14}$	$C_{2,20}$	$C_{2,24}$	$C_{2,38}$	$C_{2,44}$	$C_{2,56}$	$C_{2,62}$
$C_{3,9}$	$C_{3,16}$	$C_{3,23}$	$C_{3,28}$	$C_{3,44}$	$C_{3,49}$	$C_{3,64}$	$C_{3,71}$
$\rho(SP_{49}^+)$	$\rho(SP_{53}^+)$	$\rho(SP_{61}^+)$	$\rho(SP_{73}^+)$	$\rho(SP_{81}^+)$	$\rho(SP_{89}^+)$	$\rho(SP_{97}^+)$	$\rho(SP_{101}^+)$
$C_{1,49}$	$C_{1,53}$	$C_{1,61}$	$C_{1,73}$	$C_{1,81}$	$C_{1,89}$	$C_{1,97}$	$C_{1,101}$
$C_{2,74}$	$C_{2,78}$	$C_{2,94}$	$C_{2,110}$	$C_{2,122}$	$C_{2,134}$	$C_{2,146}$	$C_{2,152}$
$C_{3,86}$	$C_{3,91}$	$C_{3,107}$	$C_{3,126}$	$C_{3,142}$	$C_{3,154}$	$C_{3,168}$	$C_{3,175}$
$\rho(SP_{109}^+)$	$\rho(SP_{113}^+)$	$\rho(SP_{121}^+)$	$\rho(SP_{125}^+)$	$\rho(SP_{137}^+)$	$\rho(SP_{149}^+)$	$\rho(SP_{157}^+)$	$\rho(SP_{169}^+)$
$C_{1,109}$	$C_{1,113}$	$C_{1,121}$	$C_{1,125}$	$C_{1,137}$	$C_{1,149}$	$C_{1,157}$	$C_{1,169}$
$C_{2,166}$	$C_{2,168}$	$C_{2,182}$	$C_{2,188}$	$C_{2,204}$	$C_{2,224}$	$C_{2,236}$	$C_{2,254}$
$C_{3,191}$	$C_{3,195}$	$C_{3,212}$	$C_{3,219}$	$C_{3,237}$	$C_{3,260}$	$C_{3,274}$	$C_{3,296}$
$\rho(SP_{173}^+)$	$\rho(SP_{181}^+)$	$\rho(SP_{193}^+)$	$\rho(SP_{197}^+)$	$\rho(SP_{229}^+)$	$\rho(SP_{233}^+)$	...	$\rho(SP_{677}^+)$
$C_{1,173}$	$C_{1,181}$	$C_{1,193}$	$C_{1,197}$	$C_{1,229}$	$C_{1,233}$	...	$C_{1,677}$
$C_{2,258}$	$C_{2,274}$	$C_{2,290}$	$C_{2,294}$	$C_{2,346}$	$C_{2,348}$	...	$C_{2,1014}$
$C_{3,300}$	$C_{3,316}$	$C_{3,336}$	$C_{3,342}$	$C_{3,400}$	$C_{3,405}$	...	$C_{3,1182}$

By construction, for every vertex v of G, v^{-1} and v^{+1} are *antitwins*, the positive neighbors of v^{-1} are the negative neighbors of v^{+1} and vice versa. A 2-edge-colored graph is *antitwinned* if every vertex has a unique antitwin. In the remainder of the paper, $\bar{v}$ will denote the antitwin of the vertex v and $\bar{\bar{v}} = v$ (the antitwin of the antitwin of v is v itself).

Lemma 6 *([4]). Let G and H be 2-edge-colored graphs. The two following propositions are equivalent:*

- *The graph G admits a homomorphism to $\rho(H)$.*
- *The graph G, seen as a signed graph, admits a homomorphism to H.*

In other words, if a 2-edge-colored graph admits a homomorphism to an antitwinned target graph on n vertices, then the same graph seen as a signed graph also admits a homomorphism to a target graph on $\frac{n}{2}$ vertices.

Let SP_q^+ be SP_q with an additional vertex that is connected to every other vertex with a positive edge. Note that $\rho(SP_q^+)$ is known in the literature as Tromp-Paley graph; this construction has been introduced by Tromp [11]. Ochem, Pinlou, and Sen [9] proved that $\rho(SP_q^+)$ is edge-transitive.

The families $\rho(SP_q)$ and $\rho(SP_q^+)$ are interesting target graphs (especially for bounding the chromatic number of signed graphs since they are antitwinned graphs) since they have good $P_{n,k}$ properties.

First remark that, by construction, each vertex of $\rho(SP_q)$ (resp. $\rho(SP_q^+)$) has $q-1$ (resp. q) positive neighbors and $q-1$ (resp. q) negative neighbors. Thus $\rho(SP_q)$ (resp. $\rho(SP_q^+)$) has Property $P_{1,q-1}$ (resp. $P_{1,q}$). Moreover, for $n \geq 2$, we can deduce Properties $P_{n,k}$ of $\rho(SP_q)$ and $\rho(SP_q^+)$ from the properties of SP_q.

Lemma 7 *([9]). If SP_q has Property $P_{n,k}$, then $\rho(SP_q^+)$ has Property $P_{n+1,k}$ and, when $k \geq 2$, $\rho(SP_q)$ has Property $P_{n+1,k-1}$.*

Properties $C_{n,k}$ will also be useful in our proofs. We computed these properties for $n = 1, 2, 3$ for the first $\rho(SP_q^+)$ using brute force algorithm (see Table 2).

To conclude this section, let us state a lemma which will be useful in our proofs of the next sections.

Lemma 8. *Let H be a 2-edge-colored complete graph such that $\rho(H)$ has Property $P_{n,k}$. Let φ be a $\rho(H)$-coloring of a 2-edge-colored graph G. If $u \in V(G)$ has degree at most n, then there exist at least k distinct colors for u that leave unchanged $\varphi(v)$ for all $v \in V(G) \setminus \{u\}$. Moreover, the k colors form a complete subgraph in $\rho(H)$.*

Proof. First note that, since H is complete, $\rho(H)$ is isomorphic to a complete graph minus the matching $\{w\bar{w} : w \in V(\rho(H))\}$.

Let u_i denote the i^{th} neighbor of u.

Let us first show that the proof can be reduced to the case where the colors of the u_i's induce a complete subgraph in $\rho(H)$ of order at most $d_G(u)$. If $\varphi(u_i) = \varphi(u_j)$ for some $i < j$, then the signs of the edges uu_i and uu_j must coincide since by hypothesis φ is a $\rho(H)$-coloring of G. Similarly, if $\varphi(u_i) = \overline{\varphi(u_j)}$ for some $i < j$, then the signs of the edges uu_i and uu_j must differ (one is positive and the other is negative) since by hypothesis φ is a $\rho(H)$-coloring of G. In both cases, the coloring constraints are unchanged for u by removing the edge uu_j since the positive (resp. negative) neighborhood of any vertex x of $\rho(H)$ is the negative (resp. positive) neighborhood of $\bar{x}$. We can therefore assume that each pair of vertices u_i

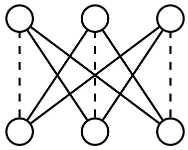


Fig. 1. A signed clique on 6 vertices and maximum degree 3.

and u_j, for $1 \le i < j \le n$, are neither colored with the same color nor with a color and its antitwin; hence the colors of the u_i's induce a complete subgraph in $\rho(H)$ of order at most $d_G(u)$.

By definition of Property $P_{n,k}$, given the n colors (or less) of the neighbors of u, there exist k distinct colors for u, namely $c_1, \dots, c_k$. Hence, there exist at least k distinct $\rho(H)$-colorings of G that coincide on each vertex $v \in V(G) \setminus \{u\}$. To show that the c_i's form a complete subgraph of $\rho(H)$, it remains to prove that $c_i \ne \overline{c_j}$ for all $1 \le i < j \le k$. By contradiction, assume w.l.o.g. that $c_2 = \overline{c_1}$, that means u can be colored with c_1 and its antitwin $\overline{c_1}$. It implies that, for any $1 \le i \le n$, the edges $\varphi(u_i)c_1$ and $\varphi(u_i)\overline{c_1}$ have the same sign in $\rho(H)$. However, by definition of $\rho(H)$, the positive neighbors of a given vertex are the negative neighbors of its antitwin. The two previous edges cannot have the same sign, contradicting the fact that $c_2 = \overline{c_1}$. Thus, among the k colors, we do not have a color and its antitwin. As $\rho(H)$ is isomorphic to a complete graph minus the matching $\{w\bar{w} : w \in V(\rho(H))\}$, it follows that the k colors induce a complete subgraph. $\square$

2. Results

The chromatic number of 2-edge-colored graphs and signed graphs of maximum degree k has already been studied for some particular values of k, namely $k = 3$ and for $k \ge 5$. The aim of this paper is to improve the known bounds and propose bounds for $k = 1, 2,$ and 4. Moreover, we want to highlight differences that appear between connected and non-connected graph families.

In the following, $\mathcal{D}_k$ (resp. $\mathcal{D}_k^c$) denotes the class of (resp. connected) 2-edge-colored or signed graphs with maximum degree k.

In the remainder of this section, we present the state of the art and preliminary results.

Maximum degree 1. Consider a 2-edge-colored graph having two vertices and one edge; it has chromatic number 2 and thus $\chi_2(\mathcal{D}_1^c) = 2$. However, a 2-edge-colored graph with two non-incident edges, one positive and one negative, has chromatic number 3 (the target graph needs a positive and a negative edge, hence at least three vertices) and thus $\chi_2(\mathcal{D}_1) = 3$. We therefore have a difference between the chromatic numbers of connected and non-connected 2-edge-colored graphs with maximum degree 1. This difference does not exist for signed graphs since a negative edge can be changed into a positive one after a switch and we thus have $\chi_s(\mathcal{D}_1^c) = \chi_s(\mathcal{D}_1) = 2$.

Maximum degree 2. For graphs of maximum degree 2, this difference (and lack thereof for signed graphs) also appears and we prove in this paper that $\chi_2(\mathcal{D}_2^c) = 5$, $\chi_2(\mathcal{D}_2) = 6$, and $\chi_s(\mathcal{D}_2^c) = \chi_s(\mathcal{D}_2) = 4$. The proofs are given in Section 4.

Maximum degree 3. Bensmail et al. [2] proved that every connected 2-edge-colored graph with maximum degree 3 has chromatic number at most 12, i.e. $\chi_2(\mathcal{D}_3^c) \le 12$. Moreover, it follows by Lemma 6 that $\chi_s(\mathcal{D}_3^c) \le 6$.

The signed clique on 6 vertices and maximum degree 3 depicted in Fig. 1 gives $\chi_s(\mathcal{D}_3) \ge \chi_s(\mathcal{D}_3^c) \ge 6$.

We improve the above-mentioned bound by showing $\chi_2(\mathcal{D}_3^c) \le 10$ and, for non necessarily connected 2-edge-colored graphs, we prove that $\chi_2(\mathcal{D}_3) \le 11$ (see Section 5).

Maximum degree at least 4. The chromatic numbers of graphs with maximum degree $k = 4$ have not been yet considered. Das, Nandi, and Sen [6] considered coloring of connected (m, n)-mixed graphs of maximum degree $k \ge 5$ (the case $m = 0$ and $n = 2$ corresponds to 2-edge-colored graphs). They proved that $2^{\frac{k}{2}} \le \chi_2(\mathcal{D}_k^c) \le (k-1)^2 \cdot 2^k + 2$ for $k \ge 5$.

We first fill the lack for $k = 4$ by proving that $\chi_2(\mathcal{D}_4) \le 30$ and $\chi_s(\mathcal{D}_4) \le 16$. The proof technique also gives specific bounds for graphs with maximum degree 5, 6, and 7. For maximum degree 5 graphs, we get $\chi_2(\mathcal{D}_5^c) \le \chi_2(\mathcal{D}_5) \le 102$ (which is better than the bound of 514 given by the above-mentioned general formula) and $\chi_s(\mathcal{D}_5^c) \le \chi_s(\mathcal{D}_5) \le 52$. For maximum degree 6 graphs, we get $\chi_2(\mathcal{D}_6^c) \le \chi_2(\mathcal{D}_6) \le 342$ (which is better than the bound of 1602 given by the general formula) and $\chi_s(\mathcal{D}_6^c) \le \chi_s(\mathcal{D}_6) \le 172$. For maximum degree 7 graphs, we get $\chi_2(\mathcal{D}_7^c) \le \chi_2(\mathcal{D}_7) \le 1358$ (which is better than the bound of 4610 given by the general formula) and $\chi_s(\mathcal{D}_7^c) \le \chi_s(\mathcal{D}_7) \le 680$. Proofs are given in Section 6.

Then, for $k \ge 8$, we adapt the proof of Das, Nandi, and Sen [6] to get an upper bound for non necessarily connected 2-edge-colored graphs, that is $\chi_2(\mathcal{D}_k) \le k^2 \cdot 2^{k+1}$ (see Corollary 25) and to get a lower bound for signed graphs, that is $\chi_s(\mathcal{D}_k) \ge 2^{\frac{k}{2}-1}$ (see Theorem 26).

Maximum degree at least 29. For $k \ge 29$, we can adapt a result of Bensmail et al. [1] on (pushable) oriented chromatic number to get better general upper bounds, namely $\chi_2(\mathcal{D}_k^c) \le (k-3) \cdot (k-1) \cdot 2^k + 2$, $\chi_s(\mathcal{D}_k^c) \le (k-3) \cdot (k-1) \cdot 2^{k-1} + 2$, and $\chi_s(\mathcal{D}_k) \le \chi_2(\mathcal{D}_k) \le (k-2) \cdot k \cdot 2^k$ (see Theorems 27 and 28 and Corollary 29).

Table 3
Chromatic number of the classes of (connected) 2-edge-colored graphs of bounded degree.

	$\chi_2(\mathcal{D}_k)$	$\chi_2(\mathcal{D}_k^c)$
$k=1$	$\chi_2(\mathcal{D}_1)=3$	$\chi_2(\mathcal{D}_1^c)=2$
$k=2$	$\chi_2(\mathcal{D}_2)=6$	$\chi_2(\mathcal{D}_2^c)=5$
$k=3$	$8\le\chi_2(\mathcal{D}_3)\le 11$	$8\le\chi_2(\mathcal{D}_3^c)\le 10$
$k=4$	$12\le\chi_2(\mathcal{D}_4^c)\le\chi_2(\mathcal{D}_4)\le 30$	
$k=5$	$16\le\chi_2(\mathcal{D}_5^c)\le\chi_2(\mathcal{D}_5)\le 102$	
$k=6$	$20\le\chi_2(\mathcal{D}_6^c)\le\chi_2(\mathcal{D}_6)\le 342$	
$k=7$	$24\le\chi_2(\mathcal{D}_7^c)\le\chi_2(\mathcal{D}_7)\le 1358$	
$8\le k\le 10$	$4(k-1)\le\chi_2(\mathcal{D}_k)\le k^2\cdot 2^{k+1}$	$4(k-1)\le\chi_2(\mathcal{D}_k^c)\le(k-1)^2\cdot 2^k+2$ [6]
$11\le k\le 28$	$2^{\frac{k}{2}}\le\chi_2(\mathcal{D}_k)\le k^2\cdot 2^{k+1}$	$2^{\frac{k}{2}}\le\chi_2(\mathcal{D}_k^c)\le(k-1)^2\cdot 2^k+2$ [6]
$29\le k$	$2^{\frac{k}{2}}\le\chi_2(\mathcal{D}_k)\le(k-2)\cdot k\cdot 2^k$	$2^{\frac{k}{2}}\le\chi_2(\mathcal{D}_k^c)\le(k-3)\cdot(k-1)\cdot 2^k+2$ [1]

Table 4
Chromatic number of the classes of (connected) signed graphs of bounded degree.

	$\chi_s(\mathcal{D}_k)$	$\chi_s(\mathcal{D}_k^c)$
$k=1$	$\chi_s(\mathcal{D}_1)=\chi_s(\mathcal{D}_1^c)=2$	
$k=2$	$\chi_s(\mathcal{D}_2)=\chi_s(\mathcal{D}_2^c)=4$	
$k=3$	$6\le\chi_s(\mathcal{D}_3)\le 7$ [2]	$\chi_s(\mathcal{D}_3^c)=6$ [2]
$k=4$	$10\le\chi_s(\mathcal{D}_4^c)\le\chi_s(\mathcal{D}_4)\le 16$	
$k=5$	$12\le\chi_s(\mathcal{D}_5^c)\le\chi_s(\mathcal{D}_5)\le 52$	
$k=6$	$14\le\chi_s(\mathcal{D}_6^c)\le\chi_s(\mathcal{D}_6)\le 172$	
$k=7$	$16\le\chi_s(\mathcal{D}_7^c)\le\chi_s(\mathcal{D}_7)\le 680$	
$8\le k\le 11$	$2(k+1)\le\chi_s(\mathcal{D}_k)\le k^2\cdot 2^{k+1}$	$2(k+1)\le\chi_s(\mathcal{D}_k^c)\le(k-1)^2\cdot 2^k+2$ [6]
$12\le k\le 28$	$2^{\frac{k}{2}-1}\le\chi_s(\mathcal{D}_k)\le k^2\cdot 2^{k+1}$	$2^{\frac{k}{2}-1}\le\chi_s(\mathcal{D}_k^c)\le(k-1)^2\cdot 2^k+2$ [6]
$29\le k$	$2^{\frac{k}{2}-1}\le\chi_s(\mathcal{D}_k)\le(k-2)\cdot k\cdot 2^k$	$2^{\frac{k}{2}-1}\le\chi_s(\mathcal{D}_k^c)\le(k-3)\cdot(k-1)\cdot 2^{k-1}+2$ [1]

Lower bounds. We also constructed k-regular 2-edge-colored cliques for $k\ge 3$ and k-regular signed cliques for $k\ge 4$. It follows $\chi_2(\mathcal{D}_k)\ge\chi_2(\mathcal{D}_k^c)\ge 4(k-1)$ and $\chi_s(\mathcal{D}_k)\ge\chi_s(\mathcal{D}_k^c)\ge 2(k+1)$ (proofs are given in Section 3). These lower bounds are the best known up to our knowledge for $k\le 11$ (resp. $k\le 12$) for 2-edge-colored graphs (resp. signed graphs) with maximum degree k. For greater k, already mentioned lower bounds are better.

Tables 3 and 4 summarize the above-mentioned results. Grey cells contain our results presented in this paper, while white cells contain already known results.

3. Lower bounds

In this section, we construct 2-edge-colored cliques and signed cliques. This gives us lower bounds for the related chromatic numbers. Note that for $k\ge 11$ (resp. $k\ge 12$), better lower bounds exist and are given in Section 6.

Theorem 9. *For every $k\ge 3$, there is a k-regular 2-edge-colored clique on $4(k-1)$ vertices.*

Proof. Let G be the 2-edge-colored graph with vertex set $V(G)=\{0,1,\dots,4(k-1)-1\}$. In this proof, every number is considered modulo $4(k-1)$. For all $u\in V(G)$:

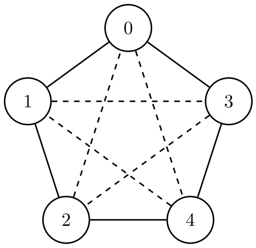
- If u is even, then u is positively adjacent to $u+2(k-1)$ and $u+2i+1$ for $0\le i\le k-3$ and negatively adjacent to $u-1$.
- If u is odd, then u is positively adjacent to $u-2i-1$ for $0\le i\le k-3$ and negatively adjacent to $u+1$ and $u+2(k-1)$.

Graph G is k-regular. We now show that every pair of vertices is either adjacent or is connected by a path of length 2 made of one positive and one negative edge in order to conclude with Lemma 2. It suffices to show that this is the case for each pair of vertices containing 0 or 1 (since adding 2 to every vertex yields an automorphism).

Vertex 0 is adjacent to $2(k-1)$, $2i+1$ for $0\le i\le k-3$, $4(k-1)-1$. The following paths are made of one positive and one negative edge: $(0,2(k-1),2(k-1)-1)$, $(0,2i+1,2i+2)$, $(0,2i+1,2(k-1)+2i+1)$, $(0,4(k-1)-1,4(k-1)-2-2i)$ for $0\le i\le k-3$. We have covered all pairs $(0,v)$ with $v\in\{2i+1,2i+2,2(k-1)-1,2(k-1),2(k-1)+2i+1,4(k-1)-2-2i,4(k-1)-1\}=V(G)\setminus\{0\}$.

Vertex 1 is adjacent to $4(k-1)-2i$ for $0\le i\le k-3$, 2 and $2(k-1)+1$. The following paths are made of one positive and one negative edge: $(1,4(k-1)-2i,4(k-1)-2i-1)$, $(1,2,2i+3)$, $(1,2(k-1)+1,2(k-1)-2i)$ and $(1,2(k-1)+1,2(k-1)+2)$ for $0\le i\le k-3$. We have covered all pairs $(1,v)$ with $v\in\{2,2i+3,2(k-1)-2i,2(k-1)+1,2(k-1)+2,4(k-1)-2i-1,4(k-1)-2i\}=V(G)\setminus\{1\}$. $\square$

Theorem 10. *For every $k\ge 4$, there is a k-regular signed clique on $2(k+1)$ vertices.*



(a) The 2-edge-colored graph SP_5.

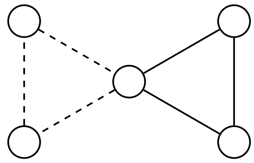

(b) The 2-edge-colored graph SB.

Fig. 2. Every connected 2-edge-colored graph with maximum degree 2 can be colored with at least one of these two graphs.

Proof. Let G be the signed graph with vertex set $V(G) = \{0, 1, ..., 2(k+1) - 1\}$. In this proof, every number is considered modulo $2(k+1)$. For all $u \in V(G)$:

- If u is even, then u is positively adjacent to $u+1$ and $u+4+2i$ for $0 \le i \le k-3$ and negatively adjacent to $u-1$.
- If u is odd, then u is negatively adjacent to $u+1$ and $u+4+2i$ for $0 \le i \le k-3$ and positively adjacent to $u-1$.

Graph G is k-regular. We now show that every pair of vertices is part of an unbalanced cycle of length 4 in order to conclude with Lemma 3. It suffices to show that this is the case for each pair of vertices containing 0 (since adding 2 to every vertex yields an automorphism and adding 1 to every vertex yields an antiautomorphism).

Cycles $(0, 1, 2, 2(k+1) - 4)$, $(0, 2(k+1) - 1, 2(k+1) - 2, 4)$, $0, 4, 3, 2(k+1) - 1)$, $(0, 4+2i, 5+2i, 1)$ and $(0, 4+2i, 3+2i, 2(k+1) - 1)$ for $0 \le i \le k-3$ are unbalanced. This covers all pairs $(0, v)$ with $v \in \{1, 2, 3, 4, 3+2i, 4+2i, 5+2i, 2(k+1) - 2, 2(k+1) - 1\} = V(G) \setminus \{0\}$. $\square$

4. Graphs with maximum degree 2

This section is devoted to 2-edge-colored and signed graphs with maximum degree 2. We prove that $\chi_2(\mathcal{D}_2^c) = 5$, $\chi_2(\mathcal{D}_2) = 6$, and $\chi_s(\mathcal{D}_2) = \chi_s(\mathcal{D}_2^c) = 4$.

4.1. Connected 2-edge-colored graphs with maximum degree 2

In this subsection, we consider the case of connected 2-edge-colored graphs with maximum degree 2 and we prove that their chromatic number is exactly 5. We obtain this result by showing that every graph $G \in \mathcal{D}_2^c$ admits a homomophism to one of the two graphs of Fig. 2.

Theorem 11 *($\chi_2(\mathcal{D}_2^c) = 5$). The class of connected 2-edge-colored graphs with maximum degree 2 has chromatic number 5 and is not optimally colorable.*

Proof. The class of connected graphs with maximum degree 2 is the set of all paths and cycles. The cycle of length 6 from Fig. 4 has chromatic number 5. We start by showing that it is not possible to color it with four colors.

Vertices v_1, v_2, and v_3 belong to a path of length 2 with one negative and one positive edge. We therefore need 3 distinct colors for these vertices and without loss of generality we color v_1, v_2, and v_3 with 1, 2, and 3 respectively. Using the same argument, v_4 cannot receive colors 2 or 3.

Suppose that we color v_4 in 1. Vertex v_5 cannot be colored 1, 2, or 3 so we color it 4. We would need a new color to color v_6.

Suppose that we color v_4 in 4. Vertex v_5 cannot be colored 3 or 4. If we color v_5 with 1, then it will not be possible to color v_6. If we color v_5 with 2, then we would need a new color to color v_6.

Therefore, it is not possible to color this graph with 4 colors. A 5-coloring exists (we color the vertices with 1, 2, 3, 4, 5, and 3 in order) so the chromatic number of this 2-edge-colored graph is 5 and the class of connected 2-edge-colored graphs with maximum degree 2 has chromatic number at least 5.

We now show that any connected 2-edge-colored graph with maximum degree two admits a homomorphism to either SP_5 (see Fig. 2a), the signed Paley graph on 5 vertices, or SB, the signed butterfly (see Fig. 2b).

Notice that any 2-edge-colored path can be colored with the graph from Fig. 3 because every vertex in this graph has at least one positive and at least one negative neighbor. This graph is a subgraph of SP_5, thus every path maps to SP_5. In the following we refer to vertices with even or odd labels as even or odd vertices. Note that in this subgraph, odd (resp. even) vertices are only connected to even (resp. odd) vertices. Also note that every odd (resp. even) vertex of this subgraph is linked with a positive (resp. negative) edge to 0 in SP_5.

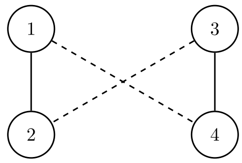
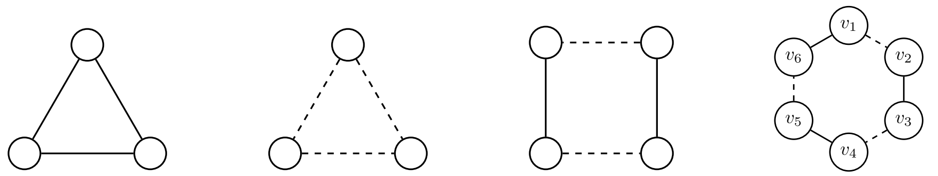


Fig. 3. Target graph that can color any 2-edge-colored path.



(a) All positive triangle. (b) All negative triangle. (c) Alternating C_4 (d) Alternating C_6

Fig. 4. Connected 2-edge-colored graphs are not optimally colorable.

Let $G = (V, E, s)$ be a 2-edge-colored cycle with $V(G) = \{v_0, v_1, ..., v_{n-1}\}$ and $E(G) = \{v_i v_j | i - j \equiv 1 \mod n\}$. We now create a homomorphism φ from G to SP_5 or SB.

Suppose that n is even.

Suppose there is a vertex which is incident to two positive edges. Without loss of generality, let v_0 be this vertex. We create $\varphi : G \to SP_5$ as follows. Set $\varphi(v_0) = 0$. We then color the path $\{v_1, v_2, ..., v_{n-1}\}$ with the subgraph from Fig. 3. Since $s(v_0 v_1) = +1$, $\varphi(v_1)$ has to be an odd color. Since every odd (resp. even) vertex of the subgraph is only adjacent to even (resp. odd) vertices, we alternate between odd and even colors along the path $\{v_2, v_3, ..., v_{n-1}\}$. Hence v_{n-1} is colored with an odd color. That completes the homomorphism since $s(v_{n-1} v_0) = +1$.

Similarly, if there is a vertex which is incident to two negative edges, then we can also create a homomorphism $\varphi : G \to SP_5$.

We can now assume that the cycle alternates between positive and negative edges. Without loss of generality let $s(v_0 v_1) = -1$. We create $\varphi : G \to SP_5$ as follows:

$$\varphi(v_i) = \begin{cases} 0 & \text{if } i = 0, \\ 4 & \text{if } i = 1, \\ 2 & \text{if } i \equiv 2 \mod 4, \\ 3 & \text{if } i \equiv 3 \mod 4, \\ 4 & \text{if } i \equiv 0 \mod 4 \text{ and } i \neq 0, \\ 1 & \text{if } i \equiv 1 \mod 4 \text{ and } i \neq 1. \end{cases}$$

The color of v_{n-1} will thus be an odd color. That completes the homomorphism since $s(v_{n-1} v_0) = +1$.

Suppose that n is odd.

Suppose there is a vertex which is incident to one positive and one negative edge. Without loss of generality let $s(v_{n-1} v_0) = -1$ and $s(v_0 v_1) = +1$. We create $\varphi : G \to SP_5$ as follows. Set $\varphi(v_0) = 0$. We then color the path $\{v_1, v_2, ..., v_{n-1}\}$ with the subgraph from Fig. 3. Since $s(v_0 v_1) = +1$, $\varphi(v_1)$ has to be an odd color. Since every odd (resp. even) vertex of the subgraph is only adjacent to even (resp. odd) vertices, we alternate between odd and even colors along the path $\{v_1, v_3, ..., v_{n-1}\}$. Hence v_{n-1} is colored with an even color. That completes the homomorphism since $s(v_{n-1} v_0) = -1$.

Suppose now G is all positive or all negative. If G is an all positive (resp. negative) cycle of odd length, then we can color it with the all positive (resp. negative) triangle of SB.

It follows that the chromatic number of connected 2-edge-colored graphs with maximum degree 2 is at most 5.

We show that there is no unique graph on 5 vertices that can color the four graphs from Fig. 4 and therefore that connected 2-edge-colored graphs with maximum degree 2 are not optimally colorable.

The first three graphs of Fig. 4 are 2-edge-colored cliques so they need to be subgraphs of the target graph. There is only one way, up to isomorphisms, to have the two triangles as subgraphs of a 5 vertex graph with a minimal number of edges: the graph SB (Fig. 2b). There is only one way, up to isomorphisms, to add edges to SB so that it admits the alternating C_4 as a subgraph (see Fig. 5).

However, the candidate target graph (Fig. 5) cannot color the alternating C_6. Indeed, let v_1, v_2,..., v_6 be the vertices of the alternating C_6 (see Fig. 4). Suppose that color 2 is used in the coloring. Without loss of generality let v_1 be colored 2. Since the only negative neighbor of 2 in the target graph is 3, v_2 needs to be colored 3. Similarly, v_3 needs to be colored 5, v_4 in 4 and v_5 in 2. Since v_6 is both a positive and a negative neighbor of vertices colored 2, the graph cannot be colored

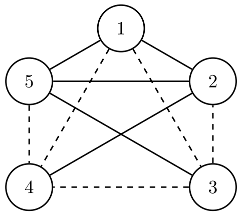


Fig. 5. Candidate target graph on 5 vertices for 2-edge-colored graphs with maximum degree 2.

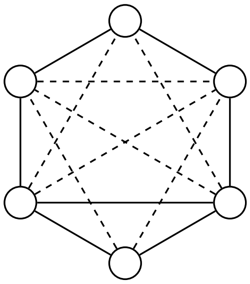


Fig. 6. Target graph for 2-edge-colored graphs with maximum degree 2.

by using the color 2. However, recall that the alternating C_6 has chromatic number 5, so it does not admit a homomorphism to the candidate target graph.

Thus, connected 2-edge-colored graphs are not optimally colorable. □

4.2. 2-edge-colored graphs with maximum degree 2

While 5 colors are enough in the case of *connected* 2-edge-colored graphs with maximum degree 2 (see Theorem 11), 6 colors are needed when the graphs are not necessarily connected and this bound is tight.

Theorem 12 *($\chi_2(\mathcal{D}_2) = 6$). The class of 2-edge-colored graphs with maximum degree 2 has chromatic number 6 and is optimally colorable by the target graph depicted in Fig. 6.*

Proof. The class of graphs with maximum degree 2 is the set of disjoint unions of paths and cycles.

Notice that the graph depicted in Fig. 6 admits the two graphs from Fig. 2, SP_5 and SB, as subgraphs. Therefore, this graph can color any connected 2-edge-colored graph with maximum degree 2 so it can color any 2-edge-colored graph with maximum degree 2.

Thus $\mathcal{D}_2$ has chromatic number at most 6, is colorable, and is complete: so it is optimally colorable by Lemma 4. By Theorem 11, the class of connected 2-edge-colored graphs with maximum degree 2 has chromatic number 5 and is not optimally colorable. Therefore, there is no single 2-edge-colored graph on 5 vertices that can color every 2-edge-colored paths and cycles. Thus, the class of graphs with maximum degree 2 has chromatic number 6. □

4.3. Signed graphs with maximum degree 2

Recall that $\chi_s(\mathcal{C}) \le \chi_2(\mathcal{C})$ for any class of graphs $\mathcal{C}$. Therefore, $\chi_s(\mathcal{D}_2^c) \le 5$ by Theorem 11 and $\chi_s(\mathcal{D}_2) \le 6$ by Theorem 12. We prove that 4 colors are enough in both cases (connected or non-connected) and it is tight:

Theorem 13 *($\chi_s(\mathcal{D}_2) = 4$). The class of signed graphs with maximum degree 2 has chromatic number 4 and is optimally colorable by the target graph depicted in Fig. 7.*

Proof. An unbalanced C_4 is a signed clique by Lemma 3 so the chromatic number of signed graphs with maximum degree 2 is at least 4.

We consider the target graph T depicted in Fig. 7.

The class of graphs with maximum degree 2 is the set of disjoint unions of paths and cycles. Any signed path is switching equivalent to the all positive path of the same length by Theorem 1 and every positive path admits a homomorphism to a positive edge. Therefore, a signed path has chromatic number 2.

A cycle of length n is either balanced or unbalanced.

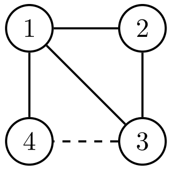


Fig. 7. Target graph T for signed graphs with maximum degree 2.

If it is balanced, then it is switching equivalent to an all positive cycle by Theorem 1. An all positive cycle of even (resp. odd) length can be colored with a positive edge (resp. positive triangle).

If it is unbalanced, then it is switching equivalent to a cycle with exactly one negative edge by Theorem 1. Such a cycle of even (resp. odd) length can be colored with the cycle $(1, 2, 3, 4)$ (resp. $(1, 3, 4)$) of T. □

5. Graphs with maximum degree 3

First note that the complete graph K_4 on 4 vertices admits a homomorphism to itself and therefore $\chi_2(K_4) = \chi_s(K_4) = 4$. Bensmail et al. [2] proved that every connected signed graph with maximum degree 3 except the all positive and the all negative K_4 admits a homomorphism to SP_5^+. This implies that connected signed graphs with maximum degree 3 have chromatic number at most 6, i.e. $\chi_s(\mathcal{D}_3^c) \le 6$. Here, *connected* is mandatory since we do not have a universal target graph on 6 vertices but three distinct target graphs to color the connected signed graphs with maximum degree 3: the all positive K_4 to color itself, the all negative K_4 to color itself, and SP_5^+ to color the remaining connected signed graphs with maximum degree 3. Note that, by Lemma 6, connected 2-edge-colored graphs with maximum degree 3 except the all positive K_4 and the all negative K_4 admit a homomorphism to $\rho(SP_5^+)$, and thus have chromatic number at most 12. Since all positive (resp. negative) K_4 admits a homomorphism to itself, we have $\chi_2(\mathcal{D}_3^c) \le 12$. Note that, in his Ph.D. thesis, Duffy [7] proved that 11 colors are sufficient, that is $\chi_2(\mathcal{D}_3^c) \le 11$.

It is easy to add one universal vertex u to $\rho(SP_5^+)$ to construct the graph $\rho(SP_5^+)^*$ having the all positive K_4 and the all negative K_4 as subgraphs. This graph is thus a universal target graph for class $\mathcal{D}_3$ and thus $\chi_2(\mathcal{D}_3) \le 13$. Now, one can add a new vertex $\bar{u}$ to $\rho(SP_5^+)^*$ to get the antitwinned $\rho(SP_5^+)^{**}$ which has 14 vertices and is clearly universal of the class $\mathcal{D}_3$. We thus have $\chi_s(\mathcal{D}_3) \le 7$ by Lemma 6.

We improve the above-mentioned upper bounds, that is $\chi_2(\mathcal{D}_3^c) \le 10$ and $\chi_2(\mathcal{D}_3) \le 11$.

Note that, by Theorem 9, we have $\chi_2(\mathcal{D}_3) \ge \chi_2(\mathcal{D}_3^c) \ge 8$. Also, the graph depicted in Fig. 1 is a signed clique that gives $\chi_s(\mathcal{D}_3) \ge \chi_s(\mathcal{D}_3^c) \ge 6$.

5.1. 2-edge-colored graphs with maximum degree 3

In this section, we prove that $\chi_2(\mathcal{D}_3) \le 11$. Graph SP_9 is depicted in Fig. 8 and has vertex set $\mathbb{F}_9 = \{0, 1, 2, x, x+1, x+2, 2x, 2x+1, 2x+2\}$. It has Properties $P_{1,4}$ and $P_{2,1}$ by Lemma 5. Despite of SP_9 does not have Property $P_{2,2}$, it has a property which we call $P_{2,2}^*$:

Lemma 14 *(Property $P_{2,2}^*$ of SP_9). Given two vertices u and v of SP_9 and two signs $(s_1, s_2) \in \{-1, +1\}^2$ such that $|\{s(uv), s_1, s_2\}| > 1$, there are two vertices w_1 and w_2 of SP_9 such that $s(uw_1) = s(uw_2) = s_1$ and $s(vw_1) = s(vw_2) = s_2$.*

Proof. Since SP_9 is edge-transitive and antiautomorphic by Lemma 5, it suffices to consider the case $u = 0$ and $v = 1$. Since 01 is a positive edge, we have two cases to consider:

- Either $s_1 = s_2 = -1$ and we can have $w_1 = x + 2$ and $w_2 = 2x + 2$;
- Or $s_1 = +1$, $s_2 = -1$ and we can have $w_1 = x$ and $w_2 = 2x$. □

Let K_4^{s+} (resp. K_4^{s-}) be the all positive (resp. negative) complete graph on 4 vertices with one edge subdivided into a path of length 2 with one negative edge and one positive edge. We say that a 2-edge-colored graph is a K_4^s if it is either K_4^{s+} or K_4^{s-}. See Fig. 9.

A graph is said to be *k-degenerate* if each of its subgraphs contains at least one vertex of degree at most k.

Lemma 15 *([7]). Every 2-degenerate 2-edge-colored graph with maximum degree 3 that does not contain a K_4^s as a subgraph admits a homomorphism to SP_9.*

Consider the graph SP_9^* obtained from SP_9 by adding two new vertices $0'$ and $1'$ as follows. Take the two vertices 0 and 1 of SP_9 (note that $s(01) = +1$), and link $0'$ and $1'$ to the vertices of SP_9 in the same way as 0 and 1 are, respectively; add an edge $0'1'$ with $s(0'1') = -1$; finally we add edges $00'$ and $11'$ with $s(00') = -1$ and $s(11') = +1$.

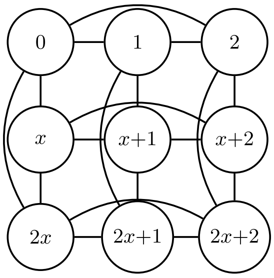


Fig. 8. The graph SP_9. Drawn edges are positive and non-edges are negative.

Fig. 9. The two K_4^s graphs.

Lemma 16. *Every 3-regular 2-edge-colored graph with no K_4^s as a subgraph admits a homomorphism to SP_9^*.*

Proof. Let G be a 3-regular 2-edge-colored graph with no K_4^s as a subgraph. The vertices $0, 1, 2, 1'$ of SP_9^* induce an all positive K_4. Therefore, if G is an all positive 2-edge-colored graph, then it admits an homomorphism φ to SP_9^* ($\forall v \in G, \varphi(v) \in \{0, 1, 2, 1'\}$).

We can now assume that G contains at least one negative edge and let uv be this negative edge. The graph $G - uv$ is 2-degenerate 2-edge-colored graph with maximum degree 3 and no K_4^s as a subgraph. It thus admits a homomorphism φ to SP_9 by Lemma 15.

If $\varphi(u)\varphi(v)$ is a negative edge in SP_9, then φ is already a homomorphism from G to SP_9^*.

If $\varphi(u)\varphi(v)$ is a positive edge in SP_9, then by the edge-transitivity of SP_9 there exists a homomorphism φ' from G to SP_9 such that $\varphi'(u) = 0$ and $\varphi'(v) = 1$. The following application φ'' is a homomorphism from G to SP_9^* because $0'1'$ is a negative edge and $0'$ and $1'$ have the same positive and negative neighbors in SP_9^* as 0 and 1 in SP_9.

$$\varphi''(w) = \begin{cases} 0' & \text{if } w = u, \\ 1' & \text{if } w = v, \\ \varphi'(w) & \text{otherwise.} \end{cases}$$

If $\varphi(u) = \varphi(v)$, then by the vertex-transitivity of SP_9 there exists a homomorphism φ' from G to SP_9 such that $\varphi'(u) = \varphi'(v) = 0$. The following application φ'' is a homomorphism from G to SP_9^* because $0'0$ is a negative edge and $0'$ has the same positive and negative neighbors in SP_9^* as 0 in SP_9.

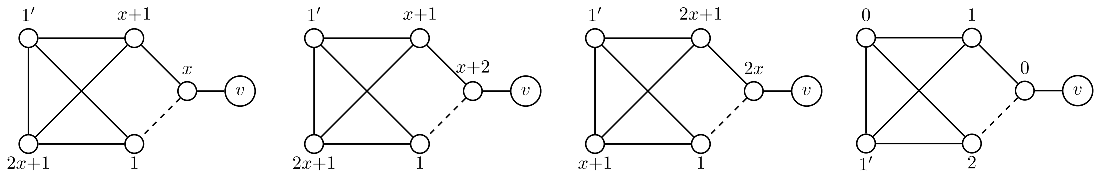
$$\varphi''(w) = \begin{cases} 0' & \text{if } w = u, \\ \varphi'(w) & \text{otherwise.} \end{cases} \quad \square$$

Theorem 17 *($\chi_2(\mathcal{D}_3) \le 11$). The class of 2-edge-colored graphs with maximum degree 3 has chromatic number at most 11 and is optimally colorable by SP_9^*.*

Proof. Let G be a 2-edge-colored graph with maximum degree 3 and let C be a component of G. It suffices to prove that every component admits a homomorphism to SP_9^*.

Suppose C is 2-degenerate or contains a K_4^s as a subgraph. Let C' be obtained from C after removing all its K_4^s. The component C' is thus 2-degenerate with maximum degree 3 and does not contain a K_4^s. By Lemma 15, C' admits a homomorphism φ' to SP_9. We can extend φ' to a SP_9^*-coloring φ of C as follows. We set $\varphi(u) = \varphi'(u)$ for all $u \in V(C')$. Then for every K_4^{s+} linked to the rest of the graph by a positive edge, Fig. 10 shows how to color it depending of the color of the pending vertex (denoted v in Fig. 10). Since $0'$ (resp. $1'$) has the same neighborhood as 0 (resp 1) in SP_9, SP_9 is antiautomorphic, and K_4^{s+} is antiisomorphic to K_4^{s-} (it is isomorphic to K_4^{s-} after replacing each positive edge by a negative one and vice versa), this can also be done for a K_4^{s-} or if the edge linking a K_4^s to the rest of the graph is negative.

If C is not 2-degenerate and does not contain K_4^s as a subgraph, then we can conclude by using Lemma 16. $\square$



(a) $\varphi(v) \in \{0, x+2, 2x\}$ (b) $\varphi(v) \in \{0, x+2, 2x\}$ (c) $\varphi(v) = 2x+1$ (d) $\varphi(v) = 1$

Fig. 10. How to color K_4^{s+} depending on the color of the pending vertex v.

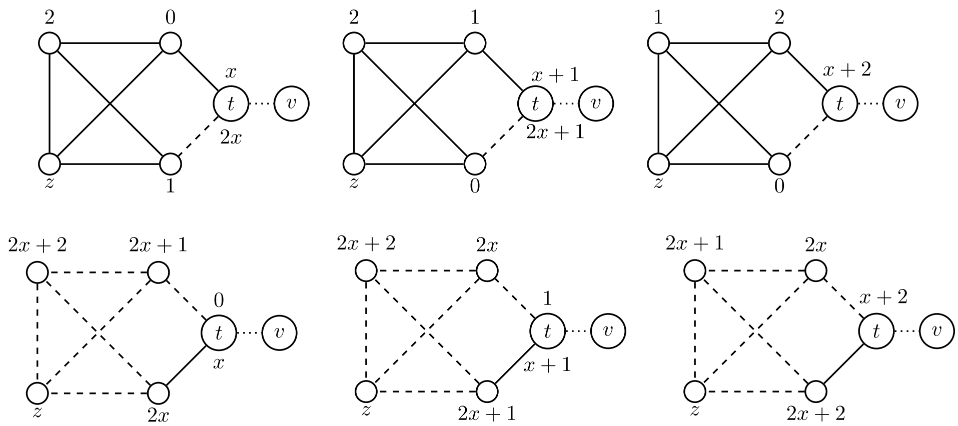


Fig. 11. How to color a K_4^s with $SP_9^\dagger$.

5.2. Connected 2-edge-colored graphs with maximum degree 3

In this subsection, we consider the *connected* 2-edge-colored graphs with maximum degree 3. In the previous subsection, we proved that 11 colors are enough (when the graphs are not necessarily connected) by proving the existence of a universal target graph on 11 vertices, that is SP_9^*. In the connected case, we decrease the upper bound to 10 by using multiple target graphs on 10 vertices.

Recall that, by Theorem 9, the class of connected 2-edge-colored graphs with maximum degree 3 has chromatic number at least 8.

Theorem 18 *($\chi_2(\mathcal{D}_3^c) \le 10$). The class of connected* 2*-edge-colored graphs with maximum degree* 3 *has chromatic number at most* 10*.*

Proof. By contradiction, let G be a connected 2-edge-colored graph with maximum degree 3 such that $\chi_2(G) > 10$.

Claim 1: G contains no copy of K_4^s.

Assume otherwise. Let $SP_9^\dagger$ be the 2-edge-colored graph formed from SP_9 by adding a new vertex z so that there are a positive edge za for all $a \in \{0, 1, 2\}$ and a negative edge zb for all $b \in \{2x, 2x+1, 2x+2\}$.

Let G' be the graph obtained from G after removing the five vertices of every K_4^s. Graph G' is 2-degenerate and by Lemma 15, there exists a homomorphism $\varphi' : G' \to SP_9$. We now extend φ' into a homomorphism $\varphi : G \to SP_9^\dagger$. We first set $\varphi(u) = \varphi'(u)$ for all $u \in V(G')$.

Note that every vertex of SP_9 admits a positive (resp. negative) neighbor in $\{x, x+1, x+2, 2x, 2x+1\}$ and a positive (resp. negative) neighbor in $\{0, 1, x, x+1, x+2\}$.

We independently color each K_4^s as follows and let K be one of them. Let v be the vertex of G' connecting K (see Fig. 11, the dotted edge tv can be either a positive or negative edge). According to the signature of K, the color of v, and the sign of the edge tv, we extend the coloring using Fig. 11.

We obtain a 10-coloring of G, a contradiction. ◇

By Claim 1 and Lemma 15, G is 3-regular.

Claim 2: G contains no bridge.

By contradiction, let uv be a positive (resp. negative) bridge of G. By Claim 1, Lemma 15, and vertex transitivity of SP_9, each component C_1 and C_2 of $G - uv$ ($u \in V(C_1)$ and $v \in V(C_2)$) admits a homomorphism φ_1 and φ_2 to SP_9 such that $\varphi_1(u) = 0$ and $\varphi_2(v) = 1$ (resp. $\varphi_2(v) = x+1$). The union of the φ_i's is a SP_9-coloring of G, a contradiction. ◇

Claim 3: No vertex of G is incident to three positive or three negative edges.

By contradiction, let v be a vertex of G with neighbors u_1, u_2, u_3 such that all of vu_1, vu_2, vu_3 have the same sign. By Claim 1, G contains no K_4^s. Therefore, by Lemma 15, there is a homomorphism $\varphi : G - v \to SP_9$. We extend φ to a 10-coloring of G by coloring v with a 10^{th} color. $\diamond$

Claim 4: G contains no triangle.

By contradiction, let $u, v, w \in V(G)$ induce a triangle in G. Let u' (resp. v', w') be the remaining neighbor of u (resp. v, w).

By Claim 1, G contains no K_4^s. By Claim 3, we may assume that the edges uu' and uv have opposite signs. By Lemma 15 there is a homomorphism $\varphi : G - \{u, v, w\} \to SP_9$. By Property $P_{1,4}$, we can color v with a color compatible with v' and distinct from $\varphi(u')$ and $\varphi(w')$. Since uu' and uv have opposite signs, we can color u with a color compatible with v and u' and distinct from $\varphi(w')$ by Property $P^*_{2,2}$ of SP_9 (Lemma 14). We can finally color w with a 10^{th} color, a contradiction. $\diamond$

By Claim 3, we partition the vertices of G into two sets P and N where vertices in P are incident with exactly two positive edges and vertices in N are incident with exactly two negative edges.

Claim 5: There is no edge between a vertex of P and a vertex of N.

By contradiction, consider two adjacent vertices $u \in P$ and $v \in N$. Suppose first uv is a negative edge. Let u_1 and u_2 be the two other neighbors of u. Since $u \in P$, the edges uu_1 and uu_2 are both positive. By Claim 4, G does not contain a triangle, so u_1 and u_2 are not adjacent. Let w be the neighbor of v such that vw is positive. Let G' be the graph obtained from G by removing u and adding the negative edge u_1u_2.

Note that K_4^s contains three vertices incident with only positive or negative edges. By Claim 3, G does not contain such vertices. Adding the edge u_1u_2 may create at most two vertices in G' incident with 3 negative edges. Therefore, G' does not contain a K_4^s. Moreover, since G does not contain a bridge by Claim 2, G' is 2-degenerate. Therefore, by Lemma 15, there is a homomorphism $\varphi' : G' \to SP_9$. By Property $P^*_{2,2}$ of SP_9 (Lemma 14), we can extend φ' to u so that $\varphi'(u) \neq \varphi'(w)$. Coloring v with a 10^{th} color yields a 10-coloring of G, a contradiction.

The case where uv is a positive edge is similar. $\diamond$

By Claim 5, either P or N is empty since G is connected.

Claim 6: G does not exist.

Assume that N is empty, i.e. every vertex is incident to exactly two positive edges (the case where P is empty is similar). Let u and v be two vertices adjacent with a positive edge. Let G' be the graph obtained from G by removing uv. Graph G' is 2-degenerate and contains no K_4^s (since by Claim 4 G' contains no triangle). By Lemma 15, there is a homomorphism $\varphi' : G' \to SP_9$. Let u' be the negative neighbor of u. If $\varphi'(u') = \varphi'(v)$, then by Property $P^*_{2,2}$ of SP_9 (Lemma 14) we recolor v so that $\varphi'(u') \neq \varphi'(v)$ in G'. We extend φ' to 10-coloring φ of G by recoloring u in a 10^{th} color, a contradiction. $\diamond$ $\square$

6. Graphs with maximum degree $k \geq 4$

In this section, we present two general theorems that work for any maximum degree $k \geq 4$. The first one requires us to first find for each k a target graph that has some special properties while the second one gives us directly an upper bound for every k (at the cost of giving a looser upper bound).

For the first results of this section, we will consider $\rho(SP_q^+)$ as target graphs (recall that such the construction of such graphs has been defined in Section 1.3).

Lemma 19. *Let H be a 2-edge-colored complete graph such that $\rho(H)$ has Properties $P_{k-1,k-2}$ and $C_{k-2,n-\frac{n-1}{k-1}}$, where $n = |\rho(H)|$. Every $(k-1)$-degenerate 2-edge-colored graph with maximum degree k admits a homomorphism to $\rho(H)$, for $k \geq 3$.*

Proof. Let G be a $(k-1)$-degenerate 2-edge-colored graph with maximum degree k.

We proceed by induction on the number of edges of G. The lemma is clearly true when G has no edges (its vertices map to any vertex of $\rho(H)$). Assume that the lemma is true for any such graph with m edges and consider now G has $m+1$ edges.

Let $v \in V(G)$ be a vertex of degree $k' \leq k-1$, and let $v_1, v_2, \ldots, v_{k'}$ be its neighbors.

Suppose first that the neighborhood of v contains an edge and, w.l.o.g., assume that this edge is v_1v_2. Let G' be the graph obtained from G by removing the edge vv_1. The graph G' has m edges and by the induction hypothesis it admits a homomorphism φ to $\rho(H)$.

The degree of v_1 is at most $k-1$ in G', so by Lemma 8 and Property $P_{k-1,k-2}$, v_1 admits a set S of $k-2$ available colors inducing a complete subgraph in $\rho(H)$. The set S cannot thus contain a color c and its antitwin $\bar{c}$. Thus we can assign to v_1 a color of S that is distinct from $\varphi(v_i)$ and $\overline{\varphi(v_i)}$ for $3 \leq i \leq k'$. Note that since v_1v_2 is an edge of G', we have $\{\varphi(v_2), \overline{\varphi(v_2)}\} \cap S = \emptyset$. If $\varphi(u_i) = \overline{\varphi(u_j)}$ for some $i < j$, then the signs of the edges vv_i and vv_j must differ (one is positive and the other is negative) since by hypothesis φ is a $\rho(H)$-coloring of G'. The coloring constraints are unchanged for v by removing the edge vv_j since the positive (resp. negative) neighborhood of any vertex x of $\rho(H)$ is the negative (resp. positive) neighborhood of $\bar{x}$. We can thus assume that the set $\{\varphi(v_1), \varphi(v_2), \ldots, \varphi(v_{k'})\}$ does not contain a color and its antitwin and so it induces a complete subgraph in $\rho(H)$.

By Property $P_{k-1,k-2}$, one can color vertex v ($k-2$ available colors), extending coloring φ to G.

Assume now there is no edges in the neighborhood of v. The graph $G' = G \setminus v$ has $m - k'$ edges and by the induction hypothesis admits a homomorphism φ to $\rho(H)$.

The degree of v_1 in G' is at most $k-1$. By Lemma 8 and Property $P_{k-1,k-2}$, we have $k-2$ possible colors for v_1 and these $k-2$ colors induce a complete subgraph. By Property $C_{k-2,n-\frac{n-1}{k-1}}$, given these $k-2$ possible colors for v_1, there are at least $n - \frac{n-1}{k-1}$ choices of colors for v. Thus v_1 forbids at most $\frac{n-1}{k-1}$ colors for v. Since the neighborhood of v contains no edges, the previous arguments hold for each v_i independently. That is, every v_i forbids at most $\frac{n-1}{k-1}$ colors for v. Therefore, at most $\frac{n-1}{k-1}k' \leq n-1$ colors are forbidden. So there exists at least one available color for v and φ can be extended to a $\rho(H)$-coloring of G. □

Lemma 20. *If all the $(k-1)$-degenerate 2-edge-colored graphs with maximum degree k admit a homomorphism to a single edge-transitive target graph on n vertices, then all the graphs in $\mathcal{D}_k$ admit a homomorphism to a single target graph on $n+2$ vertices.*

Proof. Let T be an edge-transitive target graph on n vertices that can color every $(k-1)$-degenerate 2-edge-colored graph with maximum degree k and let s be the signature of T. Let xy be a positive edge of T. Consider the graph T^* obtained from T by adding two new vertices x' and y' as follows. For each edge xv (resp. yv), $v \neq y$ (resp. $v \neq x$), add an edge $x'v$ (resp. $y'v$) such that $s(x'v) = s(xv)$ (resp $s(y'v) = s(yv)$). Then add an edge $x'y'$ with $s(x'y') = -1$; finally we add edges xx' and yy' with $s(xx') = -1$ and $s(yy') = +1$. To prove that every graph from $\mathcal{D}_k$ admits a homomorphism to T^* it suffices to show that every connected k-regular graph admits a homomorphism to T^*.

Let G be a k-regular 2-edge-colored graph. Since T can color every $(k-1)$-degenerate graph with maximum degree k, T contains an all positive K_k as a subgraph. Since T is edge-transitive, it is in particular vertex-transitive and there exists an all positive K_k, say $\{y, y_1, y_2, ..., y_{k-1}\}$ (that contains y). Since y and y' have the same neighborhoods in T and they are adjacent with a positive edge, $\{y, y', y_1, y_2, ..., y_{k-1}\}$ is an all positive K_{k+1}. If G is all positive, then it can be colored using this all positive K_{k+1}. We can now assume that G contains at least one negative edge.

Let uv be a negative edge. The graph $G - uv$ is $(k-1)$-degenerate graph and admits a homomorphism φ to T.

If $\varphi(u)\varphi(v)$ is a negative edge in T, then φ is already a homomorphism from G to T^*.

If $\varphi(u)\varphi(v)$ is a positive edge in T, then by the edge-transitivity of T there exists a homomorphism φ' from G to T such that $\varphi'(u) = x$ and $\varphi'(v) = y$. The following application φ'' is a homomorphism from G to T^* because $x'y'$ is a negative edge and x' and y' have the same positive and negative neighbors in T^* as x and y in T.

$$\varphi''(w) = \begin{cases} x' & \text{if } w = u, \\ y' & \text{if } w = v, \\ \varphi'(w) & \text{otherwise.} \end{cases}$$

If $\varphi(u) = \varphi(v)$, then by the vertex-transitivity of T there exists a homomorphism φ' from G to T such that $\varphi'(u) = \varphi'(v) = x$. The following application φ'' is a homomorphism from G to T^* because $x'x$ is a negative edge and x' has the same positive and negative neighbors in T^* as x in T.

$$\varphi''(w) = \begin{cases} x' & \text{if } w = u, \\ \varphi'(w) & \text{otherwise.} \end{cases} \quad \square$$

Theorem 21. *If there exists a 2-edge-colored graph $\rho(SP_q^+)$ with Properties $P_{k-1,k-2}$ and $C_{k-2,2q+2-\frac{2q+1}{k-1}}$, then the class of (connected) 2-edge-colored graphs with maximum degree $k \geq 3$ has chromatic number at most $2q+4$.*

Proof. This follows from Lemmas 19 and 20. □

Corollary 22. *If there exists a 2-edge-colored graph $\rho(SP_q^+)$ with Property $P_{k-1,k-2}$ and $C_{k-2,2q+2-\frac{2q+1}{k-1}}$, then the class of (connected) signed graphs with maximum degree k has chromatic number at most $q+3$.*

Proof. By Lemma 19, $\rho(SP_q^+)$ can color all $(k-1)$-degenerate 2-edge-colored graphs with maximum degree k. We apply the construction described along the proof of Lemma 20 to get the target graph $\rho(SP_q^+)^*$ that can color every graph in $\mathcal{D}_k$. This graph is not antitwinned since two vertices x' and y' do not have antitwins. We add the missing antitwins in order to get an antitwinned signed target graph on $2q+6$ vertices. By Lemma 6 we get that every signed graph in $\mathcal{D}_k$ admits a homomorphism to a single target graph on $q+3$ vertices. □

We therefore need to get edge-transitive target graphs $\rho(H)$ with Properties $P_{k-1,k-2}$ and $C_{k-2,|\rho(H)|-\frac{|\rho(H)|-1}{k-1}}$ to color graphs with maximum degree k.

Note that $\rho(SP_{13}^+)$ (resp. $\rho(SP_{49}^+)$, $\rho(SP_{169}^+)$, $\rho(SP_{677}^+)$) is edge-transitive, has Property $P_{3,2}$ (resp. $P_{4,3}$, $P_{5,4}$, $P_{6,5}$) by Table 1 and Lemma 7, and has Property $C_{2,19}$ (resp. $C_{3,76}$, $C_{4,273}$, $C_{5,1131}$) by Table 2 and the fact that given two integers n and k, Property $C_{n,k}$ implies Property $C_{n',k'}$ for any n' and k' such that $n' \geq n$ and $k' \leq k$. We can thus deduce the following bounds using Theorem 21 and Corollary 22:

- $12 \leq \chi_2(\mathcal{D}_4^c) \leq \chi_2(\mathcal{D}_4) \leq 30$ (by Theorems 9 and 21 using $\rho(SP_{13}^+)$);
- $10 \leq \chi_s(\mathcal{D}_4^c) \leq \chi_s(\mathcal{D}_4) \leq 16$ (by Theorem 10 and by Corollary 22 using $\rho(SP_{13}^+)$);
- $16 \leq \chi_2(\mathcal{D}_5^c) \leq \chi_2(\mathcal{D}_5) \leq 102$ (by Theorems 9 and 21 using $\rho(SP_{49}^+)$);
- $12 \leq \chi_s(\mathcal{D}_5^c) \leq \chi_s(\mathcal{D}_5) \leq 52$ (by Theorem 10 and Corollary 22 using $\rho(SP_{49}^+)$);
- $20 \leq \chi_2(\mathcal{D}_6^c) \leq \chi_2(\mathcal{D}_6) \leq 342$ (by Theorems 9 and 21 using $\rho(SP_{169}^+)$);
- $14 \leq \chi_s(\mathcal{D}_6^c) \leq \chi_s(\mathcal{D}_6) \leq 172$ (by Theorem 10 and Corollary 22 using $\rho(SP_{169}^+)$);
- $24 \leq \chi_2(\mathcal{D}_7^c) \leq \chi_2(\mathcal{D}_7) \leq 1358$ (by Theorems 9 and 21 using $\rho(SP_{677}^+)$);
- $16 \leq \chi_s(\mathcal{D}_7^c) \leq \chi_s(\mathcal{D}_7) \leq 680$ (by Theorem 10 and Corollary 22 using $\rho(SP_{677}^+)$);

We voluntarily stop at maximum degree $k = 7$, but we could have continued by computing the properties of the bigger SP_q and $\rho(SP_q^+)$ graphs to fill in Tables 1 and 2. However, even if these upper bounds are the best known, it seems that they are far from the optimal and it has not great interest to continue for greater values of k.

We now present a general upper bound for the chromatic number of 2-edge-colored and signed graphs with maximum degree $k \geq 5$ that does not require computations of Properties $P_{n,k}$ and $C_{n,k}$.

An (m,n)*-colored-mixed graph* is a graph in which each pair of vertices can either be connected by an edge, of which there are n types, or an arc, of which there are m types. Note that a 2-edge-colored graph is therefore a $(0,2)$-colored-mixed graph and an oriented graph is a $(1,0)$-colored-mixed graph.

Das, Nandi, and Sen [6] proved the following general theorem on (m,n)-colored-mixed graphs using probabilistic arguments:

Theorem 23 *([6]). The chromatic number of a connected (m,n)-colored-mixed graph with maximum degree $k \geq 5$ and $2m+n \geq 2$ is at most $2(k-1)^{2m+n}(2m+n)^{k-\min(2m+n,3)+2}+2$ and at least $(2m+n)^{\frac{k}{2}}$.*

It follows for 2-edge-colored graphs ($m = 0$, $n = 2$) that:

Corollary 24 *($2^{\frac{k}{2}} \leq \chi_2(\mathcal{D}_k^c) \leq 2^{k+1}(k-1)^2+2$). The chromatic number of connected 2-edge-colored graphs with maximum degree $k \geq 5$ is at most $2^{k+1}(k-1)^2+2$ and at least $2^{\frac{k}{2}}$.*

The upper bound also applies trivially to connected signed graphs. Note that the lower bound given by Theorem 9 is better than $2^{\frac{k}{2}}$ for $k \leq 10$.

Lemma 25 *($2^{\frac{k}{2}} \leq \chi_2(\mathcal{D}_k) \leq k^2 2^{k+2}$). The chromatic number of 2-edge-colored graphs with maximum degree $k \geq 5$ is at most $k^2 2^{k+2}$ and at least $2^{\frac{k}{2}}$.*

Proof. The lower bound from Corollary 24 also applies trivially to disconnected graphs.

To prove Theorem 23, Das, Nandi, and Sen proposed a construction allowing to obtain a complete (m,n)-mixed-colored graph with $2k^p p^{k-\min(p,3)+3}$ vertices where $p = 2m+n \geq 2$, $k \geq 5$, and having Property $P_{k,k}$. Thus, there exists a complete 2-edge-colored graph H with $k^2 2^{k+2}$ vertices for $k \geq 5$ and having Property $P_{k,k}$. We prove in the remainder that every 2-edge-colored graph with maximum degree k admits a homomorphism to H, for $k \geq 1$.

Let G be a 2-edge-colored graph with maximum degree k. Let s be the signature of G.

We proceed by induction on the number of edges of G. The lemma is clearly true when G has no edges (its vertices map to any vertex of H). Assume that the lemma is true for any graph with m edges and consider now G has $m+1$ edges.

Let $v \in V(G)$ be a vertex of degree k' with $1 \leq k' \leq k$, and let $v_1, v_2, \ldots, v_{k'}$ be its neighbors. By the induction hypothesis, $G' = G - vv_1$ has m edges and it thus admits a homomorphism φ' to H. Observe that if $\varphi'(v_i) = \varphi'(v_j)$, for $2 \leq i < j \leq k'$, then it means that $s(vv_i) = s(vv_j)$ and thus the colors of v_i and v_j induce the same coloring constraints on v. Hence, one can forget one of them. The remaining significant colors on the v_i's, that is the set $\{\varphi'(v_i), 2 \leq i \leq k'\}$, form a complete subgraph in H.

The vertex v_1 has degree $k'' \leq k-1$ in G'.

Recall that if a graph T has Property $P_{a,b}$, then T has Property $P_{a',b'}$ for any $a' \leq a$ and $b' \leq b$. Thus H has Properties $P_{k'',k'}$ and $P_{k',1}$.

By Lemma 8 and Property $P_{k'',k'}$, there exists k' available colors for v_1 in G' and we can recolor v_1 with a color distinct from those of v_i's, $2 \le i \le k'$. Now, the color of v_1 and the significant colors on the v_i's, $2 \le i \le k'$, form a complete subgraph in H. We can therefore apply Property $P_{k',1}$ to recolor v and extend φ' to an H-coloring of G. □

The upper bound given in Corollary 25 also applies trivially to signed graphs.

The following theorem gives a lower bound for the chromatic number of signed graphs with maximum degree $k \ge 5$.

Theorem 26 *($2^{\frac{k}{2}-1} \le \chi_s(\mathcal{D}_k)$). The chromatic number of signed graphs with maximum degree $k \ge 5$ is at least $2^{\frac{k}{2}-1}$.*

Proof. We adapt the proof of the lower bound of Theorem 23 for signed graphs.

Let G be a labeled connected simple graph and let $\chi_s(G)$ (resp. $\chi_2(G)$) denotes the maximum of the chromatic numbers of all the signed (resp. 2-edge-colored) graphs with underlying graph G.

The number of labeled signed graphs with underlying graph G is $2^{|E(G)|}$ since each edge of G can either be positive or negative.

For each of these signed graphs, there are $2^{|V(G)|-1}$ ways to switch its vertices (note that switching a subset of vertices or its complement yields the same signed graph).

Each of these signed graphs has chromatic number at most $\chi_s(G)$ so it admits a homomorphism to at least one complete signed graph on $\chi_s(G)$ vertices. There are $2^{\binom{\chi_s(G)}{2}}$ complete labeled signed graphs on $\chi_s(G)$ vertices.

There are $\chi_s(G)^{|V(G)|}$ applications from the vertex set of a graph on $|V(G)|$ vertices to the vertex set of a graph on $\chi_s(G)$ vertices.

For each of the labeled signed graphs with underlying graph G, for at least one of its switching equivalent graphs, at least one of the applications from the vertex set of this graph to the vertex set of at least one of the complete signed graphs on $\chi_s(G)$ vertices is a homomorphism. Therefore we have:

$$2^{|V(G)|-1} \cdot \chi_s(G)^{|V(G)|} \cdot 2^{\binom{\chi_s(G)}{2}} \ge 2^{|E(G)|}$$

Remark: Let G^1, G^2 ($G^1 \ne G^2$) be two of the $2^{|E(G)|}$ labeled signed graphs with underlying graph G. Graphs G^1 and G^2 have a signature that is different on a least one edge and therefore an application from the vertex set of G to a given complete signed graph on $\chi_s(G)$ after switching the same subset of vertices in G^1 and G^2 cannot be a homomorphism for both G^1 and G^2.

We raise each side to $\frac{1}{|V(G)|}$:

$$2^{\frac{|V(G)|-1}{|V(G)|}} \cdot \chi_s(G)^{\frac{|V(G)|}{|V(G)|}} \cdot 2^{\frac{\binom{\chi_s(G)}{2}}{|V(G)|}} \ge 2^{\frac{|E(G)|}{|V(G)|}}$$

$$\chi_s(G) \ge \frac{2^{\frac{|E(G)|}{|V(G)|}}}{2^{\frac{|V(G)|-1}{|V(G)|}} \cdot 2^{\frac{\binom{\chi_s(G)}{2}}{|V(G)|}}}$$

We choose G k-regular:

$$\chi_s(G) \ge \frac{2^{\frac{k}{2}}}{2^{\frac{|V(G)|-1}{|V(G)|}} \cdot 2^{\frac{\binom{\chi_s(G)}{2}}{|V(G)|}}}$$

Since $\chi_s(G)$ is bounded (by Lemma 25 and the fact that $\chi_s(G) \le \chi_2(G)$ for any G), the right side approaches $2^{\frac{k}{2}-1}$ as $|V(G)|$ goes to infinity. □

Finally, the following theorems improve the upper bound on the chromatic number for connected 2-edge-colored and signed graphs when the maximum degree is at least 29.

Theorem 27 *($\chi_2(\mathcal{D}_k^c) \le (k-3)(k-1)2^k + 2$). The chromatic number of connected 2-edge-colored graphs with maximum degree $k \ge 29$ is at most $(k-3)(k-1)2^k + 2$.*

Theorem 28 *($\chi_s(\mathcal{D}_k^c) \le (k-3)(k-1)2^{k-1} + 2$). The chromatic number of connected signed graphs with maximum degree $k \ge 29$ is at most $(k-3)(k-1)2^{k-1} + 2$.*

These results can be established by following the exact same lines as the proof from [1] on homomorphisms of (pushable) oriented graphs. They rely on the existence of a 2-edge-colored graph with Property $P_{k-1,k-1}$ of order $(k-3)(k-1)2^{k-1}$ for all $k \ge 29$. Therefore, using the same arguments as in the proof of Lemma 25, one can deduce the following corollary which also applies trivially to signed graphs:

Corollary 29 *($\chi_2(\mathcal{D}_k) \leq k(k-2)2^k$). The chromatic number of 2-edge-colored graphs with maximum degree $k \geq 29$ is at most $k(k-2)2^k$.*

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Data availability

No data was used for the research described in the article.

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