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Homomorphisms of 2-edge-colored graphs

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ABSTRACT

In this paper, we study homomorphisms of 2-edge-colored graphs, that is graphs with edges colored with two colors. We consider various graph classes (outerplanar graphs, partial 2-trees, partial 3-trees, planar graphs) and the problem is to find, for each class, the smallest number of vertices of a 2-edge-colored graph H such that each graph of the considered class admits a homomorphism to H.

1. Introduction

Several papers related to the study of homomorphisms as a generalization of colorings have been done in the context of oriented graphs (see the monograph [6] by Nešetřil and Hell for more details).

Our general aim is to study homomorphisms of (n, m)-mixed graphs. This notion was introduced by Nešetřil and Raspaud [8] as a generalization of the notion of homomorphisms of edge-colored graphs (see e.g. [1]) and the notion of oriented coloring (see e.g. [12]).

A *mixed graph* G is a graph in which some pair of vertices are linked by edges and some are linked by arcs, and such that the underlying graph is simple (no multiple edges/arcs or loops). A mixed graph G is usually denoted by an ordered triple $G = (V(G); A(G), E(G))$ with $V(G)$ its vertex set, $E(G)$ its edge set, and $A(G)$ its arc set. Oriented and undirected graphs are special cases of mixed graphs.

An (n, m)-*mixed graph* G is a generalization of a mixed graph where vertices are linked by arcs $A(G)$ and edges $E(G)$ such that the arcs are colored with n colors and the edges are colored with m colors. In other words, there is a partition $A(G) = A_1(G) \cup \cdots \cup A_n(G)$ of the set of arcs of G, where $A_i(G)$ contains all arcs with color i and a partition $E(G) = E_1(G) \cup \cdots \cup E_m(G)$ of the edges of G, where $E_j(G)$ contains all edges with color j. Therefore, there exist $2n+m$ possibilities of adjacency between

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two vertices of G. For a vertex $u \in V(G)$, we define $N_i^+(u)$ (resp. $N_i^-(u)$) to be the set of vertices v of G such that $\overrightarrow{uv} \in A_i(G)$ (resp. $\overrightarrow{vu} \in A_i(G)$). Similarly, we define $N_i(u)$ to be the set of vertices v of G such that $uv \in E_i(G)$. Then, for two adjacent vertices u and v, we can define the *adjacency type* of the ordered pair (u, v) as:

$$t(u, v) = \begin{cases} i^+ & \text{if } v \in N_i^+(u) \\ i^- & \text{if } v \in N_i^-(u) \\ i & \text{if } v \in N_i(u). \end{cases}$$

Let $u \sim v$ denote that u and v are adjacent in G, that is either $uv \in E(G)$, or $\overrightarrow{uv} \in A(G)$, or $\overrightarrow{vu} \in A(G)$.

A *k-coloring* of an (n, m)-mixed graph G is a mapping f from $V(G)$ to a set of k colors such that (1) $f(u) \neq f(v)$ whenever $u \sim v$, and (2) $f(u) \neq f(x)$ whenever $u \sim v$ and $x \sim y$ with $f(v) = f(y)$ and $t(u, v) \neq t(x, y)$. In other word, a k-coloring of G is a partition of the vertices of G into k stable sets $S_1, S_2, \ldots, S_k$ such that there are only edges of the same color, or only arcs of the same orientation and the same color between any pair of stable sets S_i and S_j.

Let $G = (V(G); \bigcup_{i=1}^n A_i(G), \bigcup_{j=1}^m E_j(G))$ and $H = (V(H); \bigcup_{i=1}^n A_i(H), \bigcup_{j=1}^m E_j(H))$ be two (n, m)-mixed graphs. A *homomorphism* from G to H is a mapping $h : V(G) \to V(H)$ such that $\overrightarrow{h(u), h(v)} \in A_i(H)$ whenever $\overrightarrow{uv} \in A_i(G)$ (for every $i \in \{1, \ldots n\}$), and $h(u)h(v) \in E_j(H)$ whenever $uv \in E_j(G)$ (for every $j \in \{1, \ldots m\}$).

A k-coloring of G can be equivalently viewed as a homomorphism from G to H, where H is an (n, m)-mixed graph of order k (the *order* of a graph is its number of vertices). The existence of such a homomorphism from G to H is denoted by $G \to H$. The vertices of H are called *colors*, and we say that G is *H-colorable*. Given an (n, m)-mixed graph G, the problem is to find the smallest number of colors needed to color G, or, in other words, to find the smallest number of vertices of an (n, m)-mixed graph H such that $G \to H$. This number is denoted by $\chi_{(n,m)}(G)$ and is called the *chromatic number* of the (n, m)-mixed graph G. For a simple graph G, the (n, m)*-mixed chromatic number* is the maximum of the chromatic numbers taken over all the possible (n, m)-mixed graphs having G as underlying graph. Note that $\chi_{(0,1)}(G)$ is the ordinary chromatic number $\chi(G)$, and $\chi_{(1,0)}(G)$ is the oriented chromatic number $\chi_o(G)$. Given a family $\mathcal{F}$ of simple graphs, we denote by $\chi_{(n,m)}(\mathcal{F})$ the maximum of $\chi_{(n,m)}(G)$ taken over all members G in $\mathcal{F}$.

In this paper, we mainly study three particular classes of (0, 2)-mixed graphs: outerplanar graphs, partial k-trees, and planar graphs. This paper is organised as follows. Section 2 is dedicated to state some known results and to give some preliminary results on the (n, m)-mixed chromatic number of partial k-trees. In Section 3, we describe the constructions of the target graphs we use in the next sections. In Section 4, we give the complete classification for the (0, 2)-mixed chromatic number of outerplanar graphs for every girth. Section 5 is devoted to partial 2-trees with given girth and partial 3-trees. Finally, we obtain upper bounds for the (0, 2)-mixed chromatic number of planar graphs with given girth in Section 6.

Note that a complexity result of Edwards and McDiarmid [4] on the harmonious chromatic number implies that to find the (0, 2)-mixed chromatic number of a graph is in general an NP-complete problem.

Notation. We use the following notions. Let G be an (n, m)-mixed graph. For a vertex v of G, we denote by $d_G(v)$ its degree (subscripts are omitted when the considered graph is clearly identified from the context). A vertex of degree k (resp. at least k, at most k) is called a *k-vertex* (resp. $^{\geq}k$*-vertex*, $^{\leq}k$*-vertex*). We denote by $\delta(G)$ the smallest degree of a vertex in G. A path $P_k = [u, v_1, \ldots, v_{k-1}, w]$ of length k (i.e. formed by k edges) is called a *k-path*. If all internal vertices v_i of P_k are vertices of degree d, then P_k is a (k, d)*-path*. If two graphs G and H are isomorphic, we denote it by $G \cong H$.

We denote by $\mathcal{L}$ (resp. $\mathcal{F}$, $\mathcal{O}$, $\mathcal{T}^{(k)}$, $\mathcal{P}$) the class of (n, m)-mixed paths (resp. forests, outerplanar graphs, partial k-trees, planar graphs).

Drawing conventions. In Sections 5 *and* 6, we adopt the following drawing conventions for a *configuration* C contained in a graph G. If u and v are two vertices of C, then they are adjacent in G if and only if they are adjacent in C. Moreover, the neighbors of a *white* vertex in G are exactly its neighbors in C, whereas a *black* vertex may have neighbors outside of C. Two or more black vertices in C may coincide in a single vertex in G, provided they do not share a common white neighbor. Finally, an edge between two vertices u and v will mean that $u \sim v$ (i.e. u and v are linked by either an edge or an arc of any type).

2. Preliminary results

In this section, we state some known lower and upper bounds for the (n, m)-mixed chromatic number of outerplanar graphs, partial k-trees, and planar graphs, for every nonnegative n and m. We then improve lower bounds for the (n, m)-mixed chromatic number of partial k-trees and planar graphs for every m even.

Recall that an *acyclic coloring* of a simple graph G, is a proper vertex-coloring satisfying that every cycle of G receives at least three colors. The *acyclic chromatic number* of G, denoted by $\chi_a(G)$, is the smallest k such that G admits an acyclic vertex-coloring. The class of graphs with acyclic chromatic number at most k is denoted by $\mathcal{A}_k$. Nešetřil and Raspaud [8] proved that the families of graphs with bounded acyclic chromatic number have bounded (n, m)-mixed chromatic number. More precisely:

Theorem 1 *([8]).* $\chi_{(n,m)}(\mathcal{A}_k) \leq k(2n + m)^{k-1}$.

Remark that this result implies the result of Raspaud and Sopena [11] for oriented graphs ($\chi_{(1,0)}(\mathcal{A}_k) \leq k \cdot 2^{k-1}$) and the result of Alon and Marshall [1] for the m-edge-colored graphs ($\chi_{(0,m)}(\mathcal{A}_k) \leq k \cdot m^{k-1}$). Recently, Huemer et al. [5] proved that this bound is tight for $k \geq 3$: $\chi_{(n,m)}(\mathcal{A}_k) = k(2n + m)^{k-1}$.

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Combining Theorem 1 with the well-known result of Borodin [2] (every planar graph has an acyclic chromatic number at most five), we get the next result:

Corollary 2 (*[8]*). $\chi_{(n,m)}(\mathcal{P}) \leq 5(2n+m)^4$.

A *k-tree* is a simple graph obtained from the complete graph K_k by repeatedly inserting new vertices adjacent to an existing clique of size k. A *partial k-tree* is a subgraph of a k-tree. It is not difficult to see that every partial k-tree has acyclic chromatic number at most $(k+1)$: starting with a proper k-coloring of the complete graph K_k, every newly inserted vertex has exactly k neighbors and can be thus colored using a $(k+1)$st color. Moreover, this coloring is clearly acyclic since all the neighbors of a newly inserted vertex have pairwise distinct colors. Therefore, by Theorem 1 we get:

Corollary 3. *For every $n \geq 0$, $m \geq 0$, and $k \geq 1$, $\chi_{(n,m)}(\mathcal{T}^{(k)}) \leq (k+1)(2n+m)^k$.*

Concerning the class of partial 1-trees (which are usual forests), the Theorem 1 gives $\chi_{(n,m)}(\mathcal{F}) \leq 2(2n+m)$. Nešetřil and Raspaud [8] gave the exact (n,m)-mixed chromatic number of (n,m)-mixed forests.

Theorem 4 (*[8]*). *$\chi_{(n,m)}(\mathcal{F}) = 2n+m+\epsilon$ where $\epsilon = 1$ for m odd or $m = 0$, and $\epsilon = 2$ for $m > 0$ even.*

Huemer et al. [5] recently provided lower bounds for the (n,m)-mixed chromatic number of some graph classes, such as paths, outerplanar graphs, partial k-trees, or planar graphs. More precisely:

Theorem 5 (*[5]*). *For every $n \geq 0$ and $m \geq 0$,*

$$\chi_{(n,m)}(\mathcal{T}^{(k)}) \geq (2n+m)^k + \epsilon(2n+m)^{k-1} + \frac{(2n+m)^{k-1}-1}{(2n+m)-1}$$

where $\epsilon = 1$ for m odd or $m = 0$, and $\epsilon = 2$ for $m > 0$ even.

Theorem 6 (*[5]*). $\chi_{(n,m)}(\mathcal{L}) = \chi_{(n,m)}(\mathcal{F})$.

This proves that the lower bound of Theorem 4 is reached with paths. By means of this result, one can get the following lower bounds for outerplanar graphs and planar graphs:

Theorem 7 (*[5]*). *Let $\epsilon = 1$ for m odd or $m = 0$, and $\epsilon = 2$ for $m > 0$ even.*
(1) $\chi_{(n,m)}(\mathcal{O}) \geq (2n+m)^2 + \epsilon(2n+m) + 1$
(2) $\chi_{(n,m)}(\mathcal{P}) \geq (2n+m)^3 + \epsilon(2n+m)^2 + (2n+m) + 1$.

We shall prove that the lower bound for the (n,m)-mixed chromatic number of partial k-trees given by Theorem 5 can be improved by one when m is even and $k \geq 3$. This will allow us to get a tight bound for the $(0,2)$-mixed chromatic number of partial 3-trees in Section 5.

Theorem 8. *For every $k \geq 3$, $n \geq 0$ and $m > 0$ even,*

$$\chi_{(n,m)}(\mathcal{T}^{(k)}) \geq (2n+m)^k + 2(2n+m)^{k-1} + \frac{(2n+m)^{k-1}-1}{(2n+m)-1} + 1.$$

Proof. Let $a(x) = (2n+m)^x + 2(2n+m)^{x-1} + \frac{(2n+m)^{x-1}-1}{(2n+m)-1}$. To prove our result, we construct an (n,m)-mixed partial k-tree T' which admits no homomorphism to any (n,m)-mixed complete graph of order $a(k)$.

Theorem 5 ensures that there exists an (n,m)-mixed partial $(k-1)$-tree T such that $\chi_{(n,m)}(T) \geq a(k-1)$. Let T' be the (n,m)-mixed partial k-tree obtained by taking $2n+m$ disjoint copies of T, namely $T_1^+, T_2^+, \ldots, T_n^+, T_1^-, T_2^-, \ldots, T_n^-, T_1, T_2, \ldots, T_m$, and a universal vertex u in such a way that $t(u,v) = i^+$ (resp. $t(u,v) = i^-$, $t(u,v) = i$) for every $v \in V(T_i^+)$ (resp. $v \in V(T_i^-)$, $v \in V(T_i)$). Such a construction clearly guaranties that T' is a partial k-tree. By construction, a color c cannot appear in two different copies of T. Moreover, the vertex u must be assigned a color distinct from those assigned to all other vertices. Hence, the number of colors needed to color T' is at least $(2n+m) \times a(k-1) + 1 = a(k)$.

Suppose that T' has ℓ vertices and let $V(T') = \{v_1, v_2, \ldots, v_\ell\}$. Now consider the (n,m)-mixed partial k-tree T'' obtained as follows. We take $\ell+1$ copies of T', a first one named T'_0, and the k remaining one named $T'_{v_1}, T'_{v_2}, \ldots, T'_{v_l}$ (one copy per vertex of T'). Then, for each vertex v of T'_0, we glue the universal vertex of T'_v with the vertex v of T'_0. The graph T'' is clearly a partial k-tree. Suppose that T'' is $K_{a(k)}$-colorable for some (n,m)-mixed complete graph $K_{a(k)}$ of order $a(k)$. Since $\chi_{(n,m)}(T'_0) \geq a(k)$, each of the $a(k)$ colors appears on at least one vertex of $V(T'_0)$. Therefore, since each vertex $v \in V(T'_0)$ is a universal vertex of a copy of T', we necessarily have, for every vertex $w \in V(K_{a(k)})$, $|N_i^+(w)| \geq a(k-1)$, $|N_i^-(w)| \geq a(k-1)$, and $|N_j(w)| \geq a(k-1)$ for every $1 \leq i \leq n$ and $1 \leq j \leq m$. Since $K_{a(k)}$ has $a(k)$ vertices, its maximum degree is $a(k)-1 = (2n+m) \times a(k-1)$, that implies that $|N_i^+(w)| = a(k-1)$, $|N_i^-(w)| = a(k-1)$, and $|N_j(w)| = a(k-1)$ for every $1 \leq i \leq n$ and $1 \leq j \leq m$. Let us now consider the subgraph K' of $K_{a(k)}$ induced by the edges uv such that $t(u,v) = 1$ (since $m > 0$, there exist edges of type 1, and thus K' is non-empty). This subgraph has $a(k)$ vertices and is $a(k-1)$-regular. Then, we obviously have $\sum_{v \in V(K')} d(v) = a(k) \times a(k-1)$. However, the sum of the degrees should be even, but $a(k) \times a(k-1)$ is odd since m is even. Then, the graph $K_{a(k)}$ does not exist. □

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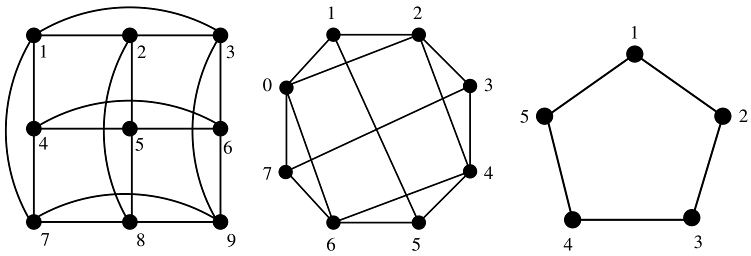

(a) $C_3 \times C_3$. (b) (c) C_5.

Fig. 1. Construction of the three target graphs T_9, T_8, and T_5.

Note that for planar partial 3-trees, the partial 3-tree T'' constructed in the previous proof is also a planar graph. Therefore, this also improves the bound of Theorem 7(2) for every $m \geq 0$ even.

Corollary 9. *For every $n \geq 0$ and $m > 0$ even,*

$$\chi_{(n,m)}(\mathcal{T}^{(3)} \cap \mathcal{P}) \geq (2n+m)^3 + 2(2n+m)^2 + (2n+m) + 2.$$

3. The target graphs

In the rest of this paper, we focus on the class of (0, 2)-mixed graphs, that is 2-edge-colored graphs. Therefore, the target graphs provided in this section are 2-edge-colored graphs.

When studying homomorphisms, to get upper bounds for the (n, m)-mixed chromatic number of a graph class $\mathcal{C}$, one often tries to find an *universal* target graph for $\mathcal{C}$, that is a target graph H such that all the graphs of $\mathcal{C}$ admits a homomorphism to H. To prove that a target graph is universal for a graph class, we need "useful" properties of this target graph. In this section, we construct five (0, 2)-mixed target graphs which will be used to get upper bounds for the (0, 2)-mixed chromatic number. Their useful properties are given below.

First consider the three graphs depicted in Fig. 1(a), (b) and (c). These graphs are all self-complementary (i.e. isomorphic to their complement). Thus, let T_9 (resp. T_8, T_5) be the complete (0, 2)-mixed graphs on 9 (resp. 8, 5) vertices where the edges of each color induce an isomorphic copy of the graph depicted in Fig. 1(a) (resp. 1(b), (c)). In other words, the edges of the graph depicted in Fig. 1(a) (resp. 1(b), (c)) are the edges of type 1 of T_9 (resp. T_8, T_5) and the non-edges are the edges of type 2 of T_9 (resp. T_8, T_5).

It is not difficult to check that T_9 and T_5 are *vertex-transitive* and *colored-edge-transitive* (i.e. for every two edges uv and $u'v'$ of the same color, there exists an automorphism that maps u to u' and v to v').

A *type-vector* of size n (or a *n-type-vector*) is a sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \{1, 2\}^n$ of n elements. Let $S = (v_1, v_2, \ldots, v_n)$ be a sequence of n vertices of a (0, 2)-mixed graph $T = (V(T); \bigcup_{i=1}^{2} E_i(T))$ which induces an n-clique subgraph; a vertex u is said to be an *α-neighbor of* S if for every i, $1 \leq i \leq n$, we have $uv_i \in E_{\alpha_i}(T)$.

In the remainder, we say that a (0, 2)-mixed graph T has *property* $\mathcal{P}_{n,k}$ if, for every $1 \leq \ell \leq n$, for every sequence S of ℓ vertices of T which induces an ℓ-clique subgraph, and any ℓ-type-vector α, there exist at least k α-neighbors of S.

Proposition 10. *The graph T_9 has properties $\mathcal{P}_{1,4}$ and $\mathcal{P}_{2,1}$.*

Proof. Property $\mathcal{P}_{1,4}$ is trivial.

Recall that T_9 is colored-edge-transitive, and that the two graphs induced by the edges of each color in T_9 are isomorphic. Therefore, it is enough to show that the sequence of vertices $S = (1, 2)$ has at least one α-neighbor for any $\alpha \in \{1, 2\}^2$. If $\alpha = (1, 1)$ (resp. (1, 2), (2, 1), (2, 2)), then 3 (resp. 4, 5, 6) is an α-neighbor of S. □

Proposition 11. *The graph T_8 (resp. T_5) has property $\mathcal{P}_{1,3}$ (resp. $\mathcal{P}_{1,2}$).*

Proof. It directly follows from $|N_i(v)| \geq 3$ (resp. $|N_i(v)| \geq 2$) for every $i \in \{1, 2\}$ and every $v \in V(T_8)$ (resp. $V(T_5)$). □

Let $P = [v_0, v_1, v_2, \ldots, v_k]$ be a (0, 2)-mixed k-path, G a (0, 2)-mixed graph, and u a vertex of G. We denote $N_P(G, u) = \{v \in G, \exists h : P \to G \text{ with } h(v_0) = u \text{ and } h(v_k) = v\}$.

Proposition 12. *For every* (0, 2)*-mixed 3-path P:*

- *for every $u \in T_5$, $V(T_5) \setminus \{u\} \subseteq N_P(T_5, u)$*
- *for every $u \in T_8$, $N_P(T_8, u) = V(T_8)$.*

Proof. (1) Recall that T_5 is colored-edge-transitive, and the two graphs induced by the edges of each color in T_5 are isomorphic. Therefore, it is enough to prove that there exist the eight possible 3-paths linking 1 and 2. This easy case study is left to the reader.

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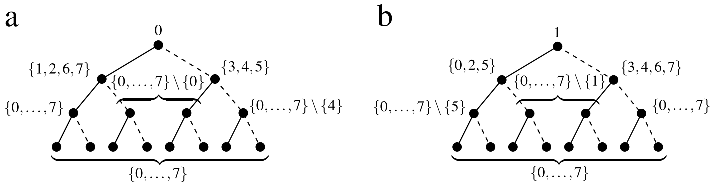

Fig. 2. The vertices of T_8 reachable from 0 and 1 by each $(0, 2)$-mixed k-path with $1 \le k \le 3$.

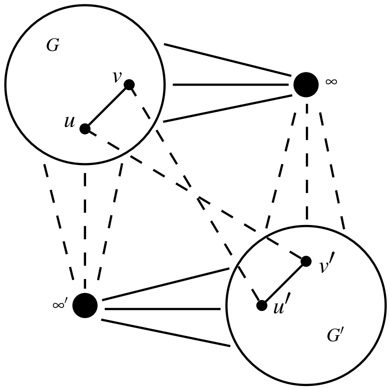

Fig. 3. The $(0, 2)$-mixed tromp graph $\mathrm{Tr}(G)$.

(2) Observe that there are two kinds of vertices in T_8: the even vertices $\{0, 2, 4, 6\}$, and the odd vertices $\{1, 3, 5, 7\}$. It is not difficult to check that for every two even (resp. odd) vertices u and v, there exists an automorphism of T_8 that maps u to v. Thus is sufficient to prove that there exist the eight $(0, 2)$-mixed 3-paths joining 0 (resp. 1) to every vertex of T_8. Fig. 2(a) and (b) show the sets of vertices of T_8 reachable from 0 and 1 by each $(0, 2)$-mixed k-path with $1 \le k \le 3$. □

Proposition 13. *For every $(0, 2)$-mixed 4-path P and for every $u \in T_5$, $N_P(T_5, u) = V(T_5)$.*

Proof. This follows directly from property $\mathcal{P}_{1,2}$ and Proposition 12. □

Proposition 14. *For every vertex u of T_8 and for a every $(0, 2)$-mixed 2-path P, there exists one vertex $v \in \{u, u + 4 \pmod 8\}$ such that $V(T_8) \setminus \{v\} \subseteq N_P(T_8, u)$.*

Proof. This can be easily checked on Fig. 2(a) and (b). □

Proposition 15. *For every vertex u of T_5 and for every $(0, 2)$-mixed 2-path P, $|N_P(T_5, u)| \ge 3$.*

Proof. This can be easily checked. □

The last two target graphs we provide are obtained by using *Tromp's construction* [14] extended to mixed graphs. Let $G = (V(G); \bigcup_{i=1}^{2} E_i(G))$ be a $(0, 2)$-mixed graph and G' be an isomorphic copy of G. The Tromp graph $\mathrm{Tr}(G) = (V(\mathrm{Tr}(G)); \bigcup_{i=1}^{2} E_i(\mathrm{Tr}(G)))$ has $2|V(G)| + 2$ vertices and is defined as follows:

- $V(\mathrm{Tr}(G)) = V(G) \cup V(G') \cup \{\infty, \infty'\}$;
- $\forall u \in V(G)\colon u\infty, u'\infty' \in E_1(\mathrm{Tr}(G))$ and $u'\infty, u\infty' \in E_2(\mathrm{Tr}(G))$;
- $\forall uv \in E_i(G)\colon uv, u'v' \in E_i(\mathrm{Tr}(G))$ and $u'v, uv' \in E_{3-i}(\mathrm{Tr}(G))$.

Fig. 3 illustrates the construction of $\mathrm{Tr}(G)$. We can observe that, for every $u \in V(G) \cup \{\infty\}$, there is no edge between u and u'. Such pairs of vertices will be called *twin vertices*, and we denote by $\mathrm{twin}(u)$ the twin vertex of u. Remark that $\mathrm{twin}(\mathrm{twin}(u)) = u$.

By construction, the graph $\mathrm{Tr}(G)$ satisfies the following property:

$$\forall u \in \mathrm{Tr}(G) : N_1(u) = N_2(\mathrm{twin}(u)) \quad \text{and} \quad N_2(u) = N_1(\mathrm{twin}(u)).$$

This construction was already used to construct target graphs to bound the oriented chromatic number, i.e. the $(1, 0)$-mixed chromatic number (see e.g. [10,13]).

In the remainder, let $T_{12} = \mathrm{Tr}(T_5)$ and $T_{20} = \mathrm{Tr}(T_9)$ be the Tromp graphs obtained from T_5 and T_9, respectively. The vertex set of T_{12} (resp. T_{20}) is $V(T_5) \cup V(T_5') \cup \{\infty, \infty'\} = \{1, 2, 3, 4, 5\} \cup \{1', 2', 3', 4', 5'\} \cup \{\infty, \infty'\}$ (resp. $V(T_9) \cup V(T_9') \cup \{\infty, \infty'\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \cup \{1', 2', 3', 4', 5', 6', 7', 8', 9'\} \cup \{\infty, \infty'\}$). These two graphs have remarkable symmetry and some useful properties given below. It is not difficult to check that T_{12} and T_{20} are vertex-transitive.

Proposition 16. *The graph T_{12} has properties $\mathcal{P}_{1,5}$ and $\mathcal{P}_{2,2}$.*

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Table 1
Property $\mathcal{P}_{2,2}$ of T_{12}.

(α_1, α_2) \ (∞, x)	$(\infty, 1)$	$(\infty, 1')$
$(1, 1)$	$2; 5$	$1; 3$
$(1, 2)$	$3; 4$	$4; 5$
$(2, 1)$	$3'; 4'$	$4'; 5'$
$(2, 2)$	$2'; 5'$	$1'; 3'$

Table 2
Property $\mathcal{P}_{2,4}$ of T_{20}.

(α_1, α_2) \ (∞, x)	$(\infty, 1)$	$(\infty, 2)$	$(\infty, 3)$	$(\infty, 4)$	$(\infty, 5)$	$(\infty, 6)$	$(\infty, 7)$	$(\infty, 8)$	$(\infty, 9)$
$(1, 1)$	$2; 3; 4; 7$	$1; 3; 5; 8$	$1; 2; 6; 9$	$1; 5; 6; 7$	$2; 4; 6; 8$	$3; 4; 5; 9$	$1; 4; 8; 9$	$2; 5; 7; 9$	$3; 6; 7; 8$
$(1, 2)$	$5; 6; 8; 9$	$4; 6; 7; 9$	$4; 5; 7; 8$	$2; 3; 8; 9$	$1; 3; 7; 9$	$1; 2; 7; 8$	$2; 3; 5; 6$	$1; 3; 4; 6$	$1; 2; 4; 5$
$(2, 1)$	$5'; 6'; 8'; 9'$	$4'; 6'; 7'; 9'$	$4'; 5'; 7'; 8'$	$2'; 3'; 8'; 9'$	$1'; 3'; 7'; 9'$	$1'; 2'; 7'; 8'$	$2'; 3'; 5'; 6'$	$1'; 3'; 4'; 6'$	$1'; 2'; 4'; 5'$
$(2, 2)$	$2'; 3'; 4'; 7'$	$1'; 3'; 5'; 8'$	$1'; 2'; 6'; 9'$	$1'; 5'; 6'; 7'$	$2'; 4'; 6'; 8'$	$3'; 4'; 5'; 9'$	$1'; 4'; 8'; 9'$	$2'; 5'; 7'; 9'$	$3'; 6'; 7'; 8'$

(α_1, α_2) \ (∞, x)	$(\infty, 1')$	$(\infty, 2')$	$(\infty, 3')$	$(\infty, 4')$	$(\infty, 5')$	$(\infty, 6')$	$(\infty, 7')$	$(\infty, 8')$	$(\infty, 9')$
$(1, 1)$	$5; 6; 8; 9$	$4; 6; 7; 9$	$4; 5; 7; 8$	$2; 3; 8; 9$	$1; 3; 7; 9$	$1; 2; 7; 8$	$2; 3; 5; 6$	$1; 3; 4; 6$	$1; 2; 4; 5$
$(1, 2)$	$2; 3; 4; 7$	$1; 3; 5; 8$	$1; 2; 6; 9$	$1; 5; 6; 7$	$2; 4; 6; 8$	$3; 4; 5; 9$	$1; 4; 8; 9$	$2; 5; 7; 9$	$3; 6; 7; 8$
$(2, 1)$	$2'; 3'; 4'; 7'$	$1'; 3'; 5'; 8'$	$1'; 2'; 6'; 9'$	$1'; 5'; 6'; 7'$	$2'; 4'; 6'; 8'$	$3'; 4'; 5'; 9'$	$1'; 4'; 8'; 9'$	$2'; 5'; 7'; 9'$	$3'; 6'; 7'; 8'$
$(2, 2)$	$5'; 6'; 8'; 9'$	$4'; 6'; 7'; 9'$	$4'; 5'; 7'; 8'$	$2'; 3'; 8'; 9'$	$1'; 3'; 7'; 9'$	$1'; 2'; 7'; 8'$	$2'; 3'; 5'; 6'$	$1'; 3'; 4'; 6'$	$1'; 2'; 4'; 5'$

Table 3
Property $\mathcal{P}_{3,1}$ of T_{20}.

	$u, v, w \in V(T_9)$				$u, v \in V(T_9), w \in V(T'_9)$				$u, v \in V(T_9)$, $w \in \{\infty, \infty'\}$		$u \in V(T_9)$, $v \in V(T'_9)$, $w \in \{\infty, \infty'\}$	
α \ (u, v, w)	$(1, 2, 3)$	$(1, 2, 4)$	$(1, 2, 5)$	$(1, 2, 6)$	$(1, 2, 3')$	$(1, 2, 4')$	$(1, 2, 5')$	$(1, 2, 6')$	$(1, 2, \infty)$	$(1, 2, \infty')$	$(1, 2', \infty)$	$(1, 2', \infty')$
$(1, 1, 1)$	∞'	∞'	∞'	∞'	$6'$	3	3	$9'$	3	$6'$	4	$5'$
$(1, 1, 2)$	$6'$	3	3	$9'$	∞'	∞'	∞'	∞'	$6'$	3	$5'$	4
$(1, 2, 1)$	$5'$	7	5	4	4	$5'$	7	$5'$	4	$5'$	3	$6'$
$(1, 2, 2)$	4	$5'$	7	$5'$	$5'$	7	5	4	$5'$	4	$6'$	3
$(2, 1, 1)$	$4'$	5	$7'$	5	5	$7'$	$4'$	$4'$	5	$4'$	6	$3'$
$(2, 1, 2)$	5	$7'$	$4'$	$4'$	$4'$	5	$7'$	5	$4'$	5	$3'$	6
$(2, 2, 1)$	6	$3'$	$3'$	9	∞	∞	∞	∞	6	$3'$	5	$4'$
$(2, 2, 2)$	∞	∞	∞	∞	6	$3'$	$3'$	9	$3'$	6	$4'$	5

Proof. Property $\mathcal{P}_{1,5}$ is trivial.

For property $\mathcal{P}_{2,2}$, since T_{12} is vertex-transitive, it is enough to check that for every sequence of vertices $S = (\infty, x)$, $x \in V(T_5) \cup V(T'_5)$, and every 2-type-vector α, there exist at least two α-neighbors of S. However, there exists an obvious automorphism $h\colon V(T_{12}) \rightarrow V(T_{12})$ that fixes ∞ and ∞' (i.e. $h(\infty) = \infty$, $h(\infty') = \infty'$) with orbits $(1, 2, 3, 4, 5)$ and $(1', 2', 3', 4', 5')$. Therefore, we only need to consider the sequences $(\infty, 1)$ and $(\infty, 1')$. Table 1 gives, for each above-mentioned case, the α-neighbor of S. □

Proposition 17. *The graph T_{20} has properties $\mathcal{P}_{1,9}$, $\mathcal{P}_{2,4}$, and $\mathcal{P}_{3,1}$.*

Proof. Property $\mathcal{P}_{1,9}$ follows from $|N_i(v)| = 9$ for every $i \in \{1, 2\}$ and every $v \in V(T_{20})$.

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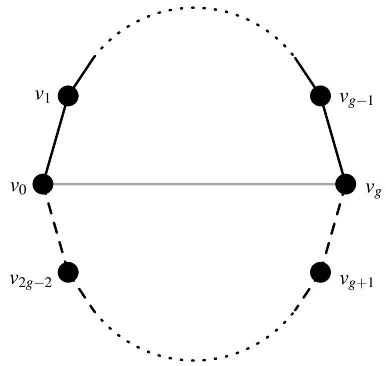

Fig. 4. An outerplanar graph with girth $g \geq 4$ and $(0, 2)$-mixed chromatic number 5.

Since T_{20} is vertex-transitive, it is enough to check that for every sequence of vertices $S = (\infty, x)$, $x \in V(T_9) \cup V(T_9')$, and every 2-type-vector α, there exist at least four α-neighbors of S. Table 2 gives, for each sequence $S = (\infty, x)$ and each type-vector α, the four α-neighbors of S.

To prove property $\mathcal{P}_{3,1}$, we have to check that for every triangle $[u, v, w]$ and every 3-type-vector α, there exists at least one α-neighbor of $S = (u, v, w)$. It is not difficult to verify that it suffices to consider the following four cases:

Case 1: $u, v, w \in V(T_9)$ (since T_9 is arc-transitive, we just consider the triangles [1, 2, 3], [1, 2, 4], [1, 2, 5], and [1, 2, 6]).
Case 2: $u, v \in V(T_9)$ and $w \in V(T_9')$ (by symmetry of T_9, we just consider the triangles $[1, 2, 3']$, $[1, 2, 4']$, $[1, 2, 5']$, and $[1, 2, 6']$).
Case 3: $u, v \in V(T_9)$ and $w \in \{\infty, \infty'\}$ (by symmetry of T_9, we just consider the triangles $[1, 2, \infty]$ and $[1, 2, \infty']$).
Case 4: $u \in V(T_9)$, $v \in V(T_9')$, and $w \in \{\infty, \infty'\}$ (by symmetry of T_9, we just consider the triangles $[1, 2', \infty]$ and $[1, 2', \infty']$).

Table 3 gives, for each above-mentioned case and each 3-type-vector α, the α-neighbor of S. □

Proposition 18. *For every vertex u of T_{20} (resp. T_{12}) and for every $(0, 2)$-mixed 2-path, there exists at most one vertex $v \in V(T_{20})$ (resp. $V(T_{12})$) such that u and v are not joined by such a $(0, 2)$-mixed 2-path. Moreover, $v \in \{u, \text{twin}(u)\}$.*

Proof. This follows directly from property $\mathcal{P}_{2,4}$ for T_{20} and property $\mathcal{P}_{2,2}$ for T_{12}. □

4. Outerplanar graphs

Let $\mathcal{O}_g$ be the class of $(0, 2)$-mixed outerplanar graphs with girth at least g. Outerplanar graphs form a strict subclass of partial 2-trees (also known as series–parallel graphs or K_4 minor-free graphs); therefore, Corollary 3 and Theorem 7 imply that $9 \leq \chi_{(0,2)}(\mathcal{O}_3) \leq 12$. We improve this result and give the exact $(0, 2)$-mixed chromatic number of outerplanar graphs for all girths:

Theorem 19. (1) $\chi_{(0,2)}(\mathcal{O}_3) = 9$*;*
(2) $\chi_{(0,2)}(\mathcal{O}_g) = 5$ *for* $g \geq 4$*.*

To prove Theorem 19, we need the following proposition:

Proposition 20. *Every outerplanar graph G with girth g and $\delta(G) \geq 2$ contains, for some $\ell \geq g - 1$, a $(\ell, 2)$-path whose end-vertices are adjacent.*

Proof. Consider a 2-connected component H of G which is linked to the rest of the graph by at most one cut-vertex. If H is a cycle, it is clear that the required $(\ell, 2)$-path exists. Otherwise, H has at least two faces. Then consider a face which contains at most two $^{\geq}3$-vertices. These two vertices are necessarily adjacent and therefore, since G has girth g, it contains the required $(\ell, 2)$-path. □

Proof of Theorem 19(1). We first prove that $\chi_{(0,2)}(\mathcal{O}_3) \leq 9$ by showing that every $(0, 2)$-mixed outerplanar graph admits a T_9-coloring. Let H be a minimal (with respect to the subgraph order) $(0, 2)$-mixed outerplanar having no T_9-coloring. We show that H contains neither a 1-vertex nor a $(k, 2)$-path, for any $k \geq 2$, in which the end-vertices are adjacent.

- Suppose that H contains a 1-vertex u. By minimality of H, the outerplanar graph $H \setminus \{u\}$ admits a T_9-coloring. This coloring extends to H by property $\mathcal{P}_{1,4}$.
- Suppose that H contains a $(2, 2)$-path $P = [v_0, v_1, v_2]$ in which the end-vertices v_0 and v_2 are adjacent. By minimality of H, the outerplanar graph $H \setminus \{v_1\}$ admits a T_9-coloring. By property $\mathcal{P}_{2,1}$, this coloring can be extended to a T_9-coloring of H. It is then clear that H contains no $(k, 2)$-paths for any $k \geq 3$.

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We thus get a contradiction by Proposition 20. The graph H does not exist.

To complete the proof, observe that Theorem 7 gives $\chi_{(0,2)}(\mathcal{O}_3) \geq 9$. □

Proof of Theorem 19(2). We prove that $\chi_{(0,2)}(\mathcal{O}_4) \leq 5$ by showing that every (0, 2)-mixed outerplanar graph with girth at least 4 admits a T_5-coloring. The proof is almost the same as the previous one and can be obtained by replacing $k \geq 2$ by $k \geq 3$, use the fact that T_5 has property $\mathcal{P}_{1,2}$, and use Proposition 12.

To prove that $\chi_{(0,2)}(\mathcal{O}_4) \geq 5$, we construct, for every $g \geq 4$, a (0, 2)-mixed outerplanar graph G_g with girth g and (0, 2)-mixed chromatic number 5. The graph G_g consists of a cycle $[v_0, v_1, \ldots, v_{2g-2}, v_0]$ with a chord $v_0 v_g$; see Fig. 4. Moreover, the adjacency types are given below:

- $t(v_i, v_{i+1}) = 1$ for every $0 \leq i \leq g - 1$ (solid lines in Fig. 4);
- $t(v_i, v_{i+1}) = 2$ for every $g \leq i \leq 2g - 1$ (dashed line in Fig. 4);
- $t(v_0, v_{2g-2}) = 2$ (dashed lines in Fig. 4);
- $t(v_0, v_g) = 1$ if g is even, $t(v_0, v_g) = 2$ otherwise (grayed line in Fig. 4);

If g is even (resp. odd), the graph induced by the edges of type 1 (resp 2) is an odd cycle, and the graph induced by the edges of type 2 (resp. 1) is an odd path. Suppose that there exists a (0, 2)-mixed graph T_4 on four vertices such that G_g admits a T_4-coloring. Then, T_4 must contain an odd cycle whose all edges are of type 1 (resp. 2): this is necessarily a triangle. Moreover, T_4 must contains an odd path linking $h(v_0)$ and $h(v_g)$ whose all edges are of type 2 (resp. 1). We can check that this is impossible. □

5. Partial 2-trees and partial 3-trees

In this section, we study the (0, 2)-mixed chromatic number of partial 2-trees (also known as series–parallel graphs or K_4 minor-free graphs) and partial 3-trees.

We first describe suitable target graphs to color partial k-trees. This will allow us to get bounds for the (0, 2)-mixed chromatic number of partial 2-trees and partial 3-trees.

Theorem 21. *Every (n, m)-mixed k-tree admits a T-coloring where T is an (n, m)-mixed graph having property $\mathcal{P}_{k,1}$.*

Proof. We show that every (n, m)-mixed partial k-tree admits a T-coloring for every (n, m)-mixed graph T having property $\mathcal{P}_{k,1}$. We proceed by induction on the number ℓ of vertices of an (n, m)-mixed k-tree G (observe that it suffices to consider k-trees, since partial k-trees are subgraphs of k-trees). If $\ell = k$ then G is an (n, m)-mixed clique. A graph having property $\mathcal{P}_{k,1}$ necessarily contains every (n, m)-mixed k-clique as subgraph. Therefore, any (n, m)-mixed k-clique admits a T-coloring. Suppose now that every (n, m)-mixed k-tree of order ℓ admits a T-coloring and let G be any (n, m)-mixed k-tree of order $\ell + 1$. The graph G necessarily contains a vertex v with degree k whose neighbors induce a k-clique and whose deletion leads to a k-tree G'. The induction hypothesis ensures that G' admits a T-coloring f and property $\mathcal{P}_{k,1}$ ensures that we can extend f to T-coloring of G. □

We then give the exact (0, 2)-mixed chromatic number of partial 2-trees for all girths ($\mathcal{T}_g^2$ denotes the class of partial 2-trees with girth at least g):

Theorem 22. (1) $\chi_{(0,2)}(\mathcal{T}_3^2) = 9$;
(2) $\chi_{(0,2)}(\mathcal{T}_g^2) = 8$ *for* $4 \leq g \leq 5$;
(3) $\chi_{(0,2)}(\mathcal{T}_g^2) = 5$ *for* $g \geq 6$.

Proof of Theorem 22(1). Since T_9 has property $\mathcal{P}_{2,1}$, Theorem 21 ensures that every partial 2-tree admits a T_9-coloring.

By Theorem 19(1), there exist outerplanar graphs with (0, 2)-mixed chromatic number 9. Since an outerplanar graph is a partial 2-tree, that completes the proof. □

Concerning the class of partial 2-trees with given girth, Ochem and Pinlou [9] proved the following structural lemma which is a generalization of a previous result proposed by Lih, Wang, and Zhu [7]. For a graph G with girth at least g and a vertex $v \in V(G)$, we denote:

$$S_g^G(v) = \Big\{ u \in V(G), d(u) \geq 3, \text{ such that there exist a unique path of 2-vertices linking} u \text{ and } v,$$
$$\text{or } u \text{ and } v \text{ are the end points of at least a } \left(\left\lceil \frac{g}{2} \right\rceil, 2\right) \text{-path} \Big\}.$$

Then, we denote $D_g^G(v) = |S_g^G(v)|$.

Lemma 23 (*[9]*). *Let G be a partial 2-tree with girth g such that $\delta(G) \geq 2$. Then, one of the following holds:*

(1) *there exist a* $(\lceil \frac{g}{2} \rceil + 1, 2)$*-path;*

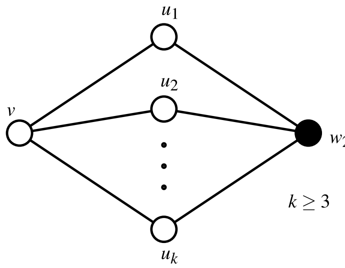

Fig. 5. A vertex v with $d(v) \geq 3$ and $D_4^H(v) = 1$.

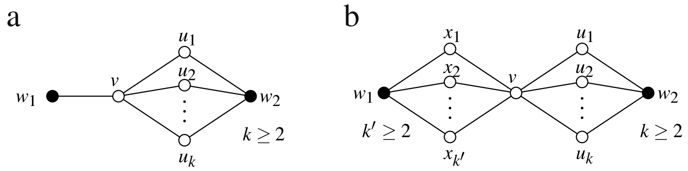

Fig. 6. A vertex v with $d(v) \geq 3$ and $D_4^H(v) = 2$.

(2) *there exist a* $\geq$3*-vertex* v *such that* $D_g^G(v) \leq 2$.

As a corollary, one can deduce the following:

Corollary 24 ([9]). *Every partial* 2*-tree with girth* $g \geq 3$ *contains either a* 1*-vertex or a* $\left(\left\lceil \frac{g}{2} \right\rceil, 2\right)$*-path.*

Proof of Theorem 22(2). We show that every partial 2-tree with girth at least 4 admits a T_8-coloring. Note that is sufficient to consider the case $g = 4$. Let H be a minimal (with respect to the subgraph order) (0, 2)-mixed partial 2-tree with girth 4 having no T_8-coloring. It is assumed that H is connected, as different components can be independently T_8-colored. We show that H contains neither a 1-vertex, nor a (3, 2)-path, nor a vertex v with $D_4^H(v) \leq 2$.

(1) Suppose that H contains a 1-vertex v. Due to the minimality of H, the graph $H \setminus \{v\}$ admits a T_8-coloring f. The coloring f can be extended to H by property $\mathcal{P}_{1,3}$.
(2) Suppose that H contains a (3, 2)-path $[u, v_1, v_2, w]$. Due to the minimality of H, the graph $H \setminus \{v_1, v_2\}$ admits a T_8-coloring f. Proposition 12 ensures that f can be extended to a T_8-coloring of H.
(3) Suppose that H contains a vertex v such that $d(v) \geq 3$ and $D_4^H(v) = 1$; see Fig. 5. Then $S_4^H(v) = \{w\}$ and since H does not contains a (3, 2)-path, there exist at least three (2, 2)-paths linking v and w. Due to the minimality of H, the graph $H \setminus \{v, u_1, \ldots, u_k\}$ admits a T_8-coloring f. We then set $f(v) \notin \{f(w), f(w) + 4 \pmod 8\}$. Proposition 14 ensures that f can be extended to H.
(4) Suppose that H contains a vertex v such that $d(v) \geq 3$ and $D_4^H(v) = 2$; see Fig. 6. Then $S_4^H(v) = \{w_1, w_2\}$.

Suppose first that, for some $i \in \{1, 2\}$, say $i = 1$, the arc vw_i exists; see Fig. 6(a). In this case, since H does not contain any (3, 2)-path, the arc vw_1 is the only path linking u and w_1. Then, since $d(v) \geq 3$, there are at least two (2, 2)-paths linking v and w_2. Due to the minimality of H, the graph $H \setminus \{v, u_1, \ldots, u_k\}$ admits a T_8-coloring f. Then, by property $\mathcal{P}_{1,3}$, w_1 allows three colors for v while w_2 forbids two colors for v, namely $f(w_1)$ and $f(w_1) + 4 \pmod 8$, by Proposition 14. The coloring f can be extended to H.

Suppose finally that there exist at least two (2, 2)-paths linking v and w_1 (resp. w_2); see Fig. 6(b). Due to the minimality of H, the graph $H \setminus \{v, u_1, \ldots, u_k, x_1, \ldots, x_{k'}\}$ admits a T_8-coloring f. Then, by Proposition 14, w_1 and w_2 each forbids two colors for v, namely $f(w_1), f(w_1) + 4 \pmod 8, f(w_2)$, and $f(w_2) + 4 \pmod 8$. We thus have four available colors for v and thus f can be extended to H.

We thus get a contradiction by Lemma 23.

To complete the proof, we construct a partial 2-tree with girth 5 and (0, 2)-mixed chromatic number 8.

The size of a universal graph is at least 5 by Theorem 19(2). We used a computer program to rule out target graphs on 5, 6, and 7 vertices, in that order. Let us construct the family G_t, $t \geq 0$, of series–parallel graphs of girth 5 as follows:

- G_0 consists in two non-adjacent vertices u and x.
- G_{t+1} consists in two non-adjacent vertices u and x joined by the eight possible (3, 2)-paths of the form $[u, v_i, w_i, z]$, $1 \leq i \leq 8$ (recall that each edge can have two different types), and eight copies of G_t such that the vertex u of G_{t+1} is identified with the vertex u of every copy of G_t, and every vertex w_i of G_{t+1} is identified with the vertex x of one copy of G_t.

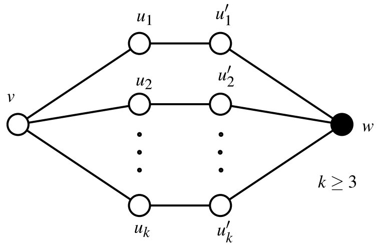

Fig. 7. A vertex v with $d(v) \geq 3$ and $D_6^H(v) = 1$.

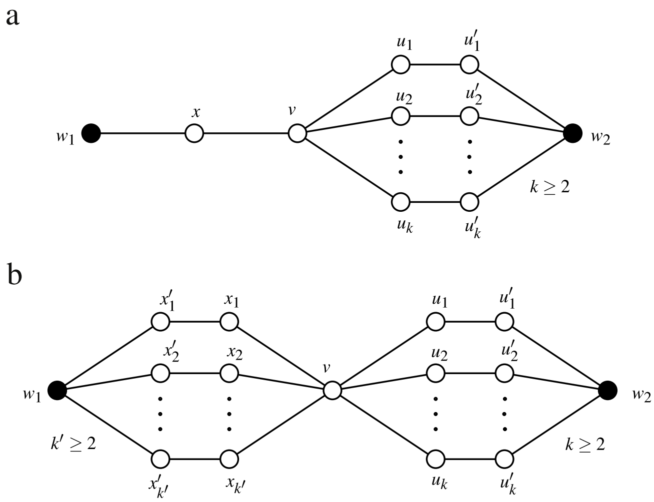

Fig. 8. A vertex v with $d(v) \geq 3$ and $D_6^H(v) = 2$.

Consider a $(0, 2)$-mixed complete graph T of order n. We define a family M_t, $t \geq 0$, of Boolean square matrices of order n as follows: for $1 \leq i, j \leq n$; $M_t[i, j]$ corresponds to the existence of a T-coloring of G_t such that its vertex u gets color i and its vertex x gets color j.

If there exists $t \geq 0$ and $1 \leq i \leq n$ such that $M_t[i, j]$ is false for every $1 \leq j \leq n$, then T is not universal. Indeed, since every potential target graph of order less than n has been previously ruled out, there exists a series–parallel graph W such that all n colors appear in every T-coloring of W. Now, if we identify each vertex of W with the vertex u of a copy of G_t, the vertex x of a copy of G_t attached to a vertex of W colored i cannot be colored. □

Proof of Theorem 22(3). We show that every partial 2-tree with girth at least 6 admits a T_5-coloring. Note that is sufficient to consider the case $g = 6$. Let H be a minimal (with respect to the subgraph order) $(0, 2)$-mixed partial 2-tree with girth 6 having no T_5-coloring. It is assumed that H is connected, as different components can be independently T_5-colored. We show that H contains neither a 1-vertex, nor a $(4, 2)$-path, nor a $^{\geq}3$-vertex v such that $D_6^H(v) \leq 2$.

(1) Suppose that H contains a 1-vertex v. Due to the minimality of H, the graph $H \setminus \{v\}$ admits a T_5-coloring f. The coloring f can be extended to H by property $\mathcal{P}_{1,2}$.

(2) Suppose that H contains a $(4, 2)$-path $[u, v_1, v_2, v_3, w]$. Due to the minimality of H, the graph $H \setminus \{v_1, v_2, v_3\}$ admits a T_5-coloring f. Proposition 13 ensures that f can be extended to a T_5-coloring of H.

(3) Suppose that H contains a vertex v such that $d(v) \geq 3$ and $D_6^H(v) = 1$; see Fig. 7. Then $S_6^H(v) = \{w\}$ and since H does not contains a $(4, 2)$-path, there exist at least three $(3, 2)$-paths linking v and w. Due to the minimality of H, the graph $H \setminus \{v, u_1, \ldots, u_k, u'_1, \ldots, u'_k\}$ admits a T_5-coloring f. We then set $f(v) \neq f(w)$. Proposition 12 ensures that f can be extended to H.

(4) Suppose that H contains a vertex v such that $d(v) \geq 3$ and $D_6^H(v) = 2$; see Fig. 8. Then $S_6^H(v) = \{w_1, w_2\}$.

Suppose first that, for some $i \in \{1, 2\}$, say $i = 1$, there is $(\ell, 2)$-path P linking v and w_i with $\ell < 3$; see Fig. 8(a). In this case, since H does not contain any $(4, 2)$-path, P is the only path linking u and w_1. Then, since $d(v) \geq 3$, there are at least two $(3, 2)$-paths linking v and w_2. Due to the minimality of H, the graph $H \setminus \{v, u_1, \ldots, u_k, u'_1, \ldots, u'_k\}$ admits a T_5-coloring f. Then, by property $\mathcal{P}_{1,2}$, w_1 allows at least two colors for v while w_2 forbids one color for v, namely $f(w_1)$, by Proposition 12. The coloring f can be extended to H.

Suppose finally that there exist at least two $(3, 2)$-paths linking v and w_1 (resp. w_2); see Fig. 8(b). Due to the minimality of H, the graph $H \setminus \{v, u_1, \ldots, u_k, u'_1, \ldots, u'_k, x_1, \ldots, x_{k'}, x'_1, \ldots, x'_{k'}\}$ admits a T_5-coloring f. Then, by Proposition 12,

w_1 and w_2 each forbids one color for v, namely $f(w_1)$ and $f(w_2)$. We thus have three available colors for v and thus f can be extended to H.

We thus get a contradiction by Lemma 23.

Note that Theorem 19 states that there exists, for every girth $g \geq 4$, an outerplanar graph G with girth g such that $\chi_{(0,2)}(G) \geq 5$. Since the class of outerplanar graphs is a strict subclass of the class of partial 2-trees, that completes the proof. □

The last result of this section concerns partial 3-trees. Corollary 9 shows that $\chi_{(0,2)}(\mathcal{T}^3) \geq 20$. We prove that this bound is tight:

Theorem 25. $\chi_{(0,2)}(\mathcal{T}^3) = 20$.

Proof. Since T_{20} has property $\mathscr{P}_{3,1}$, Theorem 21 ensures that every partial 3-tree admits a T_{20}-coloring. □

6. Planar graphs

Finally, we bound in this section the (0, 2)-mixed chromatic number of sparse graphs. The *average degree* of a graph G, denoted by $\text{ad}(G)$, is defined as $\text{ad}(G) = \frac{2|E(G)|}{|V(G)|}$. The *maximum average degree* of G, denoted by $\text{mad}(G)$, is then defined as the maximum of the average degrees taken over all subgraphs of G:

$$\text{mad}(G) = \max_{H \subseteq G}\{\text{ad}(H)\}.$$

Theorem 26. *Let G be a (0, 2)-mixed graph.*

(1) *If* $\text{mad}(G) < \frac{10}{3}$, *then* $\chi_{(0,2)}(G) \leq 20$.
(2) *If* $\text{mad}(G) < 3$, *then* $\chi_{(0,2)}(G) \leq 12$.
(3) *If* $\text{mad}(G) < \frac{8}{3}$, *then* $\chi_{(0,2)}(G) \leq 8$.
(4) *If* $\text{mad}(G) < \frac{7}{3}$ *and G does not contain a triangle whose edges are of the same type, then* $\chi_{(0,2)}(G) \leq 5$. *Moreover, this bound is tight.*

Bounds for the (0, 2)-mixed chromatic number of planar graphs can be deduced from the previous theorem since the maximum average degree and the girth of planar graphs are linked by the following well-known relation:

Claim 27 (*[3]*). *Let G be a planar graph with girth g. Then* $\text{mad}(G) < 2 + \frac{4}{g-2}$.

By means of the previous claim, we get the following result, where $\mathcal{P}_g$ denotes the class of (0, 2)-mixed planar graphs with girth at least g:

Theorem 28. (1) $\chi_{(0,2)}(\mathcal{P}_5) \leq 20$.
(2) $\chi_{(0,2)}(\mathcal{P}_6) \leq 12$.
(3) $\chi_{(0,2)}(\mathcal{P}_8) \leq 8$.
(4) $\chi_{(0,2)}(\mathcal{P}_{14}) = 5$.

Our proof technique is based on the well-known method of reducible configurations and discharging procedure. We consider a minimal counterexample H to the considered theorem. We prove that H does not contain a set S of configurations. Then we prove, using a discharging procedure, that every graph containing none of the configurations of S has maximum average degree greater than that required by the theorem, that contradicting that H is a counterexample.

6.1. Graphs with maximum average degree less than $\frac{10}{3}$

In this subsection, we prove that every (0, 2)-mixed graph with maximum average degree less than $\frac{10}{3}$ admits a T_{20}-coloring.

Let us define the partial order $\preceq$. Let $n_3(G)$ be the number of $^{\geq}3$-vertices in G. For any two graphs G_1 and G_2, we have $G_1 = G_2$ if and only if G_1 and G_2 are isomorphic; moreover, we have $G_1 \prec G_2$ if and only if at least one of the following conditions hold:

- G_1 is a proper subgraph of G_2;
- $n_3(G_1) < n_3(G_2)$.

Note that this partial order is well defined, since if G_1 is a proper subgraph of G_2, then $n_3(G_1) \leq n_3(G_2)$. So $\preceq$ is a partial linear extension of the subgraph poset.

Lemma 29. *A minimal counterexample (according to $\prec$) to Theorem 26(1) does not contain the following configurations:*

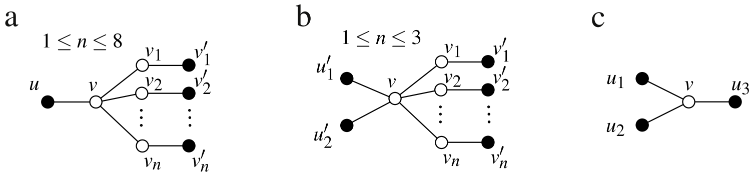

Fig. 9. Configurations of Lemma 29(2), (3), and (4).

(1) *a* 1*-vertex;*
(2) *the configuration depicted in* Fig. 9(a)*;*
(3) *the configuration depicted in* Fig. 9(b)*;*
(4) *the configuration depicted in* Fig. 9(c)*;*

Proof. Let H be a minimal (with respect to $\prec$) $(0, 2)$-mixed graph with $\text{mad}(H) < \frac{10}{3}$ which does not admit a T_{20}-coloring.It is assumed that H is connected, as different components can be independently T_{20}-colored.

(1) Suppose that H contains a 1-vertex v. Due to the minimality of H, the graph $H \setminus \{v\}$ admits a T_{20}-coloring f. property $\mathcal{P}_{1,9}$ ensures that f can be extended to a T_{20}-coloring of H.
(2) Suppose that H contains the configuration depicted in Fig. 9(a). Due to the minimality of H, the graph $H \setminus \{v, v_1, \ldots, v_n\}$ admits a T_{20}-coloring f. By property $\mathcal{P}_{1,9}$, u allows nine colors for v, while each $v_1', \ldots, v_n'$ forbids only one color for v by Proposition 18. Thus, f can be extended to a T_{20}-coloring of H since $n \leq 8$.
(3) Suppose that H contains the configuration depicted in Fig. 9(b). Due to the minimality of H, the graph $H \setminus \{v_1, \ldots, v_n, v_1', \ldots, v_n'\}$ admits a T_{20}-coloring f. By property $\mathcal{P}_{2,4}$, u_1' and u_2' allows four colors for v while each of $v_1', \ldots, v_n'$ forbids only one color for v by Proposition 18. Thus, f can be extended to a T_{20}-coloring of H since $n \leq 3$.
(4) Suppose that H contains the configuration depicted in Fig. 9(c). Since H contains neither a $^{\leq}1$-vertex, nor configuration of Fig. 9(b), u_1, u_2, and u_3 are $^{\geq}3$-vertices. Let H' be the graph obtained from $H \setminus \{v\}$ by adding, for every $1 \leq i < j \leq 3$, a 2-path joining u_i to u_j in such a way that its type is the same type as the path $[u_i, v, u_j]$ in H. We have $H' \prec H$ since $n_3(H') = n_3(H) - 1$, and one can check that $\text{mad}(H') < \frac{10}{3}$ [10]. Any T_{20}-coloring f of H' induces a coloring of $H \setminus \{v\}$ that can be extended to H by property $\mathcal{P}_{3,1}$. □

Proof of Theorem 26(1). Let H be a minimal (with respect to $\prec$) $(0, 2)$-mixed graph with $\text{mad}(H) < \frac{10}{3}$ which does not admit a T_{20}-coloring. We define the weight function ω by $\omega(v) = d(v)$ for every $v \in V(H)$ and a discharging rule (R) as follows:

(R) Each $^{\geq}4$-vertex gives $\frac{2}{3}$ to each adjacent 2-vertex.

Let v be a k-vertex of H. By Lemma 29(1) and (4), we have $k \geq 2$ and $k \neq 3$.

- If $k = 2$, then by Lemma 29(2), every 2-vertex of H has two neighbors of degree at least 3. Therefore, v receives $\frac{2}{3}$ from each neighbor and hence, $\omega^*(v) = 2 + 2 \times \frac{2}{3} = \frac{10}{3}$.
- If $4 \leq k \leq 5$, then by Lemma 29(3), v has at most $k - 3$ neighbors of degree 2. Therefore, v gives at most $\frac{2(k-3)}{3}$, and hence, $\omega^*(v) \geq k - \frac{2(k-3)}{3} \geq \frac{10}{3}$.
- If $6 \leq k \leq 9$, then by Lemma 29(2), v has at most $k - 2$ neighbors of degree 2. Therefore, v gives at most $\frac{2(k-2)}{3}$, and hence, $\omega^*(v) \geq k - \frac{2(k-2)}{3} \geq \frac{10}{3}$.
- If $k \geq 10$, then v gives at most $\frac{2k}{3}$ and hence $\omega^*(v) \geq k - \frac{2k}{3} \geq \frac{10}{3}$.

Then, for all $v \in V(G)$, $\omega^*(v) \geq \frac{10}{3}$ once the discharging is completed. Hence $\text{mad}(H) \geq \frac{10}{3}$, a contradiction. □

6.2. Graphs with maximum average degree less than 3

In this subsection, we prove that every $(0, 2)$-mixed graph with maximum average degree less than 3 admits a T_{12}-coloring.

Lemma 30. *A minimal counterexample to* Theorem 26(2) *does not contain the following configurations:*

(1) *a* 1*-vertex;*
(2) *the configuration depicted in* Fig. 10(a)*;*
(3) *the configuration depicted in* Fig. 10(b)*;*

Proof. Let H be a minimal (with respect to subgraph the order) $(0, 2)$-mixed graph with $\text{mad}(H) < 3$ which does not admit a T_{12}-coloring. It is assumed that H is connected, as different components can be independently T_{12}-colored.

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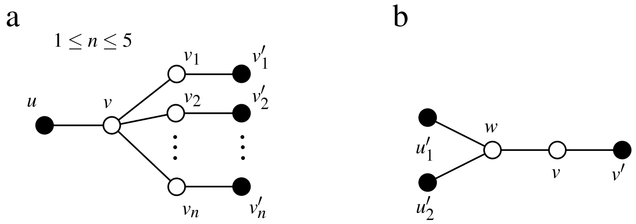

Fig. 10. Configurations of Lemma 30(2) and (3).

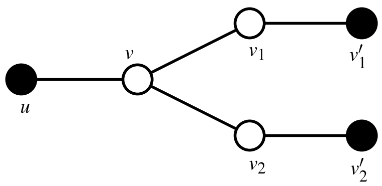

Fig. 11. Configuration of Lemma 31(3).

(1) Suppose that H contains a 1-vertex v. Due to the minimality of H, the graph $H \setminus \{v\}$ admits a T_{12}-coloring f. property $\mathcal{P}_{1,5}$ ensures that f can be extended to a T_{12}-coloring of H.
(2) Suppose that H contains the configuration depicted in Fig. 10(a). Due to the minimality of H, the graph $H \setminus \{v, v_1, \ldots, v_n\}$ admits a T_{12}-coloring f. By property $\mathcal{P}_{1,5}$, u allows five colors for v, while each $v'_1, \ldots, v'_n$ forbids only one color for v by Proposition 18. Thus, f can be extended to a T_{12}-coloring of H since $n \le 4$.
(3) Suppose that H contains the configuration depicted in Fig. 10(b). Due to the minimality of H, the graph $H \setminus \{v\}$ admits a T_{12}-coloring f. By property $\mathcal{P}_{2,2}$, u'_1 and u'_2 allows two colors for w while v' forbids only one color for w by Proposition 18. Thus, f can be extended to a T_{12}-coloring of H. □

Proof of Theorem 26(2). Let H be a minimal (with respect to subgraph the order) (0, 2)-mixed graph with $\text{mad}(H) < 3$ which does not admit a T_{12}-coloring. We define the weight function ω by $\omega(v) = d(v)$ for every $v \in V(H)$ and a discharging rule (R) as follows:

(R) Each $^{\geq}3$-vertex gives $\frac{1}{2}$ to each adjacent 2-vertex.

Let v be a k-vertex of H. By Lemma 30(1), we have $k \ge 2$.

- If $k = 2$, then by Lemma 30(2), every 2-vertex of H has two neighbors of degree at least 3. Therefore, v receives $\frac{1}{2}$ from each neighbor and hence, $\omega^*(v) = 2 + 2 \times \frac{1}{2} = 3$.
- If $k = 3$, then by Lemma 30(3), v is not adjacent to a 2-vertex. Hence, $\omega^*(v) = \omega(v) = 3$.
- If $4 \le k \le 5$, then by Lemma 30(2), v has at most $k - 2$ neighbors of degree 2. Therefore, v gives at most $\frac{k-2}{2}$ and hence $\omega^*(v) \ge k - \frac{k-2}{2} \ge 3$.
- If $k \ge 6$, then v gives at most $\frac{k}{2}$ and hence $\omega^*(v) \ge k - \frac{k}{2} \ge 3$.

Then, for all $v \in V(G)$, $\omega^*(v) \ge 3$ once the discharging is completed. Hence $\text{mad}(H) \ge 3$, a contradiction. □

6.3. Graphs with maximum average degree less than $\frac{8}{3}$

In this subsection, we prove that every (0, 2)-mixed graph with maximum average degree less than $\frac{8}{3}$ admits a T_8-coloring.

Lemma 31. *A minimal counterexample to Theorem 26(3) does not contain the following configurations:*

(1) *a 1-vertex;*
(2) *a (3, 2)-path $P_3 = [u, v_1, v_2, w]$;*
(3) *the configuration depicted in Fig. 11.*

Proof. Let H be a minimal (with respect to subgraph the order) (0, 2)-mixed graph with $\text{mad}(H) < \frac{8}{3}$ which does not admit a T_8-coloring. It is assumed that H is connected, as different components can be independently T_8-colored.

(1) Suppose that H contains a 1-vertex v. Due to the minimality of H, the graph $H \setminus \{v\}$ admits a T_8-coloring f. property $\mathcal{P}_{1,3}$ ensures that f can be extended to a T_8-coloring of H.
(2) Suppose that H contains a (3, 2)-path $P_3 = [u, v_1, v_2, w]$. Due to the minimality of H, the graph $H \setminus \{v_1, v_2\}$ admits a T_8-coloring f. Proposition 12 ensures that f can be extended to a T_8-coloring of H.

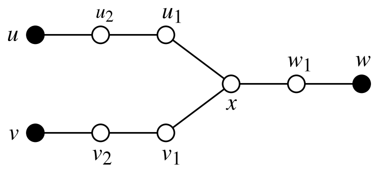

Fig. 12. Configuration of Lemma 32(3).

(3) Suppose that H contains the configuration depicted in Fig. 11. Due to the minimality of H, the graph $H\setminus\{v, v_1, v_2\}$ admits a T_8-coloring f. By property $\mathcal{P}_{1,3}$, the vertex u allows three colors for v, while each of v_1' and v_2' forbids only one color for v by Proposition 14. The coloring f can be extended to a T_8-coloring of H. □

Proof of Theorem 26(3). Let H be a minimal (with respect to the subgraph order) (0, 2)-mixed graph with $\text{mad}(H) < \frac{8}{3}$ which does not admit a T_8-coloring. We define the weight function ω by $\omega(v) = d(v)$ for every $v \in V(H)$ and a discharging rule (R) as follows:

(R) Each $^{\geq}3$-vertex gives $\frac{1}{3}$ to each adjacent 2-vertex.

Let v be a k-vertex of H. By Lemma 31(1), we have $k \geq 2$.

- If $k = 2$, then by Lemma 31(2) every 2-vertex of H has two neighbors of degree at least 3. Therefore, v receives $\frac{1}{3}$ from each neighbor and hence, $\omega^*(v) = 2 + 2 \times \frac{1}{3} = \frac{8}{3}$.
- If $k = 3$, then by Lemma 31(3), v has at most one neighbor of degree 2. Therefore, v gives at most $\frac{1}{3}$, and hence, $\omega^*(v) \geq 3 - \frac{1}{3} = \frac{8}{3}$.
- If $k \geq 4$, then v gives at most $\frac{k}{3}$ and hence $\omega^*(v) \geq k - \frac{k}{3} \geq \frac{8}{3}$.

Then, for all $v \in V(G)$, $\omega^*(v) \geq \frac{8}{3}$ once the discharging is completed. Hence $\text{mad}(H) \geq \frac{8}{3}$, a contradiction. □

6.4. Graphs with maximum average degree less than $\frac{7}{3}$

In this subsection, we prove that every (0, 2)-mixed graph with maximum average degree less than $\frac{7}{3}$ admits a T_5-coloring.

Lemma 32. *A minimal counterexample to Theorem 26(4) does not contain the following configurations:*

(1) *a 1-vertex;*
(2) *a (4, 2)-path $P_4 = [u, v_1, v_2, v_3, w]$;*
(3) *the configuration of Fig. 12.*

Proof. Let H be a minimal (with respect to the subgraph order) (0, 2)-mixed graph with $\text{mad}(H) < \frac{7}{3}$ which does not contains a triangle whose edges are of the same type and which does not admit a T_5-coloring. It is assumed that H is connected, as different components can be independently T_5-colored.

(1) Suppose that H contains a 1-vertex v. Due to the minimality of H, the graph $H \setminus \{v\}$ admits a T_5-coloring f. property $\mathcal{P}_{1,2}$ ensures that f can be extended to a T_5-coloring of H.
(2) Suppose that H contains a (4, 2)-path $P_4 = [u, v_1, v_2, v_3, w]$. Due to the minimality of H, the graph $H\setminus\{v_1, v_2, v_3\}$ admits a T_5-coloring f. By property $\mathcal{P}_{1,2}$, u allows one color for v_1 which is distinct from $f(w)$. Then, Proposition 12 ensures that f can be extended to a T_5-coloring of H.
(3) Suppose that H contains the configuration of Fig. 12.
Suppose first that u (resp. v) does not coincide with x. By the minimality of H, the graph $H \setminus \{x, u_1, u_2, v_1, v_2, w_1\}$ admits a T_5-coloring f. By Proposition 15, w forbids two colors for x, while each of u and v forbids only one color for x by Proposition 12. Thus, we have at least one color for x and f can be extended to a T_5-coloring of H.
Suppose now that u and v coincide with x (therefore, $u_1 = v_2$ and $u_2 = v_1$). By the minimality of H, the graph $H \setminus \{x, u_1, u_2, w_1\}$ admits a T_5-coloring f. By Proposition 15, w allows three colors for x; choose one color for x among these three available colors. It is then clear that, since H does not contains a triangle whose edges are of the same type, f can be extended to a T_5-coloring of H. □

Proof of Theorem 26(4). Let H be a minimal (with respect to subgraph order) (0, 2)-mixed graph with $\text{mad}(H) < \frac{7}{3}$ which does not contains a triangle whose edges are of the same type and which does not admit a T_5-coloring. We define the weight function ω by $\omega(v) = d(v)$ for every $v \in V(H)$. A *weak* 2-vertex is a 2-vertex adjacent to a 2-vertex, while a *strong* 2-vertex is a 2-vertex not adjacent to a 2-vertex. The discharging rules (R1) and (R2) are defined as follows:

(R1) Each $^{\geq}3$-vertex gives $\frac{1}{3}$ to each adjacent weak 2-vertex.
(R2) Each $^{\geq}3$-vertex gives $\frac{1}{6}$ to each adjacent strong 2-vertex.

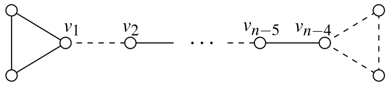

Fig. 13. A graph G on n vertices with mad(G) which tends to 2 as n tends to infinity, containing triangles whose edges are of the same type and with $\chi_{(0,2)}(G) = 6$.

Let v be a k-vertex of H. By Lemma 32(1), we have $k \geq 2$.

- If $k = 2$, then by Lemma 32(2), if v is weak, it is nevertheless adjacent to a $^{\geq}3$-vertex. Thus, it receives $\frac{1}{3}$ by (R1). If v is strong, then it receives $\frac{1}{6}$ from each of its two neighbors of degree at least 3 by (R2). Hence, $\omega^*(v) = 2+\min\{\frac{1}{3}; 2\times\frac{1}{6}\} = \frac{7}{3}$.
- If $k = 3$, then by Lemma 32(3), v gives at most $\max\{2 \times \frac{1}{3}; 2 \times \frac{1}{6} + \frac{1}{3}; 3 \times \frac{1}{6}\} = \frac{2}{3}$. Hence, $\omega^*(v) \geq 3 - \frac{2}{3} = \frac{7}{3}$.
- If $k \geq 4$, then v gives at most $\frac{k}{3}$ and hence $\omega^*(v) \geq k - \frac{k}{3} > \frac{7}{3}$.

Then, for all $v \in V(G)$, $\omega^*(v) \geq \frac{7}{3}$ once the discharging is completed. Hence mad$(H) \geq \frac{7}{3}$, a contradiction.

To prove that this bound is tight, consider the outerplanar graph G depicted in Fig. 4. This graph has mad$(G) < \frac{7}{3}$ and $\chi_{(0,2)}(G) = 5$. □

One can note that the restriction to graphs containing no triangles whose edges are of the same type cannot be dropped. Indeed, for every fixed odd $n \geq 7$, we provide the $(0, 2)$-mixed graph G on n vertices depicted in Fig. 13 (the path $v_1, v_2, \ldots, v_{n-5}, v_{n-4}$ is an alternated path of edges of both types): this graph contains triangles whose edges are of the same type, has mad(G) which tends to 2 as n tends to infinity, and has $\chi_{(0,2)}(G) = 6$.

Suppose that $\chi_{(0,2)}(G) \leq 5$ and let H be a complete $(0, 2)$-mixed graph on five vertices ($V(H) = \{1, 2, 3, 4, 5\}$) such that there exists an H-coloring f of G. Since G contains a triangle of type 1 (solid edges in Fig. 13) and a triangle of type 2 (dashed edges in Fig. 13), H must contain a triangle of type 1 and a triangle type 2. W.l.o.g., the edges 12, 23, and 13 (resp. 34, 45, and 35) of H are of type 1 (resp. type 2). Then it is clear that $f(v_1) \in \{1, 2, 3\}$; this implies that, for every even i (resp. odd i), $f(v_i) \in \{4, 5\}$ (resp. $f(v_i) \in \{1, 2\}$). Thus, $f(v_{n-4}) \in \{1, 2\}$ since n is odd; this is impossible since the vertices 1 and 2 of H do not belong to a triangle of type 2.

7. Concluding remarks

In this paper, we investigated the chromatic number of 2-edge-colored graphs, which are a special case of (n, m)-mixed graphs with $n = 0$ and $m = 2$. In addition, several results are known for the chromatic number of $(0, 1)$-mixed graphs (i.e. simple graphs) and $(1, 0)$-mixed graphs (i.e. oriented graphs). For the other values of n and m, we only obtained estimates (see Section 2).

A natural question is whether these general bounds are tight for some specific values of n and m. In particular, the cases $(n, m) = (1, 1)$ and $(n, m) = (2, 0)$ seem to be challenging cases to consider. One possible way to get bounds would might be to construct target graphs with property $\mathcal{P}_{k,1}$, for some k. We recently succeeded to construct a $(1, 1)$-mixed graph of order 21 and a $(2, 0)$-mixed graph of order 28 having both property $\mathcal{P}_{2,1}$. By means of Theorem 21, this gives that every $(1, 1)$-mixed 2-tree has chromatic number at most 21 and that every $(2, 0)$-mixed 2-tree has chromatic number at most 28.

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