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# Locally identifying coloring in bounded expansion classes of graphs[☆]

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ABSTRACT

A proper vertex coloring of a graph is said to be *locally identifying* if the sets of colors in the closed neighborhood of any two adjacent non-twin vertices are distinct. The lid-chromatic number of a graph is the minimum number of colors used by a locally identifying vertex-coloring. In this paper, we prove that for any graph class of bounded expansion, the lid-chromatic number is bounded. Classes of bounded expansion include minor closed classes of graphs. For these latter classes, we give an alternative proof to show that the lid-chromatic number is bounded. This leads to an explicit upper bound for the lid-chromatic number of planar graphs. This answers in a positive way a question of Esperet et al. [L. Esperet, S. Gravier, M. Montassier, P. Ochem, A. Parreau, Locally identifying coloring of graphs, Electron. J. Combin. 19 (2) (2012)].



## 1. Introduction

A vertex-coloring is said to be *locally identifying* if (i) the vertex-coloring is proper (i.e. no adjacent vertices receive the same color), and (ii) for any adjacent vertices $u$, $v$, the set of colors assigned to the closed neighborhood of $u$ differs from the set of colors assigned to the closed neighborhood of $v$ whenever these neighborhoods are distinct. The *locally identifying chromatic number* of the graph $G$ (or lid-chromatic number, for short), denoted by $\chi_{\text{lid}}(G)$, is the smallest number of colors required in any locally identifying coloring of $G$.

Locally identifying colorings of graphs have been recently introduced by Esperet et al. [6] and later studied by Foucaud et al. [7]. They are related to identifying codes [8,9], distinguishing colorings [1,3,4] and locating-colorings [5]. For example, upper bounds on lid-chromatic number have been obtained for bipartite graphs, $k$-trees, outerplanar graphs and bounded degree graphs. An open question asked by Esperet et al. [6] was to know whether $\chi_{\text{lid}}$ is bounded for the class of planar graphs. In this paper, we answer positively to this question proving more generally that $\chi_{\text{lid}}$ is bounded for any class of bounded expansion.

In Section 3, we first give a tight bound of $\chi_{\text{lid}}$ in terms of the tree-depth. Then we use the fact that any class of bounded expansion admits a low tree-depth coloring (that is a $k$-coloring such that each triplet of colors induces a graph of tree-depth 3, for some constant $k$) to prove that it has bounded lid-chromatic number.

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In Section 4, we focus on minor closed classes of graphs which have bounded expansion and give an alternative bound on the lid-chromatic number, which gives an explicit bound for planar graphs.

The next section is devoted to introduce notation and preliminary results.

## 2. Notation and preliminary results

Let $G = (V, E)$ be a graph. For any vertex $u$, we denote by $N_G(u)$ its *neighborhood* in $G$ and by $N_G[u]$ its *closed neighborhood* in $G$ ($u$ together with its adjacent vertices). The notion of neighborhood can be extended to sets as follows: for $X \subseteq V$, $N_G[X] = \{w \in V(G) \mid \exists v \in X, w \in N[v]\}$ and $N_G(X) = N_G[X] \setminus X$. When the considered graph is clearly identified, the subscript is dropped.

The *degree* of vertex $u$ is the size of its neighborhood. The *distance* between two vertices $u$ and $v$ is the number of edges in a shortest path between $u$ and $v$. For $X \subseteq V$, we denote by $G[X]$ the subgraph of $G$ *induced by* $X$.

We say that two vertices $u$ and $v$ are *twins* if $N[u] = N[v]$ (although they are often called *true twins* in the literature, we call them *twins* for convenience). In particular, $u$ and $v$ are adjacent vertices. Note that if $u$ and $v$ are adjacent but not twins, there exists a vertex $w$ which is adjacent to exactly one vertex among $\{u, v\}$, i.e. $w \in N[u] \Delta N[v]$ (where $\Delta$ is the symmetric difference between sets). We say that $w$ *distinguishes* $u$ and $v$, or simply $w$ *distinguishes* the edge $uv$. For a subset $X \subseteq V$, we say that a subset $Y \subseteq V$ *distinguishes* $X$ if for every pair $u, v$ of non-twin vertices of $X$, there exists a vertex $w \in Y$ that distinguishes the edge $uv$.

Let $c : V \to \mathbb{N}$ be a vertex-coloring of $G$. The coloring $c$ is *proper* if adjacent vertices have distinct colors. We denote by $\chi(G)$ the *chromatic number of* $G$, i.e. the minimum number of colors in a proper coloring of $G$. For any $X \subseteq V$, let $c(X)$ be the set of colors that appear on the vertices of $X$. A *locally identifying coloring* (lid-coloring for short) of $G$ is a proper vertex-coloring $c$ of $G$ such that for any two adjacent vertices $u$ and $v$ that are not twins (i.e. $N[u] \neq N[v]$), we have $c(N[u]) \neq c(N[v])$. A graph $G$ is *$k$-lid-colorable* if it admits a locally identifying coloring using at most $k$ colors and the minimum number of colors needed for any locally identifying coloring of $G$ is the *locally identifying chromatic number* (lid-chromatic number for short) denoted by $\chi_{\text{lid}}(G)$. For a vertex $u$, we say that $u$ *sees* color $a$ if $a \in c(N[u])$. For two adjacent vertices $u$ and $v$, a color that is in the set $c(N[u]) \Delta c(N[v])$ *separates* $u$ and $v$, or simply *separates* the edge $uv$. The notion of chromatic number (resp. lid-chromatic number) can be extended to a class of graphs $\mathscr{C}$ as follows: $\chi(\mathscr{C}) = \sup\{\chi(G), G \in \mathscr{C}\}$ (resp. $\chi_{\text{lid}}(\mathscr{C}) = \sup\{\chi_{\text{lid}}(G), G \in \mathscr{C}\}$).

The following theorem is due to Bondy [2]:

**Theorem 1** (*Bondy's Theorem [2]*). .
*Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be a collection of $n$ distinct subsets of a finite set $X$. There exists a subset $X'$ of $X$ of size at most $n - 1$ such that the sets $A_i \cap X'$ are all distinct.*

**Corollary 2.** *Let $C$ be a $n$-clique subgraph of $G$. There exists a vertex subset $S(C) \subseteq V(G)$ of size at most $n - 1$ that distinguishes all the pair of non-twin vertices of $C$.*

**Proof.** Let $C$ be a $n$-clique subgraph of $G$ induced by the vertex set $V(C) = \{v_1, v_2, \ldots, v_n\}$. Let $\mathcal{A} = \{N[v_i] \mid v_i \in V(C)\}$ be a collection of distinct subsets of the finite set $X = \bigcup_{1 \le i \le n} N[v_i]$. Note that some $v_i$'s might be twins in $G$ (i.e. $N[v_i] = N[v_j]$ for some $v_i, v_j \in V(C)$) and therefore $|\mathcal{A}|$ could be smaller than $n$. By Bondy Theorem, there exists $S(C) \subseteq X$ of size at most $|\mathcal{A}| - 1 \le n - 1$ such that for any distinct elements $A_1, A_2$ of $\mathcal{A}$, we have $A_1 \cap S(C) \neq A_2 \cap S(C)$.

Let us prove that $S(C)$ is a set of vertices that distinguish all the pairs of non-twin vertices of $C$. For a pair of non-twin vertices $v_i, v_j$ of $C$, we have $N[v_i] \neq N[v_j]$. By definition of $S(C)$, we have $N[v_i] \cap S(C) \neq N[v_j] \cap S(C)$, then there exists $w \in S(C)$ that belongs to $N[v_i] \Delta N[v_j]$. Therefore, $w$ distinguishes the edge $v_i v_j$. □

## 3. Bounded expansion classes of graphs

A rooted tree is a tree with a special vertex, called the *root*. The *height* of a vertex $x$ in a rooted tree is the number of vertices on a path from the root to $x$ (hence, the height of the root is 1). The *height* of a rooted tree $T$ is the maximum height of the vertices of $T$. If $x$ and $y$ are two vertices of $T$, $x$ is an *ancestor* of $y$ in $T$ if $x$ belongs to the path between $y$ and the root. The *closure* $\text{clos}(T)$ of a rooted tree $T$ is the graph with vertex set $V(T)$ and edge set $\{xy \mid x \text{ is an ancestor of } y \text{ in } T, x \neq y\}$. The *tree-depth* $\text{td}(G)$ of a connected graph $G$ is the minimum height of a rooted tree $T$ such that $G$ is a subgraph of $\text{clos}(T)$. If $G$ is not connected, the tree-depth of $G$ is the maximum tree-depth of its connected components.

Let $p$ be a fixed integer. A *low tree-depth coloring* of a graph $G$ (relatively to $p$) is a coloring of the vertices of $G$ such that the union of any $i \le p$ color classes induces a graph of tree-depth at most $i$. Let $\chi_p^{\text{td}}(G)$ be the minimum number of colors required in such a coloring. Note that as tree-depth one graphs and tree-depth two graphs are respectively the stables and star forests, $\chi_1^{\text{td}}$ and $\chi_2^{\text{td}}$ respectively correspond to the usual chromatic number and the star chromatic number.

In the following of this section, we first give a tight bound on the lid-chromatic number in terms of tree-depth.

**Proposition 3.** *For any graph $G$, $\chi_{\text{lid}}(G) \le 2\text{td}(G) - 1$ and this is tight.*

Using this bound, we then bound the lid-chromatic number in terms of $\chi_3^{\text{td}}$.

**Theorem 4.** *For any graph G,*

$$\chi_{\text{lid}}(G) \le 6^{\binom{\chi_3^{\text{td}}(G)}{3}}.$$

Classes of graphs of bounded expansion have been introduced by Nešetřil and Ossona de Mendez [10]. These classes contain minor closed classes of graphs and any class of graphs defined by an excluded topological minor. Actually, these classes of graphs are closely related to low tree-depth colorings:

**Theorem 5** (*Theorem 7.1 [10]*). *A class of graphs $\mathscr{C}$ has bounded expansion if and only if $\chi_p^{\text{td}}(\mathscr{C})$ is bounded for any p.*

We therefore deduce the following corollary from Theorems 4 and 5:

**Corollary 6.** *For any class $\mathscr{C}$ of bounded expansion, $\chi_{\text{lid}}(\mathscr{C})$ is bounded.*

It is in particular true for a class of bounded tree-width. A consequence is that $\chi_{\text{lid}}$ is bounded for chordal graphs by a function of the clique number (which is equal to the tree-width plus 1 for a chordal graph). It is conjectured by Esperet et al. [6] that $\chi_{\text{lid}}(G) \le 2\omega(G)$ if $G$ is chordal.

We now prove Proposition 3.

**Proof of Proposition 3.** Let us first prove that the bound is tight. Consider the graph $H_n$ obtained from a complete graph, with vertex set $\{a_1, \dots, a_n\}$, by adding a pendant vertex $b_i$ to every $a_i$ but one, say for $1 \le i < n$. The tree-depth of this graph is at least $n$ as it contains a $n$-clique. Indeed, given a rooted tree $T$, two vertices at the same height are non-adjacent in $\text{clos}(T)$, we thus need at least $n$ levels. Actually the tree-depth of this graph is at most $n$ since the tree $T$ rooted at $a_1$, and such that $a_i$ has two sons $a_{i+1}$ and $b_i$, for $1 \le i < n$, has height $n$ and is such that $\text{clos}(T)$ contains $H_n$ as a subgraph.

Let us show that in any lid-coloring of $H_n$ all the vertices must have distinct colors, and thus use $2n - 1 = 2\text{td}(H_n) - 1$ colors. Indeed, two vertices $a_i$ must have different colors as the coloring is proper. A vertex $b_j$ cannot use the same color as a vertex $a_i$, as otherwise the vertex $a_j$ would only see the $n$ colors used in the clique, just as $a_n$. Similarly if two vertices $b_i$ and $b_j$ would use the same color, the vertices $a_i$ and $a_j$ would see the same set of colors.

Let us now focus on the upper bound. We prove the result for a connected graph and by induction on the tree-depth of $G$, denoted by $k$. The result is clear for $k = 1$ (the graph is a single vertex).

Let $G$ be a graph of tree-depth $k > 1$ and let $T$ be a rooted tree of height $k$ such that $G$ is a subgraph of $\text{clos}(T)$. If $T$ is a path, the result is clear since there are only $k$ vertices. So assume that $T$ is not a path, and let $r$ be the root of $T$. Let $s$ be the smallest height such that there are at least two vertices of height $s + 1$. We name $r_i$, for $i \in \{1, \dots, s\}$, the unique vertex of height $i$. Let $R = \{r_1, \dots, r_s\}$. Note that each of the vertices of $R$ is adjacent to all the vertices of $\text{clos}(T)$. Therefore, we can choose the way we label the $s$ vertices in $R$ (i.e. we can choose the height of each of them in $T$) without changing $\text{clos}(T)$.

Necessarily, $G \setminus R$ has at least two connected components. Let $G_1, \dots, G_\ell$ be its connected components and thus $\ell \ge 2$. We choose $T$ such that $s$ is minimal. It implies that for each $i \in \{1, \dots, s\}$, $r_i$ has neighbors in all the components $G_1, \dots, G_\ell$. Indeed, if it is not the case, by permuting the elements of $R$ (this is possible by the above remark), we can assume without loss of generality that $r_s$ does not have a neighbor in $G_\ell$. Therefore, the set of edges $e(r_s, G_\ell) = \{r_s x : x \in V(G_\ell)\}$ of $\text{clos}(T)$ are not used by $G$. Then let $T'$ be the tree obtained from $T$ by moving the whole component $G_\ell$ one level up in such a way that the root of the subtree corresponding to $G_\ell$ is now the son of $r_{s-1}$ (instead of $r_s$ previously). Note that $\text{clos}(T')$ is isomorphic to $\text{clos}(T) \setminus e(r_s, G_\ell)$ and thus $G$ is a subgraph of $\text{clos}(T')$. This new tree $T'$ has two vertices at height $s$, contradicting the minimality of $s$.

Any connected component $G_j$ has tree-depth at most $k' = k - s < k$. By induction, for each $j \in \{1, \dots, \ell\}$, there exists a lid-coloring $c_j$ of $G_j$ using colors in $\{1, \dots, 2k' - 1\}$. For each $c_j$, there is a minimum value $s_j$ such that every vertex $r_i$ sees a color in $\{1, \dots, s_j\}$ in $G_j$. We choose a $(2k' - 1)$-lid-coloring $c_j$ of $G_j$ such that $s_j$ is minimized. Note that for each color $a \le s_j$, there exists $r_i \in R$ such that $r_i$ sees color $a$ in $G_j$ but no other color of $\{1, \dots, s_j\}$. Otherwise, after permuting colors $a$ and $s_j$, every vertex $r_i \in R$ would see a color in $\{1, \dots, s_j - 1\}$, contradicting the minimality of $s_j$. Assume without loss of generality that $s_1 \ge s_2 \ge \dots \ge s_\ell$.

We replace in $c_1$ the colors $1, 2, \dots, s_1$ by $1', 2', \dots, s_1'$. Note that now each vertex $r_i$ sees a color in $\{1', \dots, s_1'\}$ (in $G_1$) and a color in $\{1, \dots, s_2\}$ (in $G_2$). Furthermore, the other vertices of $G$ (that is the vertices in $G_1, \dots, G_\ell$) do not have this property since $s_1 \ge s_2$. Thus at this step every edge $xr_i$ with $x$ in some $G_j$ is separated.

Now we color each vertex $r_i$ with color $i^*$. Let $c : V(G) \to \{1^*, \dots, s^*\} \cup \{1', \dots, s_1'\} \cup \{1, \dots, 2k' - 1\}$ be the current coloring of $G$.

Note that now every distinguishable edge $xy$ in some $G_j$ is separated. Indeed, either $xy$ was distinguished in $G_j$ and it has been separated by $c_j$, or $xy$ is distinguished by some $r_i$ and it is separated by the color $i^*$. Note also that $c$ is a proper coloring.

It remains to deal with the edges $r_i r_j$. For that purpose we will refine some color classes. In the following lemma we show that such refinements do not damage what we have done so far.

**Claim 1.** *Consider a graph G and a coloring $\varphi : V(G) \longrightarrow \{1, \dots, k\}$. Consider any refinement $\varphi'$ of $\varphi$, obtained from $\varphi$ by recoloring with color $k + 1$ some vertices colored i, for some i. Any edge xy of G properly colored (resp. separated) by $\varphi$ is properly colored (resp. separated) by $\varphi'$.*

Indeed if $\varphi(x) \ne \varphi(y)$ then $\varphi'(x) \ne \varphi'(y)$, and if $i \in \varphi(N[x]) \Delta \varphi(N[y])$ then $i$ or $k + 1 \in \varphi'(N[x]) \Delta \varphi'(N[y])$.

Let us define a relation $\mathcal{R}$ among vertices in $R$ by $r_i \mathcal{R} r_j$ if and only if $c(N[r_i]) = c(N[r_j])$. Let $R_1, \ldots, R_{\bar{s}}$ be the equivalence classes of the relation $\mathcal{R}$ (note that each $R_i$ forms a clique since every $r_i$ has distinct colors). We have $\bar{s} \geq s_1$. Indeed, by definition of $s_1$ and the coloring $c_1$, for each color $a \in \{1', \ldots, s_1'\}$, there exists $r_i \in R$ that sees $a$ in $G_1$ but no other color of $\{1', \ldots, s_1'\}$. This vertex $r_i$ belongs to some equivalence class $R_j$ and thus all the vertices of $R_j$ sees color $a$ in $G_1$ but no other color of $\{1', \ldots, s_1'\}$.

By Corollary 2, there is a vertex set $S(R_i)$ of size at most $|R_i| - 1$ which distinguishes all pairs of non-twin vertices in $R_i$. We give to the vertices of $S(R_i)$ new distinct colors. By the previous claim, this last operation does not damage the coloring, and now all the distinguishable edges are separated.

Since for this last operation we need $s - \bar{s}$ new colors, since we used $2k' - 1$ colors $\{1, \ldots, 2k' - 1\}$, $s_1$ colors $\{1', \ldots, s_1'\}$ and $s$ colors $\{1^*, \ldots, s^*\}$, the total number of colors is $(s - \bar{s}) + (2k' - 1) + s_1 + s = 2k - 1 + s_1 - \bar{s} \leq 2k - 1$. This concludes the proof of the theorem. □

We are now ready to prove Theorem 4:

**Proof of Theorem 4.** Let $\alpha$ be a low tree-depth coloring of $G$ with parameter $p = 3$ and using $\chi_3^{td}(G)$ colors. Let $A = \{\alpha_1, \alpha_2, \alpha_3\}$ be a triplet of three distinct colors and let $H_A$ be the subgraph of $G$ induced by the vertices colored by a color of $A$. Since $H_A$ has tree-depth at most 3, by Proposition 3, $H_A$ admits a lid-coloring $c_A$ with five colors (say colors 1–5). We extend $c_A$ to the whole graph by giving color 0 to the vertices in $V(G) \setminus V(H_A)$.

Let $A_1, A_2, \ldots, A_k$ be the $k = \binom{\chi_3^{td}(G)}{3}$ distinct triplets of colors. We now construct a coloring $c$ of $G$ giving to each vertex $x$ of $G$ the $k$-uplet $(c_{A_1}(x), c_{A_2}(x), \ldots, c_{A_k}(x))$.

The coloring $c$ is using $6^k$ colors. Clearly it is a proper coloring: each pair of adjacent vertices will be in some common graph $H_A$ and will receive distinct colors in this graph. Let $x$ and $y$ be two adjacent vertices with $N[x] \neq N[y]$. Let $w$ be a vertex adjacent to only one vertex among $x$ and $y$. Let $A = \{\alpha(x), \alpha(y), \alpha(w)\}$. Vertices $x$ and $y$ are not twins in the graph $H_A$. Hence $c_A(N[x]) \neq c_A(N[y])$ and therefore, $c(N[x]) \neq c(N[y])$. □

## 4. Minor closed classes of graphs

Let $G$ and $H$ be two graphs. $H$ is a *minor* of $G$ if $H$ can be obtained from $G$ with successive edge deletions, vertex deletions and edge contractions. A class $\mathcal{C}$ is *minor closed* if for any graph $G$ of $\mathcal{C}$, for any minor $H$ of $G$, we have $H \in \mathcal{C}$. The class $\mathcal{C}$ is *proper* if it is not the class of all graphs. Let $H$ be a graph. A *$H$-minor free graph* is a graph that does not have $H$ as a minor. We denote by $\mathcal{K}_n$ the $K_n$-minor-free class of graphs. It is clear that any proper minor closed class of graphs is included in the class $\mathcal{K}_n$ for some $n$. It is folklore that any proper minor closed class of graphs $\mathcal{C}$ has a bounded chromatic number $\chi(\mathcal{C})$.

The class of graphs of bounded expansion includes all the proper minor closed classes of graphs. Thus, by Corollary 6, proper minor closed classes have bounded lid-chromatic number. In this section, we focus on these latter classes and give an alternative upper bound on the lid-chromatic number. This gives us an explicit upper bound for the lid-chromatic number of planar graphs.

Consider any proper minor closed class of graphs $\mathcal{C}$. Since $\mathcal{C}$ is proper, there exists $n$ such that $\mathcal{C}$ does not contain $K_n$, that is $\mathcal{C} \subseteq \mathcal{K}_n$. Let $\mathcal{C}^N$ be the class of graphs defined by $H \in \mathcal{C}^N$ if and only if there exists $G \in \mathcal{C}$ and $v \in G$ such that $H = G[N(v)]$. Note that $\mathcal{C}^N$ is a minor-closed class of graphs. Indeed, given any $H \in \mathcal{C}^N$, let $G \in \mathcal{C}$ and $v \in V(G)$ such that $H = G[N(v)]$. Let $H'$ be any minor of $H$. Since $\mathcal{C}$ is minor-closed and $H$ is a subgraph of $G$, there exists a minor $G'$ of $G$ such that $H' = G'[N(v)]$. Therefore, $H'$ belongs to $\mathcal{C}^N$.

We prove the following result on minor-closed classes of graphs:

**Theorem 7.** *Let $\mathcal{C}$ be a proper minor closed class of graphs and let $n \geq 3$ be such that $\mathcal{C} \subseteq \mathcal{K}_n$. Then*

$$\chi_{\mathrm{lid}}(\mathcal{C}) \leq 4 \cdot \chi_{\mathrm{lid}}(\mathcal{C}^N) \cdot \chi(\mathcal{C})^{n-3}$$

The class of trees is exactly the class $\mathcal{K}_3$. Esperet et al. [6] proved the following result.

**Proposition 8** ([6]). $\chi_{\mathrm{lid}}(\mathcal{K}_3) \leq 4$.

It is clear that $\mathcal{K}_3^N$ is the class of stable graphs and therefore, $\chi_{\mathrm{lid}}(\mathcal{K}_3^N) = 1$. Note that Theorem 7 implies Proposition 8.

Assume that $\chi_{\mathrm{lid}}(\mathcal{K}_{n-1})$ is bounded for some $n \geq 4$. It is clear that $\mathcal{K}_n^N = \mathcal{K}_{n-1}$. Then, by Theorem 7, we have $\chi_{\mathrm{lid}}(\mathcal{K}_n) \leq 4 \cdot \chi_{\mathrm{lid}}(\mathcal{K}_{n-1}) \cdot \chi(\mathcal{K}_n)^{n-3}$. Since $\chi_{\mathrm{lid}}(\mathcal{K}_{n-1})$ and $\chi(\mathcal{K}_n)$ are bounded, $\chi_{\mathrm{lid}}(\mathcal{K}_n)$ is bounded.

Esperet et al. [6] also proved the following result.

**Proposition 9** ([6]). *If $G$ is an outerplanar graph, $\chi_{\mathrm{lid}}(G) \leq 20$.*

We can then deduce from Theorem 7 and Proposition 9 the following corollary:

**Corollary 10.** *Let $\mathcal{P}$ be the class of planar graphs. Then $\chi_{\mathrm{lid}}(\mathcal{P}) \leq 1280$.*

**Proof.** Any graph $G \in \mathscr{P}$ is $\{K_{3,3}, K_5\}$-minor free and thus $\mathscr{P}$ is a proper minor closed class of graphs. Moreover, the neighborhood of any vertex of $G \in \mathscr{P}$ is an outerplanar graph. By Proposition 9, we have $\chi_{\text{lid}}(\mathscr{P}^N) \leq 20$. Furthermore, the Four-Color-Theorem gives $\chi(\mathscr{P}) = 4$. By Theorem 7, $\chi_{\text{lid}}(\mathscr{P}) \leq 4 \times 20 \times 4^2 = 1280$. □

We finally give the proof of Theorem 7.

**Proof of Theorem 7.** Let $G \in \mathscr{C}$ and let $u$ be a vertex of minimum degree. For any $i$, define $V_{u,i}$ as the set of vertices of $G$ at distance exactly $i$ from $u$ and let $G_{u,i} = G[V_{u,i}]$. Let $s$ be the largest distance from a vertex of $V$ to $u$. In other words, there are $s + 1$ nonempty sets $V_{u,i}$ (note that $V_{\{u,0\}} = \{u\}$).

For any $i$, contracting in $G$ the subgraph $G[V_{u,0} \cup V_{u,1} \cup \cdots \cup V_{u,i-1}]$ in a single vertex $x$ gives a graph $G' \in \mathscr{C}$ such that $x$ is exactly adjacent to every vertex of $G_{u,i}$. Therefore, for any $i$, $G_{u,i} \in \mathscr{C}^N$. Hence, $\chi_{\text{lid}}(G_{u,i}) \leq \chi_{\text{lid}}(\mathscr{C}^N)$ for any $i$. Moreover, $\mathscr{C}^N \subseteq \mathscr{K}_{n-1}$. Indeed, suppose that there exists $H \in \mathscr{C}^N$ that admits $K_{n-1}$ as a minor. Therefore there exists $G \in \mathscr{C}$ such that $H = G[N(v)]$ for some $v \in G$. Taking $v$ together with its neighborhood would give $K_n$ as a minor, that contradicts the fact that $\mathscr{C} \subseteq \mathscr{K}_n$. Hence, any $G_{u,i} \in \mathscr{K}_{n-1}$.

We construct a lid-coloring of $G$ using $4 \cdot \chi_{\text{lid}}(\mathscr{C}^N) \cdot \chi(\mathscr{C})^{n-3}$ colors. This coloring is constructed with three different colorings of the vertices of $G$: $c_1$ which uses 4 colors, $c_2$ which uses $\chi_{\text{lid}}(\mathscr{C}^N)$ colors and $c_3$ which is itself composed of $n - 3$ colorings with $\chi(\mathscr{C})$ colors. The final color $c(v)$ of a vertex $v$ will be the triplet $(c_1(v), c_2(v), c_3(v))$. Hence the coloring $c$ uses at most $4\chi_{\text{lid}}(\mathscr{C}^N) \cdot \chi(\mathscr{C})^{n-3}$ colors. The coloring $c_1$ is used to separate the pairs of vertices that lie in distinct sets $V_{u,i}$. The coloring $c_2$ separates the pairs of vertices that lie in the same set $V_{u,i}$ and are not twins in $G_{u,i}$. Finally, the coloring $c_3$ separates the pairs of vertices that lie in the same set $V_{u,i}$, that are twins in $G_{u,i}$ but that are not twins in $G$.

The coloring $c_1$ is simply defined by $c_1(v) \equiv i (\text{mod } 4)$ if $v \in V_{u,i}$.

To define $c_2$, we define for each $i$, $0 \leq i \leq s$, a lid-coloring $c_2^i$ of $G_{u,i}$ using colors 1 to $\chi_{\text{lid}}(\mathscr{C}^N)$. Then $c_2$ is defined by $c_2(v) = c_2^i(v)$ if $v \in V_{u,i}$.

We now define the coloring $c_3$. Let $V_{u,i}^{\text{id}}$ be the set of vertices of $V_{u,i}$ that have a twin in $G_{u,i}$:

$$V_{u,i}^{\text{id}} = \{v \in V_{u,i} \mid \exists w \in V_{u,i}, N_{G_{u,i}}[v] = N_{G_{u,i}}[w]\}.$$

Let $G_{u,i}^{\text{id}} = G_{u,i}[V_{u,i}^{\text{id}}]$. Since the relation "be twin" is transitive (i.e. if $u$ and $v$ are twins, and $v$ and $w$ are twins, then $u$ and $w$ are twins), then $G_{u,i}^{\text{id}}$ is clearly a union of cliques. In addition, since $G_{u,i} \in \mathscr{K}_{n-1}$, the connected components of $G_{u,i}^{\text{id}}$ are cliques of size at most $n - 2$.

Let $C$ be a clique of $G_{u,i}^{\text{id}}$. By Corollary 2, there exists a subset $S(C) \subseteq V(G)$ of at most $n - 3$ vertices that distinguishes all the pairs of non-twin vertices of $C$. Note that by definition of $C$, $S(C) \cap V_{u,i} = \emptyset$, and thus $S(C) \subseteq V_{u,i-1} \cup V_{u,i+1}$.

Let $\mathscr{S} = \{(v, C) \mid v \in S(C)$ and $C$ is a clique in a graph $G_{u,i}^{\text{id}}\}$. We partition $\mathscr{S}$ in $s \times (n-3)$ sets $S_i^k$, $1 \leq i \leq s$, $1 \leq k \leq n-3$, such that:

- if $(v, C) \in S_i^k$ for some $k$, then $v \in V_{u,i}$;
- if $(v, C)$ and $(w, C')$ are two elements of $S_i^k$, then $C \neq C'$.

This partition can be done because each set $S(C)$ has size at most $n - 3$.

For each $S_i^k = \{(x_1, C_1), (x_2, C_2), \ldots, (x_t, C_t)\}$, we define a graph $H_i^k$ as follows. We start from the graph induced by $V_{u,i} \cup V(C_1) \cup V(C_2) \cup \cdots \cup V(C_t)$. Then, for each $(x_j, C_j)$ in $S_i^k$, we contract $C_j$ in a single vertex $y_j$ and finally, we contract the edge $x_jy_j$ on the vertex $x_j$. Note that $V_{u,i}$ is the vertex set of $H_i^k$. Note also that $H_i^k \in \mathscr{C}$ since it is obtained from a subgraph of $G$ by successive edge-contractions. Therefore, $\chi(H_k^i) \leq \chi(\mathscr{C})$.

We now define a proper coloring $c_3^{i,k}$ of $H_i^k$ with colors 1 to $\chi(\mathscr{C})$. Let $c_3^k$ be the coloring of vertices of $G$ defined by $c_3^k(v) = c_3^{i,k}(v)$ if $v \in V_{u,i}$. Finally, $c_3$ is defined by $c_3(v) = (c_3^1(v), \ldots, c_3^{n-3}(v))$, and the final color of $v$ is $c(v) = (c_1(v), c_2(v), c_3(v))$.

We now prove that $c$ is a lid-coloring of $G$. First, $c$ is a proper coloring. Indeed, two adjacent vertices that are not in the same set $V_{u,i}$ lie in consecutive sets $V_{u,i}$ and $V_{u,i+1}$ and thus have different colors in $c_1$, and two adjacent vertices in the same set $V_{u,i}$ have different colors in $c_2$ (which induces a proper coloring on $V_{u,i}$).

Let now $x$ and $y$ be two adjacent vertices with $N[x] \neq N[y]$. We will prove that $c(N[x]) \neq c(N[y])$. We distinguish three cases.

Case: 1 $x \in V_{u,i}$ and $y \in V_{u,i+1}$.

If $x = u$, then $y$ has a neighbor $v$ in $V_{u,i+2} = V_{u,2}$. Indeed, $u$ is taken with minimum degree, so $y$ has at least as many neighbors as $u$ and does not have the same neighborhood than $u$, implying that $y$ has a neighbor in $V_{u,2}$. Then $c_1(v) = 2 \notin c_1(N[u])$ and so $c(N[x]) \neq c(N[y])$.

Otherwise, $x$ has neighbor $v$ in $V_{u,i-1}$ and $c_1(v) \equiv i - 1 (\text{mod } 4) \in c_1(N[x])$. On the other hand, all the neighbors of $y$ belong to $V_{u,i} \cup V_{u,i+1} \cup V_{u,i+2}$ and therefore $c_1(N[y]) \subseteq \{i, i+1, i+2 (\text{mod } 4)\}$. Thus, $c(N[x]) \neq c(N[y])$.

Case: 2 $x$ and $y$ belong to $V_{u,i}$ and they are not twins in $V_{u,i}$ (i.e. $N_{V_{u,i}}[x] \neq N_{V_{u,i}}[y]$).

By definition of the coloring $c_2^i$, there exists a color $a$ that separates $x$ and $y$, i.e. $a \in c_2^i(N_{V_{u,i}}[x]) \Delta c_2^i(N_{V_{u,i}}[y])$. Then we necessarily have $c(N[x]) \neq c(N[y])$.

Case: 3 $x$ and $y$ belong to $V_{u,i}$ and they are twins in $V_{u,i}$ (i.e. $N_{V_{u,i}}[x] = N_{V_{u,i}}[y]$).

In this case, vertices $x$ and $y$ are in the set $V_{u,i}^{\mathrm{id}}$. Let $C$ be the clique of $G_{u,i}$ containing $x$ and $y$. Let $v \in S(C)$ that distinguishes $x$ and $y$; thus, $v \in V_{u,j}$ for $j = i-1$ or $j = i+1$. Wlog, $v \in N[x]$ but $v \notin N[y]$. Let $S_j^k$ be the part of $\mathcal{S}$ that contains $(v, C)$. Suppose that there exists a neighbor $w$ of $y$ such that $c(v) = c(w)$. Then $w$ lies in $V_{u,j}$ because of the coloring $c_1$. However, in the graph $H_j^k$, the vertex $v$ is adjacent to all the neighbors of $y$ in $V_{u,j}$, and in particular is adjacent to $w$; therefore, $c_3^{j,k}(v) \neq c_3^{j,k}(w)$, a contradiction. Therefore, the vertex $y$ does not have any neighbor that has the same color as $v$. Hence, $c(v) \notin c(N[y])$, and $c(N[x]) \neq c(N[y])$. □

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