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ORIGINAL PAPER

Oriented Coloring of Triangle-Free Planar Graphs and 2-Outerplanar Graphs

Pascal Ochem · Alexandre Pinlou

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Abstract A graph is planar if it can be embedded on the plane without edge-crossings. A graph is 2-outerplanar if it has a planar embedding such that the subgraph obtained by removing the vertices of the external face is outerplanar (i.e. with all its vertices on the external face). An oriented k-coloring of an oriented graph G is a homomorphism from G to an oriented graph H of order k. We prove that every oriented triangle-free planar graph has an oriented chromatic number at most 40, that improves the previous known bound of 47 [Borodin, O. V. and Ivanova, A. O., *An oriented colouring of planar graphs with girth at least 4*, Sib. Electron. Math. Reports, vol. 2, 239–249, 2005]. We also prove that every oriented 2-outerplanar graph has an oriented chromatic number at most 40, that improves the previous known bound of 67 [Esperet, L. and Ochem, P. *Oriented colouring of 2-outerplanar graphs*, Inform. Process. Lett., vol. 101(5), 215–219, 2007].

Keywords Oriented coloring · Planar graph · Girth · 2-Outerplanargraph · Discharging procedure

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P. Ochem · A. Pinlou
Université Montpellier, 2, CNRS - LIRMM, 161 rue Ada, 34095 Montpellier Cedex 5, France
e-mail: Pascal.Ochem@lirmm.fr

A. Pinlou (✉)
Département de Mathématiques et Informatique Appliqués, Université Paul-Valéry, Montpellier 3, Route de Mende, 34199 Montpellier Cedex 5, France
e-mail: alexandre.pinlou@lirmm.fr

1 Introduction

Oriented graphs are directed graphs with neither loops nor opposite arcs. For an oriented graph G, we denote by $V(G)$ its set of vertices and by $A(G)$ its set of arcs. For two adjacent vertices u and v, we denote by $\overrightarrow{uv}$ the arc from u to v or simply $u \sim v$ whenever its orientation is not relevant (therefore, $u \sim v = \overrightarrow{uv}$ or $u \sim v = \overrightarrow{vu}$). The number of vertices of G is the *order* of G.

Let G and H be two oriented graphs. A *homomorphism* from G to H is a mapping $\varphi : V(G) \to V(H)$ that preserves the arcs: $\overrightarrow{\varphi(x)\varphi(y)} \in A(H)$ whenever $\overrightarrow{xy} \in A(G)$.

An oriented k-coloring of G can be defined as a homomorphism from G to H, where H is an oriented graph of order k. In other words, that corresponds to a partition of the vertices of G into k stable sets $S_1, S_2, \ldots, S_k$ such that all the arcs between any pair of stable sets S_i and S_j have the same direction (either from S_i to S_j, or from S_j to S_i). The existence of such a homomorphism from G to H is denoted by $G \to H$. The vertices of H are called *colors*, and we say that G is H-colorable. The *oriented chromatic number* of an oriented graph G, denoted by $\chi_o(G)$, is defined as the smallest order of an oriented graph H such that $G \to H$. For a graph family $\mathcal{F}$, the oriented chromatic number $\chi_o(\mathcal{F})$ of $\mathcal{F}$ is defined as the maximum of the oriented chromatic numbers taken over all members of $\mathcal{F}$ (i.e. $\chi_o(\mathcal{F}) = k$ iff every $G \in \mathcal{F}$ has $\chi_o(G) \le k$, and there exists $H \in \mathcal{F}$ such that $\chi_o(H) = k$).

Links between colorings and homomorphisms are presented in more details in the monograph [8] by Hell and Nešetřil.

A graph is *planar* if it can be embedded on the plane without edge-crossings. The *girth* of a graph is the length of a shortest cycle.

The notion of oriented coloring introduced by Courcelle [6] has been studied by several authors in the last decade and the problem of bounding the oriented chromatic number has been investigated for various graph classes: outerplanar graphs (with given minimum girth) [15,17], 2-outerplanar graphs [7], planar graphs (with given minimum girth) [2–5,12,14,16], graphs with bounded maximum average degree [4,5], graphs with bounded degree [9], graphs with bounded treewidth [13,17,18], and graph subdivisions [20].

For planar graphs in particular, bounding their oriented chromatic number is a very hard question. In 1994, Courcelle [6] proved that every planar graph admits an oriented 3^{63}-coloring by means of monadic second order logic. This bound was very quickly improved by Raspaud and Sopena [16] who have proved that every planar graph admits an oriented 80-coloring using Borodin's theorem stating that every planar graph is acyclically 5-colorable [1]. Since then, no new improvement has been found. However, everyone agrees to say that this bound is far from the optimal one. Nevertheless, nobody has dared to make any conjecture. In the mean time, Sopena [19] proved that there exist planar graphs with oriented chromatic number 16 in 2002; five years later, Marshall [10] improved this lower bound to 17 and very recently to 18 [11]. Thus, the oriented chromatic number of planar graphs lies between 18 and 80 and any improvement of these bounds seems, at least up to now, to be particularly challenging.

Therefore, several authors decided to bound the oriented chromatic number of sparse planar graphs, say planar graphs with given minimum girth.

Theorem 1 gives the current best known lower and upper bounds on oriented chromatic number of planar graphs due to Borodin, Ivanova, Kostochka, Marshall, Nešetřil, Ochem, Pinlou, Raspaud, and Sopena [2–5,11–14,16]:

Theorem 1 (Borodin, Ivanova, Kostochka, Marshall, Nešětřil, Ochem, Pinlou, Raspaud, Sopena [2–5,11–14,16])
Let $\mathcal{P}_g$ be the family of all planar graphs with girth at least g.

1. $\chi_o(\mathcal{P}_{12}) = 5$ [4].
2. $\chi_o(\mathcal{P}_{11}) \leq 6$ [13].
3. $\chi_o(\mathcal{P}_7) \leq 7$ [2].
4. $\chi_o(\mathcal{P}_6) \leq 11$ [5].
5. $6 \leq \chi_o(\mathcal{P}_5) \leq 16$ [14].
6. $11 \leq \chi_o(\mathcal{P}_4) \leq 47$ [3,12].
7. $18 \leq \chi_o(\mathcal{P}_3) \leq 80$ [11,16].

One way to upper bound the oriented chromatic number of a graph family $\mathcal{F}$ is to find a universal target graph H such that, for every graph $G \in \mathcal{F}$, we have $G \to H$. Such a result can be obtained if the target graph H has "interesting" structural properties that can be used to prove the existence of the homomorphism; thus an important part of the task is to construct such a target graph.

In this paper, we first describe the construction of the graph T_{40} in Section 2, an oriented graph on 40 vertices which has very useful properties for oriented coloring of planar graphs. These structural properties of T_{40} allow us to prove that every oriented triangle-free planar graph admits a homomorphism to T_{40}; this gives the following theorem, which improves the previous known upper bound of 47 due to Borodin and Ivanova [3] (see Theorem 1(6)).

Theorem 2 Let $\mathcal{P}_4$ be the family of triangle-free planar graphs. Then $\chi_o(\mathcal{P}_4) \leq 40$.

A graph is *2-outerplanar* if it has a planar embedding such that the subgraph obtained by removing the vertices of the external face is outerplanar (i.e. with all its vertices on the external face).

In 2007, Esperet and Ochem [7] studied the structural properties of 2-outerplanar graphs. By means of these properties, they proved the following:

Theorem 3 **(Esperet, Ochem [7])** Let G be a 2-outerplanar graph. Then $\chi_o(G) \leq 67$.

Concerning the lower bound, we know that there exists a 2-outerplanar graph with oriented chromatic number 15. This graph is obtained as follows: let G_1 be the graph with one vertex and no arcs; G_i is obtained from two copies of G_{i-1} plus a new vertex v by adding all the arcs from the vertices of the first copy towards v and all the arcs from v towards the vertices of the second copy. The graph G_4 is a 2-outerplanar graph and has oriented chromatic number 15.

The oriented graph T_{40}, that we have constructed to bound the oriented chromatic number of triangle-free planar graphs, has also suitable properties to bound the oriented chromatic number of 2-outerplanar graphs, leading us to an improvement of Theorem 3:

Theorem 4 Let G be a 2-outerplanar graph. Then $\chi_o(G) \leq 40$.

In the remainder of this paper, we use the following notions. For a vertex v of a graph G, we denote by $d_G^-(v)$ its *indegree*, by $d_G^+(v)$ its *outdegree*, and by $d_G(v)$ its degree (subscripts are omitted when the considered graph is clearly identified from the context). We denote by $N_G^+(v)$ the set of outgoing neighbors of v, by $N_G^-(v)$ the set of incoming neighbors of v and by $N_G(v) = N_G^+(v) \cup N_G^-(v)$ the set of neighbors of v. A vertex of degree k (resp. at least k, at most k) is called a *k-vertex* (resp. *$^{\geq}k$-vertex*, *$^{\leq}k$-vertex*). If a vertex u is adjacent to a k-vertex (resp. $^{\geq}k$-vertex, $^{\leq}k$-vertex) v, then v is a *k-neighbor* (resp. *$^{\geq}k$-neighbor*, *$^{\leq}k$-neighbor*) of u. A path of length k (i.e. formed by k edges) is called a *k-path*. The length of a face f of a graph G is denoted by $d_G(f)$. If $d_G(f) = k$ (resp. $d_G(f) \leq k$, $d_G(f) \geq k$), then f is called a *k-face* (resp. *$^{\leq}k$-face*, *$^{\geq}k$-face*). If two graphs G and H are isomorphic, we denote this by $G \cong H$. Given a planar graph G with its embedding in the plane and a vertex v of G, we say that a sequence $(u_1, u_2, \ldots, u_k)$ of neighbors of v are *consecutive* if $u_1, u_2, \ldots, u_k$ appear around v consecutively (clockwise or counterclockwise) in G.

The paper is organised as follows. The next section is devoted to the target graph T_{40} and some of its properties. We prove Theorem 2 in Sect. 3 and Theorem 4 in Sect. 4.

2 The Tromp graph T_{40}

In this section, we describe the construction of the target graph T_{40} used to prove Theorems 2 and 4 and give some useful properties.

Tromp (Unpublished manuscript) proposed the following construction. Let G be an oriented graph and G' be an isomorphic copy of G. The Tromp graph $Tr(G)$ has $2|V(G)| + 2$ vertices and is defined as follows:

- $V(Tr(G)) = V(G) \cup V(G') \cup \{\infty, \infty'\}$
- $\forall u \in V(G) : \overrightarrow{u\infty}, \overrightarrow{\infty u'}, \overrightarrow{u'\infty'}, \overrightarrow{\infty' u} \in A(Tr(G))$
- $\forall u, v \in V(G), \overrightarrow{uv} \in A(G) : \overrightarrow{uv}, \overrightarrow{u'v'}, \overrightarrow{vu'}, \overrightarrow{v'u} \in A(Tr(G))$

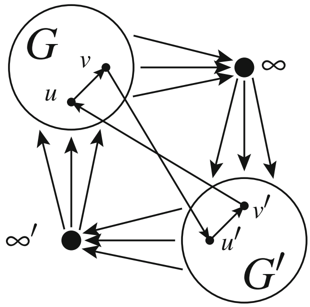
Figure 1 illustrates the construction of $Tr(G)$. We can observe that, for every $u \in V(G) \cup \{\infty\}$, there is no arc between u and u'. Such pairs of vertices will be called *twin vertices*, and we denote by $t(u)$ the twin vertex of u. Remark that $t(t(u)) = u$. This notion can be extended to sets in a standard way: for a given $W \subseteq V(G)$, $W = \{v_1, v_2, \ldots, v_k\}$, then $t(W) = \{t(v_1), t(v_2), \ldots, t(v_k)\}$.

By construction, the graph $Tr(G)$ satisfies the following property:

$$\forall u \in Tr(G) : N^+(u) = N^-(t(u)) \quad \text{and} \quad N^-(u) = N^+(t(u))$$

In the remainder, we focus on the specific graph family obtained by applying the Tromp's construction to Paley tournaments. For a prime power $p \equiv 3 \pmod 4$, the *Paley tournament* QR_p is defined as the oriented graph whose vertices are the integers modulo p and such that $\overrightarrow{uv}$ is an arc if and only if $v - u$ is a non-zero quadratic residue of p. For instance, the Paley tournament QR_{19} has vertex set

Fig. 1 The Tromp graph $Tr(G)$



$V(QR_{19}) = \{0, 1, \ldots, 18\}$ and $\overrightarrow{uv} \in A(QR_{19})$ whenever $v - u \equiv r \pmod{19}$ for $r \in \{1, 4, 5, 6, 7, 9, 11, 16, 17\}$. Note that the upper bounds of Theorems 1(3), 1(4), and 1(6) have been obtained by proving that all the graphs of the considered families admit a homomorphism to the Paley tournaments QR_7, QR_{11}, and QR_{47} respectively. Moreover, the upper bound of Theorem 1(5) has been obtained by proving that all the graphs of the considered family admit a homomorphism to the Tromp graph $Tr(QR_7)$.

Let $T_{40} = Tr(QR_{19})$ be the Tromp graph on 40 vertices obtained from QR_{19}. In the remainder of this paper, the vertex set of T_{40} is $V(T_{40}) = \{0, 1, \ldots, 18, \infty, 0', 1', \ldots, 18', \infty'\}$ where $\{0, 1, \ldots, 18\}$ is the vertex set of the first copy of QR_{19} and $\{0', 1', \ldots, 18'\}$ is the vertex set of the second copy of QR_{19}; thus, for every $u \in \{0, 1, \ldots, 18, \infty\}$, we have $t(u) = u'$. In addition, for every $u \in V(T_{40})$, we have by construction $|N^+_{T_{40}}(u)| = |N^-_{T_{40}}(u)| = 19$. The graph T_{40} has remarkable symmetry and some useful properties given below.

Proposition 1 (Marshall [10]) *For any QR_p, the graph $Tr(QR_p)$ is such that:*

$$\forall u \in V(Tr(QR_p)) : N^+(u) \cong QR_p \text{ and } N^-(u) \cong QR_p$$

Proposition 2 (Marshall [10]) *For any QR_p, if $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ span triangles t_1 and t_2 respectively in $Tr(QR_p)$ and the map ψ taking a_i to b_i ($1 \le i \le 3$) is an isomorphism $t_1 \to t_2$, then ψ can be extended to an automorphism of $Tr(QR_p)$.*

It is then clear that $Tr(QR_p)$ is vertex-transitive and arc-transitive.

For an oriented graph G and a vertex v, *pushing* v means reversing the orientation of every arc incident with v.

Proposition 3 (Push Property) *Let G be an oriented graph such that $G \to Tr(QR_p)$. Then, for any vertex v of G, the graph G' obtained from G by pushing v admits a homomorphism to $Tr(QR_p)$.*

Proof Let φ be a $Tr(QR_p)$-coloring of G. For every $w \in V(Tr(QR_p))$, we have $N^+_{Tr(QR_p)}(w) = N^-_{Tr(QR_p)}(t(w))$ and $N^-_{Tr(QR_p)}(w) = N^+_{Tr(QR_p)}(t(w))$. Therefore, the mapping $\varphi' : V(G') \to V(Tr(QR_p))$ defined by $\varphi'(u) = \varphi(u)$ for all $u \in V(G') \setminus \{v\}$ and $\varphi'(v) = t(\varphi(v))$ is clearly a $Tr(QR_p)$-coloring of G'.

An *orientation n-vector* is a sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \{0, 1\}^n$ of n elements. Let $S = (v_1, v_2, \ldots, v_n)$ be a sequence of n (not necessarily distinct) vertices of T_{40}. The vertex u is said to be an *α-successor of S* if for any i, $1 \le i \le n$, we have $\overrightarrow{uv_i} \in A(T_{40})$ whenever $\alpha_i = 1$ and $\overrightarrow{v_i u} \in A(T_{40})$ otherwise. For instance, the vertex $3'$ of T_{40} is a $(1, 1, 0, 1, 1, 0)$-successor of $(1, 2, 6', 1, \infty', 2')$ since the arcs $\overrightarrow{3'1}$, $\overrightarrow{3'2}$, $\overrightarrow{6'3'}$, $\overrightarrow{3'\infty'}$, and $\overrightarrow{2'3'}$ belong to $A(T_{40})$.

If, for a sequence $S = (v_1, v_2, \ldots, v_n)$ of n vertices of T_{40} and an orientation n-vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, there exist $i \neq j$ such that $v_i = v_j$ and $\alpha_i \neq \alpha_j$, then there does not exist any α-successor of S; indeed, T_{40} does not contain opposite arcs. In addition, if there exist $i \neq j$ such that $v_i = t(v_j)$ and $\alpha_i = \alpha_j$, then there does not exist any α-successor of S; indeed, for any pair of vertices x and y of T_{40} with $x = t(y)$, we have $N^+_{T_{40}}(x) \cap N^+_{T_{40}}(y) = \emptyset$ and $N^-_{T_{40}}(x) \cap N^-_{T_{40}}(y) = \emptyset$. A sequence $S = (v_1, v_2, \ldots, v_n)$ of n vertices of T_{40} is said to be *compatible* with an orientation n-vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ if and only if for any $i \neq j$, we have $\alpha_i \neq \alpha_j$ whenever $v_i = t(v_j)$, and $\alpha_i = \alpha_j$ whenever $v_i = v_j$. Note that if the n vertices of S induce an n-clique subgraph of T_{40} (i.e. $v_1, v_2, \ldots, v_n$ are pairwise distinct and induce a complete graph), then S is compatible with any orientation n-vector since a vertex u and its twin $t(u)$ cannot belong together to the same clique.

In the remainder, we say that T_{40} has *Property $P_{n,k}$* if, for every sequence S of n vertices of T_{40} that form an n-clique and any orientation n-vector α, there exist k α-successors of S.

Proposition 4 *If, for a fixed $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, every n-clique S of T_{40} admits k α-successors, then there exist k α'-successors of S for every $\alpha' = \{0, 1\}^n$, that is T_{40} has property $P_{n,k}$.*

Proof Assume that every n-clique admits k α-successors. Let $S = (u_1, u_2, \ldots, u_n)$ be a n-clique of T_{40} and $\alpha' = (\alpha'_1, \alpha'_2, \ldots, \alpha'_n)$ be an orientation n-vector. Then let $S' = (v_1, v_2, \ldots, v_n)$ defined such that $v_i = u_i$ if $\alpha'_i = \alpha_i$ and $v_i = t(u_i)$ otherwise. Due to the structure of T_{40} (i.e. if $x \sim y$ belongs to $A(T_{40})$, then $t(x) \sim y$, $x \sim t(y)$ and $t(x) \sim t(y)$ belongs to $A(T_{40})$), S' is an n-clique of T_{40}. By hypothesis, S' admits $k\alpha$-successors $w_1, w_2, \ldots, w_k$. Since $\overrightarrow{yt(x)} \in A(T_{40})$ if $\overrightarrow{xy} \in A(T_{40})$, we have w_i's are $k\alpha'$-successor of S.

Proposition 5 *The graph T_{40} has Properties $P_{1,19}$, $P_{2,9}$, $P_{3,4}$, and $P_{4,1}$.*

Proof By Proposition 1, we have $|N^+(u)| = |N^-(u)| = 19$ for every vertex u of T_{40}; therefore T_{40} has Property $P_{1,19}$.

It is obvious that QR_{19} has properties $P_{1,9}$ (for every vertex u of QR_{19}, we have $|N^+(u)| = |N^-(u)| = 9$). Borodin et al. [5] proved that QR_{19} has properties $P_{2,4}$ and $P_{3,1}$. We will show in the remainder of this proof that if QR_{19} has property $P_{n-1,k}$, then T_{40} has property $P_{n,k}$; that will complete the proof.

Suppose that QR_{19} has property $P_{n-1,k}$ and let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a given orientation n-vector. Let $S = (u_1, u_2, \ldots, u_{n-1}, w)$ be a induced n-clique of T_{40}. If $\alpha_n = 0$, we define $S' = (v_1, v_2, \ldots, v_{n-1}, w)$ such that $v_i = u_i$ if $\overrightarrow{u_i w}$ and $v_i = t(u_i)$ if $\overrightarrow{wu_i}$. Hence, S' is an n-clique of T_{40} such that $\bigcup_i v_i \subseteq N^-(w)$. By Proposition 1, $N^-(w) = K_{19} \cong QR_{19}$, and therefore the $(n-1)$-clique $S'' = (v_1, v_2, \ldots, v_{n-1})$

belongs to K_{19}. Then by Property $P_{n-1,k}$ of QR_{19}, there exist k $(\alpha'_1, \alpha'_2, \ldots, \alpha'_{n-1})$-successors $x_1, x_2, \ldots, x_k$ of S'' in K_{19}, with $\alpha'_i = \alpha_i$ (resp. $\alpha'_i = 1 - \alpha_i$) if $u_i = v_i$ (resp. $u_i = t(v_i)$). The x_i's are clearly in-neighbors of w and hence, they are $(\alpha'_1, \alpha'_2, \ldots, \alpha'_{n-1}, \alpha_n)$-successors S', and thus there exist k α-successors of S. Proportion 4 allows us to conclude.

The case $\alpha_n = 1$ would be treated similarly: we would have chosen $S' = (v_1, v_2, \ldots, v_{n-1}, w)$ is such a way that $\bigcup_i v_i \subseteq N^+(w)$.

3 Proof of Theorem 2

In this section, we prove Theorem 2, that is, every oriented triangle-free planar graph G admits a homomorphism to T_{40}.

Recall that Borodin and Ivanova [3] proved that every oriented triangle-free planar graph G admits a homomorphism to QR_{47}. This proof was only published in Russian. Our proof is highly inspired by this paper. Indeed, our list of forbidden configurations is designed to fit with Borodin and Ivanova's discharging procedure [3] up to a slight modification in Rule (R3).

Let us define the partial order $\preceq$. Let $n_3(G)$ be the number of $^{\geq}3$-vertices in G. For any two graphs G_1 and G_2, we have $G_1 \prec G_2$ if and only if at least one of the following conditions hold:

- $|V(G_1)| < |V(G_2)|$ and $n_3(G_1) \leq n_3(G_2)$.
- $n_3(G_1) < n_3(G_2)$.

Note that the partial order $\preceq$ is well-defined and is a partial linear extension of the induced subgraph poset.

Let H be a hypothetical minimal counterexample to Theorem 2 according to $\prec$. We first prove that H does not contain a set of ten configurations listed in Lemma 1. Then, using a discharging procedure, we show that each oriented triangle-free planar graph contains at least one of these configurations of Lemma 1, contradicting the fact that H is a triangle-free planar graph.

3.1 Structural Properties of H

In the following, H is a triangle-free planar graph given with its embedding in the plane. A *weak 7-vertex u* in H is a 7-vertex adjacent to four 2-vertices $v_1, \ldots, v_4$ and three $^{\geq}3$-vertices w_1, w_2, w_3 in such a way that the sequence of neighbors of v appear as $v_1, w_1, v_2, w_2, v_3, w_3, v_4$ (clockwise or counterclockwise).

Lemma 1 *The graph H does not contain the following configurations:*

- *(C1)* *a $^{\leq}1$-vertex;*
- *(C2)* *a k-vertex adjacent to k 2-vertices for $2 \leq k \leq 39$;*
- *(C3)* *a k-vertex adjacent to $(k-1)$ 2-vertices for $2 \leq k \leq 19$;*
- *(C4)* *a k-vertex adjacent to $(k-2)$ 2-vertices for $3 \leq k \leq 10$;*
- *(C5)* *a 3-vertex;*
- *(C6)* *a k-vertex adjacent to $(k-3)$ 2-vertices for $3 \leq k \leq 6$;*

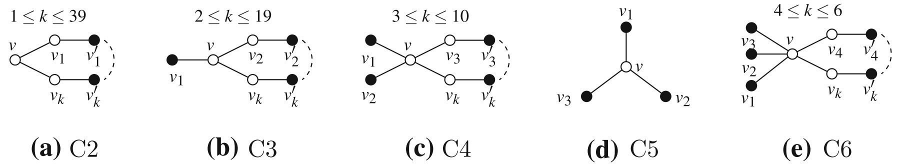


Fig. 2 Configurations $C2$–$C6$

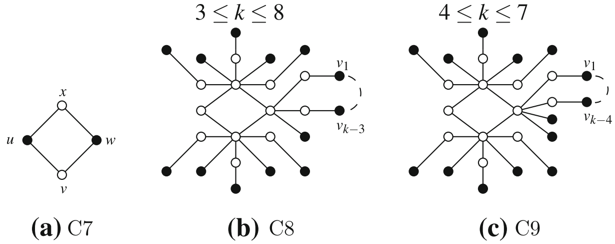


Fig. 3 Configurations $C7$–$C9$

(C7) *two vertices u and v linked by two distinct* 2*-paths, both paths having a* 2*-vertex as internal vertex;*

(C8) *a* 4*-face* $wxyz$ *such that* x *is 2-vertex,* w *and* y *are weak* 7*-vertices, and* z *is a* k*-vertex adjacent to* $(k-3)$ *2-vertices for* $3 \le k \le 8$*;*

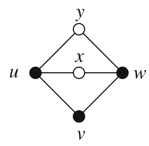
(C9) *a 4-face* $wxyz$ *such that* x *is 2-vertex,* w *and* y *are weak 7-vertices, and* z *is a* k*-vertex adjacent to* $(k-4)$ *2-vertices for* $4 \le k \le 7$*;*

The drawing conventions for a *configuration* C contained in a graph G are the following. If u and v are two vertices of C, then they are adjacent in G if and only if they are adjacent in C. Moreover, the neighbors of a *white* vertex in G are exactly its neighbors in C, whereas a *black* vertex may have neighbors outside of C. Two or more black vertices in C may coincide in a single vertex in G, provided they do not share a common white neighbor. Finally, an edge will represent an arc with any of its two possible orientations. Configurations $(C2) - (C9)$ are depicted in Figs. 2 and 3.

Let G be an oriented graph, v be a k-vertex with $N(v) = \{v_1, v_2, \dots, v_k\}$ and α be an orientation k-vector such that $\alpha_i = 0$ whenever $\overrightarrow{v_i v} \in A(G)$ and $\alpha_i = 1$ otherwise. Let φ be a T_{40}-coloring of $G \setminus \{v\}$ and $S = (\varphi(v_1), \varphi(v_2), \dots, \varphi(v_k))$. Recall that a necessary condition to have α-successors of S is that α must be compatible with S, that is for any pair of vertices v_i and v_j, $\varphi(v_i) \neq \varphi(v_j)$ whenever $\alpha_i \neq \alpha_j$ and $\varphi(v_i) \neq t(\varphi(v_j))$ whenever $\alpha_i = \alpha_j$. Hence, every vertex v_j forbids one color for each vertex v_i, $i \in [1, k]$, $i \neq j$. We define $f^{\varphi}_{v_i}(v_j)$ to be the forbidden color for v_i by $\varphi(v_j)$ (i.e. $f^{\varphi}_{v_i}(v_j) = \varphi(v_j)$ whenever $\alpha_i \neq \alpha_j$ and $f^{\varphi}_{v_i}(v_j) = t(\varphi(v_j))$ whenever $\alpha_i = \alpha_j$). Therefore, α is compatible with S if and only if we have $\varphi(v_i) \neq f^{\varphi}_{v_i}(v_j)$ for every pair i, j, $1 \le i < j \le k$. Note that if $\varphi(v_i) \neq f^{\varphi}_{v_i}(v_j)$, then we necessarily have $\varphi(v_j) \neq f^{\varphi}_{v_j}(v_i)$.

For each configuration, we suppose that H contains it and we consider a triangle-free reduction H' such that $H' \prec H$; therefore, by minimality of $H,'$ admits a

Fig. 4 Configuration $C7'$



T_{40}-coloring φ. We will then show that we can choose φ so that it can be extended to H by Proposition 5, contradicting the fact that H is a counterexample.

In the remainder, if H contains a configuration, then H^* will denote the graph obtained from H by removing all the white vertices from this configuration.

Proof of Configuration ($C1$). Trivial. □

Proof of Configuration ($C2$). Suppose that H contains the configuration depicted in Fig. 2a and let φ be a T_{40}-coloring of H^*. Let $F = \{f_v^{\varphi}(v_1'), \ldots, f_v^{\varphi}(v_k')\}$ be the set of forbidden colors for v. Any T_{40}-coloring of H^* can be extended to H since $|F| \leq 39$. □

Proof of Configuration ($C3$). Suppose that H contains the configuration depicted in Fig. 2b and let φ be a T_{40}-coloring of H^*. Let $F = \{f_v^{\varphi}(v_1'), \ldots, f_v^{\varphi}(v_k')\}$ be the set of forbidden colors for v. By Property $P_{1,19}$, φ can be extended to H since $|F| \leq 18$. □

Proof of Configuration ($C4$). Suppose that H contains the configuration depicted in Fig. 2c and let φ be a T_{40}-coloring of $H' = H \setminus \{v_3, \ldots, v_k\}$. Then, we clearly have $\varphi(v_1) \neq f_{v_1}^{\varphi}(v_2)$ since v is colored in H'. Therefore, by Property $P_{2,9}$, there exists a T_{40}-coloring φ' of H' such that $\varphi'(v) \notin \{f_v^{\varphi'}(v_3'), \ldots, f_v^{\varphi'}(v_k')\}$. The coloring φ' can be extended to H. □

Proof of Configuration ($C5$). Suppose that H contains the configuration depicted in Fig. 2d. Let H' be the graph obtained from H^* by adding, for every $1 \leq i < j \leq 3$, a 2-path joining v_i to v_j with the same orientation as the path $[v_i, v, v_j]$ in H. Since Configurations ($C1$)–($C4$) are forbidden, we have $d_H(v_i) \geq 3$ for $1 \leq i \leq 3$; we thus have $H' \prec H$ since $n_3(H') = n_3(H) - 1$, and H' is clearly triangle-free. Any T_{40}-coloring φ of H' induces a coloring of H^* such that $\varphi(v_i) \neq f_{v_i}^{\varphi}(v_j)$ for any $i, j, 1 \leq i < j \leq 3$. Then Property $P_{3,4}$ allows us to extend φ to H. □

Proof of Configuration ($C6$). Suppose that H contains the configuration depicted in Fig. 2e. Let φ be a T_{40}-coloring of $H' = H \setminus \{v_4, \ldots, v_k\}$. Then, we clearly have $\varphi(v_i) \neq f_{v_i}^{\varphi}(v_j)$, for all $1 \leq i \leq j \leq 3$, since v is colored in H'. Therefore, by Property $P_{3,4}$, there exists a T_{40}-coloring φ' of H' such that $\varphi'(v) \notin \{f_v^{\varphi'}(v_4'), \ldots, f_v^{\varphi'}(v_k')\}$. □

Proof of Configuration ($C7$). Suppose first that H contains the configuration ($C7'$) depicted in Fig. 4. Let H' be the graph obtained from H^* by adding a 2-path $uv'w$ between u and w such that $uv'w$ is directed if and only if uvw is not directed. We have that $H' \prec H$ since $|V(H')| = |V(H)| - 1$ and $n_3(H') = n_3(H)$. Due to the orientations of the 2-paths $uv'w$ and uvw, any T_{40}-coloring φ of H' ensures that $\varphi(u) \neq \varphi(w)$ and $\varphi(u) \neq t(\varphi(w))$. The coloring φ can be extended to H.

Suppose that H contains the configuration depicted in Fig. 3a. Let H' be the graph obtained from H^* by adding an edge between u and w. We have that $H' \prec H$ since $|V(H')| = |V(H)| - 2$ and $n_3(H') = n_3(H)$. Since Configuration ($C7'$) is forbidden, the vertices u and w are at distance at least 3 in H^* and H' is therefore triangle-free. Any T_{40}-coloring φ of H' ensures that $\varphi(u) \neq \varphi(w)$ and $\varphi(u) \neq t(\varphi(w))$. The coloring φ can be extended to H. □

Proof of Configurations $(C8)$ *and* $(C9)$. To prove that these two configurations are forbidden in a minimal counterexample to Theorem 2, a computer check is needed. Indeed, Properties $P_{1,19}$, $P_{2,9}$, $P_{3,4}$ and $P_{4,1}$ are not sufficient.

A computer check allows us to show that for any compatible color assignment on the black vertices (i.e. any two black vertices at distance 2 in the configuration get compatible colors) and any orientation of the arcs, the white vertices can be colored. Our computer check runs in less than two days. Therefore, that shows that H does not contain any of these two configurations. □

3.2 Discharging Procedure

To complete the proof of Theorem 2, we use a discharging procedure. We define the weight function ω by $\omega(x) = d(x) - 4$ for every $x \in V(H) \cup F(H)$. Since H is a plane graph, we have by Euler's formula ($|V(H)| - |A(H)| + |F(H)| = 2$):

$$\sum_{v \in V(H)} \omega(v) + \sum_{f \in F(H)} \omega(f) = \sum_{v \in V(H)} (d(v) - 4) + \sum_{f \in F(H)} (d(f) - 4) = -8 < 0.$$

In what follows, we will define discharging rules (R1), (R2), and (R3) and redistribute weights accordingly. Once the discharging is finished, a new weight function ω^* is produced. However, the total sum of weights is fixed by the discharging rules. Nevertheless, we can show that $\omega^*(v) \geq 0$ for every $x \in V(H) \cup F(H)$. This leads to the following obvious contradiction:

$$0 \leq \sum_{v \in V(H)} \omega^*(v) + \sum_{f \in F(H)} \omega^*(f) = \sum_{v \in V(H)} \omega(v) + \sum_{f \in F(H)} \omega(f) < 0.$$

Therefore, no such counterexample H exists.

The discharging rules are defined as follows:

(R1) Each ${}^{\geq}4$-vertex gives 1 to each of its 2-neighbors.
(R2) Each ${}^{\geq}5$-face $...axb$ such that a and b are 2-vertices gives 1 (resp. $\frac{1}{2}$) to x if x is a weak 7-vertex (resp. is not a weak 7-vertex).
(R3) Each ${}^{\geq}5$-face $f = ...awxyb$ such that a, b, x are 2-vertices and w, y are weak 7-vertices either receives $\frac{1}{2}$ from the vertex z if $wxyz$ is a 4-face, or receives 1 from the ${}^{\geq}5$-face $f' = ...cwxyd$ if c, d are ${}^{\geq}4$-vertices.

The discharging rules are illustrated in Fig. 5; white disks (resp. black disks, black squares) are 2-vertices (resp. ${}^{\geq}4$-vertices, weak 7-vertices).

3.2.1 For All Vertices v, $\omega^*(v) \geq 0$

In the following, $d_{\geq 4}(v)$ denotes the number of neighbors of v with degree at least 4. In the same way, $d_2(v)$ denotes the number of neighbors of v with degree exactly 2.

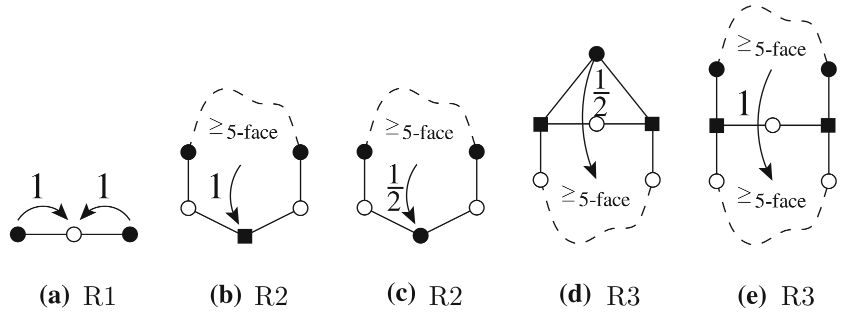


Fig. 5 Discharging rules

Then it is clear that, for every vertex v of H, we have $d(v) = d_{\geq 4}(v) + d_2(v)$ since H contains neither vertices of degree at most 1 by $(C1)$, nor 3-vertices by $(C5)$.

Let v be a k-vertex of H. Therefore, $k = d_{\geq 4}(v) + d_2(v)$. Recall that the initial charge of v is $\omega(v) = k - 4$.

If $k = 2$, then v receives 2×1 by (R1); hence, $\omega^*(v) = \omega(v) + 2 = 0$.

Clearly, in the remainder of this section, $k \geq 4$.

– if $d_{\geq 4}(v) = 0$, then $d_2(v) = k \geq 40$ by (C2). By (R1), v gives $k \times 1$. By (C7), v is incident with k $^{\geq}$5-faces, and therefore v receives $k \times \frac{1}{2}$ by (R2). Hence, $\omega^*(v) = \omega(v) - k + \frac{k}{2} \geq 16$.
– if $d_{\geq 4}(v) = 1$, then $d_2(v) = k - 1 \geq 19$ by (C3). By (R1), v gives $(k-1) \times 1$. By (C7), v is incident with $(k-2)$ $^{\geq}$5-faces each of which gives $\frac{1}{2}$ to v by (R2). Moreover, v is adjacent to at most one weak 7-vertex and therefore (R3) does not apply. Hence, $\omega^*(v) = \omega(v) - (k-1) + \frac{k-2}{2} \geq 6$.
– if $d_{\geq 4}(v) = 2$, then $d_2(v) = k - 2 \geq 9$ by (C4). By (R1), v gives $(k-2) \times 1$. By (C7), v is incident with $(k-4)$ $^{\geq}$5-faces each of which gives $\frac{1}{2}$ to v by (R2). Moreover, by (R3), v gives at most $\frac{1}{2}$ since v is adjacent to at most two weak 7-vertices. Hence, $\omega^*(v) = \omega(v) - (k-2) + \frac{k-4}{2} - \frac{1}{2} \geq 1$.
– if $d_{\geq 4}(v) = 3$, then $d_2(v) = k - 3 \geq 4$ by (C6) and so $k \geq 7$. In each subcase, by (R1), v gives $(k-3) \times 1$.
  ▷ Suppose that the three $^{\geq}$4-neighbors are consecutive. By (C7), v is incident with $(k-4)$$^{\geq}$5-faces of each of which gives $\frac{1}{2}$ to v by (R2). Moreover, by (R3), v gives at most $2 \times \frac{1}{2}$. If $k \leq 8$, then $d_2(v) < 6$ and by (C8), v gives no charge. Hence, if $k \leq 8, \omega^*(v) = \omega(v) - (k-3) + \frac{k-4}{2} \geq \frac{1}{2}$; if $k \geq 9, \omega^*(v) = \omega(v) - (k-3) + \frac{k-4}{2} - 2 \cdot \frac{1}{2} \geq \frac{1}{2}$.
  ▷ Suppose that two $^{\geq}$4-neighbors are consecutive. By (C7), v is incident with $(k-5)$ $^{\geq}$5-faces each of which gives $\frac{1}{2}$ to v by (R2). Moreover, by (R3), v gives at most $\frac{1}{2}$ if and only if $d_2(v) \geq 6$, that implies $k \geq 9$ by (C8). Hence, if $k \leq 8, \omega^*(v) = \omega(v) - (k-3) + \frac{k-5}{2} \geq 0$; if $k \geq 9, \omega^*(v) = \omega(v) - (k-3) + \frac{k-5}{2} - \frac{1}{2} \geq \frac{1}{2}$.
  ▷ Suppose that none of the $^{\geq}$4-neighbors are consecutive. By (C7), v is incident with $(k-6)$ $^{\geq}$5-faces each of which gives $\frac{1}{2}$ to v by (R2) if $d(v) \geq 8$, or

give 1 to v by (R2) if $d(v) = 7$ (i.e. v is a weak 7-vertex). Moreover, (R3) does not apply. Hence, if $d(v) = 7$, $\omega^*(v) = \omega(v) - (k-3) + 1 = 0$; if $d(v) \geq 8$, $\omega^*(v) = \omega(v) - (k-3) + \frac{k-6}{2} \geq 0$.

- If $d_{\geq 4}(v) = 4$, then $d_2(v) = k - 4$. By (C1), v gives $(k-4) \times 1$.
  Suppose that (R3) does not apply. Then, $\omega^*(v) \geq \omega(v) - (k-4) = 0$. Suppose now that (R3) applies: it applies at most twice (otherwise there would be a weak 7-vertex with three consecutive 2-neighbors). Moreover, by (C9), we have $d_2(v) \geq 4$, that implies $k \geq 8$.
  ▷ Suppose first that (R3) applies only once; then v gives $\frac{1}{2}$ to the corresponding 4-face. Moreover, by (R2), v receives $\frac{k-7}{2}$. Hence, $\omega^*(v) = \omega(v) - (k-4) + \frac{k-7}{2} - \frac{1}{2} \geq 0$.
  ▷ Suppose now that (R3) applies twice; then v gives $2 \times \frac{1}{2}$ to the corresponding 4-faces. Moreover, by (R2), v receives $\frac{k-6}{2}$. Hence, $\omega^*(v) = \omega(v) - (k-4) + \frac{k-6}{2} - 2 \times \frac{1}{2} \geq 0$.
- Suppose finally that $d_{\geq 4}(v) \geq 5$. By (C1), v gives $(k - d_{\geq 4}(v)) \times 1$. Moreover, by (R3), v gives at most $\frac{1}{2} \times \left\lfloor \frac{d_{\geq 4}(v)}{2} \right\rfloor$. Hence, $\omega^*(v) \geq \omega(v) - (k - d_{\geq 4}(v)) - \frac{1}{2} \times \left\lfloor \frac{d_{\geq 4}(v)}{2} \right\rfloor \geq 0$.

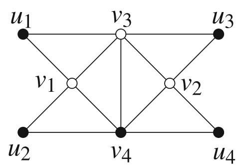
Thus, for every $v \in V(H)$, we have $\omega^*(v) \geq 0$.

3.2.2 *For All Faces f, $\omega^*(f) \geq 0$*

Let f be a k-face of H. Since H is triangle-free, we have $k \geq 4$. Recall that the initial charge of f is $\omega(f) = k - 4$.

- If $k = 4$, then no rule applies. Hence, $\omega^*(f) = \omega(f) = 0$
- If $k = 5$, then f is incident with at most two 2-vertices by (C3).
  ▷ If f has no incident 2-vertices, then $\omega^*(v) \geq \omega(f) = 1$.
  ▷ If f is incident with one 2-vertex, then only (R3) may apply and hence $\omega^*(f) \geq \omega(f) - 1 = 0$.
  ▷ If f is adjacent to two 2-vertices x and z, then f gives at most 1 to the common neighbor of x and z by (R2). Hence $\omega^*(v) \geq \omega(f) - 1 = 0$.
- If $k = 6$, then f is incident with at most three 2-vertices by (C3).
  ▷ If f has no incident 2-vertices, then $\omega^*(v) \geq \omega(f) = 2$.
  ▷ If f is incident with one 2-vertex, then only (R3) may apply and hence $\omega^*(f) \geq \omega(f) - 1 = 1$.
  ▷ Suppose that f is incident with two 2-vertices x and z. If x and z has a common neighbor, then f f gives at most 1 by (R2), and hence $\omega^*(v) \geq \omega(f) - 1 = 0$. If x and z has no common neighbor, then only (R3) may apply at most twice. Hence, $\omega^*(v) \geq \omega(f) - 2 \times 1 = 0$.
  ▷ Finally, suppose that f is adjacent to three 2-vertices.
    ◇ If f is incident with at most one weak 7-vertex, then f gives at most $1 \times 1 + 2 \times \frac{1}{2} = 2$ by (R2). Hence, $\omega^*(v) \geq \omega(f) - 2 = 0$.

Fig. 6 Unavoidable configuration in a 2-outerplanar graph containing neither a $^{\le}3$-vertex, nor two adjacent 4-vertices



◇ If f is incident with two weak 7-vertices, then f gives $2 \times 1 + 1 \times \frac{1}{2} = \frac{5}{2}$ by (R2). Moreover, f receives at least $\frac{1}{2}$ by (R3). Hence, $\omega^*(v) \ge \omega(f) - \frac{5}{2} + \frac{1}{2} = 0$.

◇ If f is incident with three weak 7-vertices, then f gives 3×1 by (R2). Moreover, f receives at least $3 \times \frac{1}{2}$ by (R3). Hence, $\omega^*(v) \ge \omega(f) - 3 + 3 \times \frac{1}{2} = \frac{1}{2}$.

– Suppose finally that $k \ge 7$, and assume that (R2) applies n times and (R3) applies m times. It is clear that f gives weights by (R2) to at most $\lfloor \frac{k}{2} \rfloor$ vertices: hence, $n \le \lfloor \frac{k}{2} \rfloor$. Moreover, we can easily check that $2n + 3m \le k$. With these constraints, we have $n + m = \frac{n+2n+3m}{3} \le \frac{\lfloor \frac{k}{2} \rfloor + k}{3}$, which implies that $n + m \le k - 4$ when $k \ge 7$. Hence, $\omega^*(v) \ge \omega(f) - n - m \ge 0$.

Thus, for every $f \in F(H)$, we have $\omega^*(v) \ge 0$.

4 Proof of Theorem 4

In this section, we prove Theorem 4, which says that every oriented 2-outerplanar graph G admits a homomorphism to T_{40}.

Esperet and Ochem [7] proved the following structural theorem for 2-outerplanar graphs.

Theorem 5 (Esperet, Ochem [7]) Let G be a 2-outerplanar graph. Then G contains either a $^{\le}3$-vertex, or two adjacent 4-vertices, or the configuration depicted in Fig. 6.

Note that the class of 2-outerplanar graphs is minor closed.

To prove Theorem 4, we will consider a minimal counterexample and prove that it cannot contain any of the configurations described in Theorem 5, to arrive at a contradiction.

Let H be a hypothetical minimal counterexample (with respect to the minor order) to Theorem 4.

– It is trivial to show that H does not contain a 1-vertex.
– Suppose that H contains a 2-vertex v adjacent to u_1 and u_2. If u_1 and u_2 are not adjacent, let H' be the graph obtained from H by contracting the arc u_1v; otherwise, let $H' = H \setminus \{v\}$. By minimality of H, the graph H' admits a T_{40}-coloring φ, and since u_1 and u_2 are adjacent in H', $\varphi(u_1) \ne \varphi(u_2)$ and $\varphi(u_1) \ne t(\varphi(u_2))$. By $P_{2,9}$, φ can be extended to H, a contradiction.
– Suppose that H contains a 3-vertex v adjacent to u_1, u_2, and u_3. If v is a sink, let $H' = H$; otherwise, let H' be the graph obtained from H by pushing u_1 and/or u_2

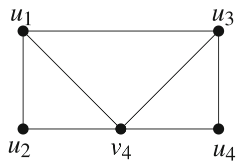


Fig. 7 Reduction of the configuration depicted in Fig. 6

and/or u_3 in such a way that v becomes a sink in H' (i.e. $\overrightarrow{u_1v}, \overrightarrow{u_2v}, \overrightarrow{u_3v} \in A(H')$). By the Push Property (Proposition 3), the graph H' is clearly a minimal counter-example to Theorem 4 since H' is T_{40}-colorable if and only of H does.
Suppose first that the subgraph induced by u_1, u_2, and u_3 in H' contains a sink, say u_1. Then, let H'' be the graph obtained from H' by contracting $\overrightarrow{u_1v}$. By minimality of H', the graph H'' admits a T_{40}-coloring φ. Since $\overrightarrow{u_2u_1}, \overrightarrow{u_3u_1} \in A(H'')$, we have that either the three vertices $\varphi(u_1), \varphi(u_2), \varphi(u_3)$ form a 3-clique in T_{40} or they form a 2-clique in T_{40} with $\varphi(u_2) = \varphi(u_3)$ (recall that $N^+(u) \cap N^+(t(u)) = \emptyset$ for every u of T_{40}). By $P_{3,4}$, the coloring φ can be extended to H'.
Suppose now that the subgraph induced by u_1, u_2, and u_3 in H' does not contain a sink; then, u_1, u_2, u_3 form a directed cycle. Let $H'' = H' \setminus \{v\}$. By minimality of H', the graph H'' admits a T_{40}-coloring φ. It is clear that $\varphi(u_1)$, $\varphi(u_2)$ and $\varphi(u_3)$ form a 3-clique in T_{40}. By $P_{3,4}$, the coloring φ can be extended to H'.
Therefore, H admits a T_{40}-coloring, a contradiction.

- Suppose that H contains two adjacent 4-vertices u and v and let $H' = H \setminus \{\overrightarrow{uv}\}$. Let u_1, u_2, u_3 (resp. v_1, v_2, v_3) denote the three neighbors of u (resp. v) distinct from v (resp. u). By minimality of H, H' admits a T_{40}-coloring φ. Then, erase the colors of u and v. By $P_{3,4}$, we can color u to get $\varphi(u) \notin \bigcup_{i=1,2,3} f_u^{\varphi}(v_i)$. Then by $P_{4,1}$, there exists a color to extend φ to H, a contradiction.
- Suppose that H contains the configuration depicted in Fig. 6. Let H' be the graph obtained from H by contracting the arcs u_1v_1, u_1v_3, and u_3v_2: we get the graph H' depicted in Fig. 7. Note that if the edge-contractions create pairs of opposite arcs in H', then we just keep the initial arc from each pair (i.e., one existing in H) and we delete the other one. By minimality of H, H' admits a T_{40}-coloring φ. Since u_1, u_3 and v_4 form a triangle in H', we have that $\varphi(u_1)$, $\varphi(u_3)$ and $\varphi(v_4)$ are compatible so that by $P_{3,4}$ we can choose one of the four available colors for v_3. At least two of these four colors are distinct from $f_{v_3}^{\varphi}(u_2)$ and $f_{v_3}^{\varphi}(u_4)$. Then, by $P_{4,1}$, we can color v_1 and v_2, a contradiction.

Therefore, H does not contain any of the configurations described in Theorem 5, a contradiction that proves Theorem 4.

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