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ScienceDirect
Electronic Notes in Discrete Mathematics 30 (2008) 33–38
Electronic Notes in DISCRETE MATHEMATICS
www.elsevier.com/locate/endm

Homomorphisms of 2-edge-colored graphs

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Abstract

In this paper, we study homomorphisms of 2-edge-colored graphs, that is graphs with edges colored with two colors. We consider various graph classes (outerplanar graphs, partial 2-trees, partial 3-trees, planar graphs) and the problem is to find, for each class, the smallest number of vertices of a 2-edge-colored graph H such that each graph of the considered class admits a homomorphism to H.

1 Introduction

Our general aim is to study homomorphisms of (n, m)-mixed graphs, that is graphs with both arcs and edges respectively colored with n and m colors. This notion was introduced by Nešetřil and Raspaud [5] as a generalization of the notion of homomorphisms of edge-colored graphs (see e.g. [1]) and the

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doi:10.1016/j.endm.2008.01.007

notion of oriented coloring (see e.g. [8]). In this paper, we focus on $(0, 2)$-mixed graphs, that is 2-edge-colored graphs.

An (n, m)-*mixed graph* is a set of vertices $V(G)$ linked by arcs $A(G)$ and edges $E(G)$, such that the underlying graph is simple (no multiple edges or loops), the arcs are colored with n colors and the edges are colored with m colors. In other words, there is a partition $A(G) = A_1(G) \cup \ldots \cup A_n(G)$ of the set of arcs of G, were $A_i(G)$ contains all arcs with color i and a partition $E(G) = E_1(G) \cup \ldots \cup E_m(G)$ of the edges of G, where $E_j(G)$ contains all edges with color j. We denote the class of (n, m)-mixed graphs by $\mathcal{G}^{(n,m)}$. Observe that $\mathcal{G}^{(0,1)}$ is the class of simple graphs and $\mathcal{G}^{(1,0)}$ is the class of oriented graphs.

Let $G = \{V(G); \bigcup_{i=1}^{n} A_i(G), \bigcup_{j=1}^{m} E_j(G)\}$ and $H = \{V(H); \bigcup_{i=1}^{n} A_i(H), \bigcup_{j=1}^{m} E_j(H)\}$ be two (n, m)-mixed graphs. A *homomorphism* from G to H is a mapping $h : V(G) \to V(H)$ such that $(h(u), h(v)) \in A_i(H)$ whenever $(u, v) \in A_i(G)$ (for every $i \in \{1, \ldots n\}$), and $h(u)h(v) \in E_j(H)$ whenever $uv \in E_j(G)$ (for every $j \in \{1, \ldots m\}$). The existence of a homomorphism from G to H is denoted by $G \to H$, and $G \not\to H$ means there is no such homomorphism.

Given an (n, m)-mixed graph G, the problem is to find the smallest number of vertices of a graph H such that $G \to H$. This number is denoted by $\chi_{(n,m)}(G)$ and is called the *chromatic number* of the (n, m)-mixed graph G. For a simple graph G, the (n, m)-*mixed chromatic number* is the maximum of the chromatic numbers taken over all the possible (n, m)-mixed graphs having G as underlying graph. Note that $\chi_{(0,1)}(G)$ is the ordinary chromatic number $\chi(G)$, and $\chi_{(1,0)}(G)$ is the oriented chromatic number $\chi_o(G)$. Given a family $\mathcal{F}$ of simple graphs, we denote by $\chi_{(n,m)}(\mathcal{F})$ the maximum of $\chi_{(n,m)}(G)$ taken over all members in $\mathcal{F}$.

Note that a complexity result of Edwards and McDiarmid [3] on the harmonious chromatic number implies that to find the $(0, 2)$-mixed chromatic number of a graph is in general an NP-complete problem.

Recall that an *acyclic coloring* of a simple graph G is a proper vertex-coloring satisfying that every cycle of G received at least three colors. The *acyclic chromatic number* of G, denoted by $\chi_a(G)$, is the smallest k such that G admits an acyclic k-vertex coloring. The class of graphs with acyclic chromatic number at most k is denoted by $\mathcal{A}_k$.

Nešetřil and Raspaud [5] proved that the families of bounded acyclic chromatic number have bounded (n, m)-mixed chromatic number. More precisely:

Theorem 1.1 *[5]* $\chi_{(n,m)}(\mathcal{A}_k) \leq k(2n + m)^{k-1}$.

Combining this result with the well-know result of Borodin [2] (every planar graph has an acyclic chromatic number at most 5), we get:

Corollary 1.2 *[5] Let $\mathcal{P}$ be the class of (n, m)-mixed planar graphs. Then $\chi_{(n,m)}(\mathcal{P}) \leq 5(2n + m)^4$.*

This last upper bound extends some previous known results on edge-colored planar graph [1] and on oriented planar graphs [6].

Nešetřil and Raspaud [5] also provided the exact (n, m)-mixed chromatic number of forests ($\mathcal{F}$ denotes the class of (n, m)-mixed forests):

Theorem 1.3 *[5] $\chi_{(n,0)}(\mathcal{F}) = 2n + 1$ and $\chi_{(n,m)}(\mathcal{F}) = 2(n + \lfloor \frac{m}{2} \rfloor + 1)$ for $m \neq 0$.*

Recently, Fabila et al. [4] studied the (n, m)-mixed chromatic number of paths. They proved that it is exactly the same as for the forests; this proves that the lower bound of Theorem 1.3 is reached with paths.

We can obtain new bounds on the (n, m)-mixed chromatic number of partial k-trees, planar graphs, and outerplanar graphs thanks to the above results.

A *k-tree* is a simple graph obtained from the complete graph K_k by repeatedly adding a new vertex adjacent to each vertex of an existing clique of size k. A *partial k-tree* is a subgraph of some k-tree. It is not difficult to see that every partial k-tree has acyclic chromatic number at most $k+1$. We then get the following from Theorem 1.1:

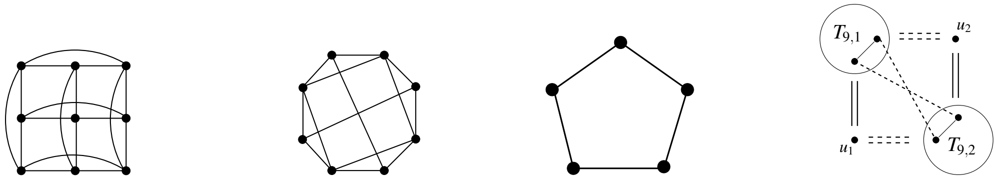
Corollary 1.4 *Let $\mathcal{T}^k$ be the class of (n, m)-mixed partial k-trees. Then $\chi_{(n,m)}(\mathcal{T}^k) \leq (k + 1)(2n + m)^k$.*

In addition, we can derive lower bounds for outerplanar graphs, planar graphs and partial 3-trees from Theorem 1.3 and the result of Fabila et al. [4]:

Corollary 1.5 *Let $\epsilon = 1$ for m odd or $m = 0$, and $\epsilon = 2$ for $m > 0$ even.*

1. *There exist outerplanar graphs G with $\chi_{(n,m)}(G) \geq (2n+m)^2+\epsilon(2n+m)+1$.*
2. *There exist planar partial 3-trees G with $\chi_{(n,m)}(G) \geq (2n + m)^3 + \epsilon(2n + m)^2 + (2n + m) + \epsilon$.*

In this extended abstract, we study the particular class of $(0, 2)$-mixed graphs. More precisely, we give the complete classification for the $(0, 2)$-mixed chromatic number of outerplanar graphs and partial 2-trees with given girth (this improves Corollary 1.4 for $k = 2$). We also provide the exact $(0, 2)$-mixed chromatic number of partial 3-trees. Finally, we obtain upper bounds for the $(0, 2)$-mixed chromatic number of the class of planar graphs with given girth.



(a) $T_9 = C_3 \times C_3$ (b) T_8 (c) $T_5 = C_5$ (d) T_{20}

Fig. 1. The four target graphs T_9, T_8, T_5, and T_{20}.

2 The target graphs

When studying homomorphisms to get bounds on the chromatic number of a graph class $\mathcal{C}$, one often tries to find an *universal* target graph for $\mathcal{C}$, that is a target graph H such that all the graphs of $\mathcal{C}$ admits a homomorphism to H. To prove that a target graph is universal for a graph class, we need "useful" properties. In this section, we construct four $(0,2)$-mixed target graphs which will be used in the sequel to get upper bounds for $(0,2)$-mixed chromatic number. Their useful properties are given below.

Consider the three graphs depicted in Figures 1(a), 1(b), and 1(c). These graphs are all self complementary (i.e. isomorphic to their complement). Thus, let T_9 (resp. T_8, T_5) be the complete $(0,2)$-mixed graphs on 9 (resp. 8, 5) vertices where the edges of each color induce an isomorphic copy of the graphs depicted in Figure 1(a) (resp. 1(b), 1(c)).

Proposition 2.1 *For every pair of distinct vertices u and v of T_9 (resp. T_5) and every $(0,2)$-mixed k-path $P_k = u_0, u_1, \ldots, u_k$, $k \geq 2$ (resp. $k \geq 3$), there exists a homomorphism h from P_k to T_9 (resp. T_5) such that $h(u_0) = u$ and $h(u_k) = v$.*

Proposition 2.2 *For each $v \in V(T_8)$ and each $(0,2)$-mixed path of length k, the number of vertices in T_8 reachable from v by such a k-path is at least 3 (resp. 7, 8) if $k = 1$ (resp. $k = 2$, $k \geq 3$).*

For a $(0,2)$-mixed graph, the edges can get two distinct colors: we will say that the edges with the first color are of *type 1* whereas the others are of *type 2*.

Let T_{20} be the complete $(0,2)$-mixed graph defined as follows (the construction is illustrated by Fig. 1(d)). Take two disjoint copies of T_9, namely $T_{9,1}$, $T_{9,2}$, and two new vertices u_1 and u_2. We put edges of type 1 (resp. of type 2) linking u_i to all vertices of $T_{9,i}$ (resp. $T_{9,3-i}$) for $1 \leq i \leq 2$. We also add an edge of type 1 (resp. type 2) between $u \in V(T_{9,1})$ and $v \in V(T_{9,2})$ whenever

$uv \in E(T_9)$ is of type 2 (resp. type 1). This construction is known as the *Tromp construction* and was already used to bound the oriented chromatic number (i.e. the $(1,0)$-mixed chromatic number) [7].

Proposition 2.3 *For every triangle u, v, w of T_{20} and every triple $(a, b, c) \in \{1, 2\}^3$, there exists a vertex t adjacent to u, v and, w such that tu (resp. tv, tw) is of type a (resp. b, c).*

3 Results

Let $\mathcal{O}_g$ be the class of $(0, 2)$-mixed outerplanar graphs with girth at least g. Outerplanar graphs form a strict subclass of partial 2-trees (also known as series-parallel graphs); therefore, Corollaries 1.4 and 1.5 implies that $9 \leq \chi_{(0,2)}(\mathcal{O}_3) \leq 12$. We improve this result and characterize the $(0, 2)$-mixed chromatic number of outerplanar graphs for all girth:

Theorem 3.1 $\chi_{(0,2)}(\mathcal{O}_3) = 9$ *and* $\chi_{(0,2)}(\mathcal{O}_g) = 5$ *for* $g \geq 4$.

These bounds are obtained by showing that every $(0, 2)$-mixed outerplanar graph with girth 3 (resp. girth at least 4) admits a homomorphism to T_9 (resp. T_5). To get the second result, we construct, for every girth $g \geq 3$, an outerplanar graph G with girth g and $\chi_{(0,2)}(G) = 5$, which proves that $\chi_{(0,2)}(\mathcal{O}) \geq 5$.

In the same vein, we find the $(0, 2)$-mixed chromatic number of partial 2-trees for all girths ($\mathcal{T}^2_g$ denotes the class of partial 2-trees with girth at least g):

Theorem 3.2 $\chi_{(0,2)}(\mathcal{T}^2_3) = 9$, $\chi_{(0,2)}(\mathcal{T}^2_g) = 8$ *for* $4 \leq g \leq 5$, *and* $\chi_{(0,2)}(\mathcal{T}^2_g) = 5$ *for* $g \geq 6$.

We get the upper bounds by showing that $(0, 2)$-mixed partial 2-trees with girth 3 (resp. 4, 6) admits a homomorphism to T_9 (resp. T_8, T_5). Each lower bound is obtained by constructing a $(0, 2)$-mixed partial 2-tree with the required girth which needs the specified number of colors.

Theorem 1.5 shows that $\chi_{(0,2)}(\mathcal{T}^3) \geq 20$. We prove that this bound is tight:

Theorem 3.3 $\chi_{(0,2)}(\mathcal{T}^3) = 20$.

We get this result by showing that every $(0, 2)$-mixed partial 3-trees admits a homomorphism to T_{20}.

Finally, we bound the $(0, 2)$-mixed chromatic number of sparse graphs. The *maximum average degree* of a simple graph G, denoted by $\text{mad}(G)$, is defined as $\text{mad}(G) = \max\left\{\frac{2|E(H)|}{|V(H)|}, H \subseteq G\right\}$, where $H \subseteq G$ means H is a subgraph of G.

Theorem 3.4 *Let G be a simple graph. If $\text{mad}(G) < \frac{8}{3}$ (resp. $\frac{7}{3}$), then $\chi_{(0,2)}(G) \leq 8$ (resp. $\chi_{(0,2)}(G) = 5$).*

Our proof technique is based on the well-know method of reducible configurations and discharging procedure. We consider a minimal counterexample H to Theorem 3.4. We prove that H does not contain a set S of configurations. Then, we prove, using a discharging procedure, that every graph containing none of the configurations of S has a maximum average degree greater than required by the theorem, that contradicts that H is a counterexample.

Let $\mathcal{P}_g$ be the class of $(0,2)$-mixed planar graphs with girth at least g.

Since every planar graph G with girth g verifies $\text{mad}(G) < \frac{2g}{g-2}$, we get the following corollary for planar graphs with given girth:

Corollary 3.5 $\chi_{(0,2)}(\mathcal{P}_8) \leq 8$ *and* $\chi_{(0,2)}(\mathcal{P}_{14}) = 5$.

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