Paths, cycles, trees and sub(di)graphs in directed graphs

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Longest paths and Cycles

- Bang-Jensen, Gutin and Yeo 1998: the hamiltonian cycle problem can be solved in polynomial time for semicomplete multipartite digraphs.

Problem 1 (Bang-Jensen, Gutin and Yeo, 1998) *Is there a polynomial algorithm for finding a longest cycle in a semicomplete multipartite digraph?*

- true for semicomplete bipartite and for extended semicomplete digraphs.
- also true for digraphs that are either locally semicomplete, quasi-transitive or path-mergeable

- A digraph D is **hamiltonian-connected** if it contains an (x,y)-hamiltonian path for every choice of distinct vertices $x, y \in V(D)$.
- Thomassen 1980: Every 4-strong semicomplete digraph is hamiltonian-connected.
- Bang-Jensen, Manoussakis and Thomassen 1992: Polynomial algorithm to test whether a given semicomplete digraph contains an (x, y)-hamiltonian path.

Problem 2 *What is the complexity of finding a longest* (x, y)*-path in a semicomplete digraph? Is there a structural characterization?*

Minimum cycle factors

- Given a digraph D with a cycle factor, what is the minimum number of cycles in a cycle factor of D?
- The minimum cycle factor problem is easy for extended semicomplete digraphs and semicomplete bipartite digraphs, but seems very difficult for general semicomplete multipartite digraphs.
- **Definition 0.1** *For every digraph, D, with at least one cycle and every non-negative integer, i, let* $\eta_i(D) = \min\{j \mid D$ *has a j-path-i-cycle factor*$\}$.

- Thus $\eta_0(D) = pc(D)$ and $\eta_i(D) = 0$ if and only if D has an i-cycle factor, so for general digraphs the computation of $\eta_i(D)$ is $\mathcal{NP}$-hard already for $i = 0, 1$.
- Calculating $\eta_0(D)$ and $\eta_1(D)$ can be done in polynomial time for quasi-transitive digraphs.
- **Theorem 0.2 (Bang-Jensen and Nielsen 2006)** *For every strong quasi-transitive digraph, D, containing a cycle factor, we have*

$$k_{\min}(D) = 1 + \sum_{i \in I(D)} \min\{j \mid \eta_j(Q_i) = m_i(D)\}.$$

Furthermore, every cycle factor of D has at least $1 + \sum_{i \in I(D)} (pc(Q_i) - m_i(D))$ cycles.

- **Theorem 0.3 (Bang-Jensen and Nielsen 2006)** *For $k \in \{2,3\}$ there exist polynomial algorithms to verify whether a quasi-transitive digraph has a cycle factor with at most k cycles.*
- **Conjecture 1 (Bang-Jensen and Nielsen 2006)** *For each fixed k there is a polynomial algorithm which determines whether a given quasi-transitive digraph D has a cycle factor with at most k cycles and, if so, finds a minimum cycle factor of D.*

Covering a digraph by cycles

- Gallai conjectured in 1964 that the vertices of every strong digraph can be covered by at most $\alpha(D)$ cycles.
- Proof of Gallai's conjecture (Bessy and Thomassé 2006):
- Let $D = (V, A)$. Given an ordering $E = v_1, \dots, v_n$ of V, we say that an arc v_iv_j is **forward** if $i < j$ and **backward** if $j < i$.
- An ordering $E = v_1, \dots, v_n$ is **elementary equivalent** to another ordering E' of V, if one of the following holds:

(i) $E' = v_n, v_1, \dots, v_{n-1}$,

(ii) $E' = v_2, v_1, v_3, \dots, v_n$ and neither v_1v_2 nor v_2v_1 is an arc of D.

- Two orderings E, E' of V are **equivalent** if there is a sequence $E = E_1, \ldots, E_k = E'$ such that E_i and E_{i+1} are elementary equivalent, for $i = 1, \ldots, k-1$.
- The classes of this equivalence relation are called the **cyclic orders** of D.
- A cycle C is **simple** w.r.t. a cyclic order $\mathcal{O}$ if C has precisely one backward arc w.r.t. $\mathcal{O}$.
- A cyclic order $\mathcal{O}$ is **coherent** if every arc of D is contained in a simple cycle.
- **Theorem 0.4 (Bessy and Thomassé)** *Every strong digraph has a coherent cyclic order and one can find such an ordering in polynomial time.*

- An independent set X of D is **cyclic independent** with respect to $\mathcal{O}$ if there exists an ordering $v_1, \ldots, v_n$ of $\mathcal{O}$ such that $X = \{v_1, \ldots, v_k\}$.
- The **cyclic independence number**, denoted $\alpha(\mathcal{O})$, of a coherent cyclic order $\mathcal{O}$ is the maximum k such that D has a cyclic independent set X with respect to $\mathcal{O}$ such that $|X| = k$.
- Observe that $\alpha(\mathcal{O})$ depends on the choice of $\mathcal{O}$
- Bessy and Thomassé proved that for every strong digraph D and every coherent cyclic order of D, the maximum cardinality of a cyclic independent set equals the minimum number of cycles needed to cover $V(D)$.
- This clearly implies Gallai's conjecture.

- A different proof due to Cameron and Edmonds (1982,1992,2008):
- A feedback arc set in a digraph $D = (V, A)$ is a set $F \subset A$ such that $D - F$ is acyclic.
- A feedback arc set F is **coherent** if every arc is contained in a cycle C such that $|C \cap F| = 1$.
- By the Bessy-Thomasse Theorem every strong digraph has a coherent feedback arc set: Just take a coherent cyclic order and let F be the backward arcs.

Theorem 0.5 (Coflow theorem) *[Cameron and Edmonds 1982] Let $D = (V, A)$, let $\omega : A \to \mathbb{Z}_0$ be a weighting of its arcs and extend ω to sets of arcs in the obvious way. Then*

$$\begin{aligned} & \max\{|S| : S \subseteq V; \forall \textit{ cycle } C, |S \cap C| \leq \omega(C)\} \\ = \; & \min\{\sum_{C \in \mathcal{C}} \omega(C) + |V - \bigcup_{C \in \mathcal{C}} V(C)| : \mathcal{C} \textit{ is a family of cycles of } D \end{aligned}$$

- Apply the coflow theorem to a strongly connected digraph D with a coherent feedback arc set F, by letting $\omega(a) = 1$ if $a \in F$ and $\omega(a) = 0$ otherwise.
- With this choice of ω every S in the formula above is independent, because every arc uv is contained in a cycle C with $\omega(C) = 1$. This C shows that S cannot contain the arc uv.
- Also note that the minimum in the theorem is attained by some family $\mathcal{C}$ of cycles which cover all of V: if v is not covered by $\mathcal{C}$ let vw be an arc and add a cycle C with $\omega(C) = 1$ to $\mathcal{C}$.

- Now we can prove Gallai's conjecture as follows:
- Max $|S|$ where S is independent is at least
- $\max\{|S| : S \subseteq V; \forall$ cycle $C, |S \cap C| \leq \omega(C)\}$
- which is equal to
- $\min\{\sum_{C \in \mathcal{C}} \omega(C) : \mathcal{C}$ is a family of cycles covering $D\}$
- which is at least the minimum cardinality of a family of cycles covering $V(D)$, because $\omega(C) \geq 1$ for every cycle.

Decompositions

Conjecture 2 (Kelly 1964) *Every regular tournament on* $2k+1$ *vertices has a decomposition into* k*-arc-disjoint hamiltonian cycles.*

Conjecture 3 (Bang-Jensen and Yeo, 2001) *Every* k*-arc-strong tournament decomposes into* k *spanning strong digraphs.*

Several results which support the conjecture

- If $D = (V, A)$ is a 2-arc-strong semicomplete digraph then it contains 2 arc-disjoint spanning strong subdigraphs except for one digraph on 4 vertices.
- The conjecture is true for every tournament (in fact semicomplete digraphs) which has a non-trivial cut (both sides of size at least 2) with precisely k arcs in one direction.
- Every k-arc-strong tournament with minimum in- and out-degree at least $37k$ contains k arc-disjoint spanning subdigraphs $H_1, H_2, \dots, H_k$ such that each H_i is strongly connected.

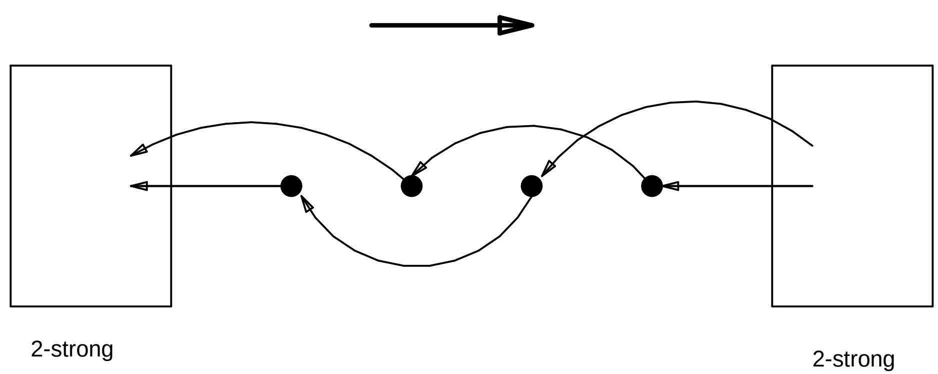
Arc-disjoint hamiltonian paths and cycles

Conjecture 4 (Thomassen, 1982) *Every 3-strong tournament contains two arc-disjoint hamiltonian cycles.*

- 2-strong tournaments may not have arc-disjoint hamiltonian cycles. See next page!

Problem 3 (Bang-Jensen, Huang and Yeo, 2001) *Which tournaments T contain a hamiltonian cycle C such that $\lambda(T-C) \geq \lambda(T)-1$?*

- It follows from the family of tournaments below that not all 2-strong tournaments satisfy this.

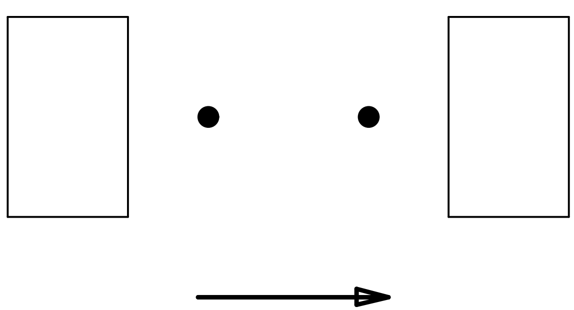


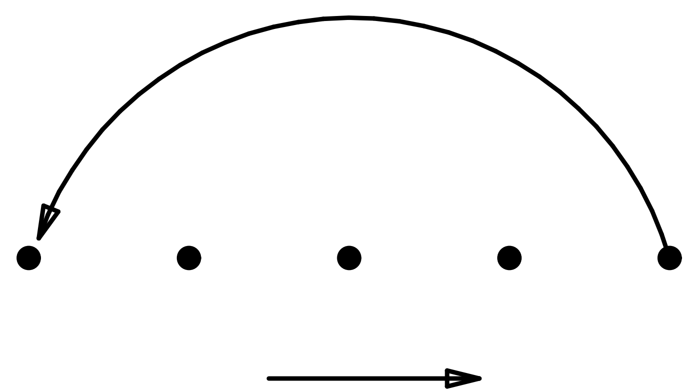
Conjecture 5 *Let T be an arbitrary tournament. Then either T contains two arc-disjoint hamiltonian cycles or T contains two arcs $a, a' \in A(T)$ such that $T - \{a, a'\}$ has no hamiltonian cycle.*

- In Figure 1 above we can destroy all hamiltonian cycles by removing two arcs, but removing one is not enough!
- By a result of Fraisse and Thomassen every k-strong tournament contains a hamiltonian cycle avoiding any prescribed set of $k-1$ arcs.
- Hence, if true, Conjecture 5 would imply Conjecture 4.

Conjecture 6 *There exists a polynomial algorithm for deciding whether a given tournament contains two arc-disjoint hamiltonian cycles.*

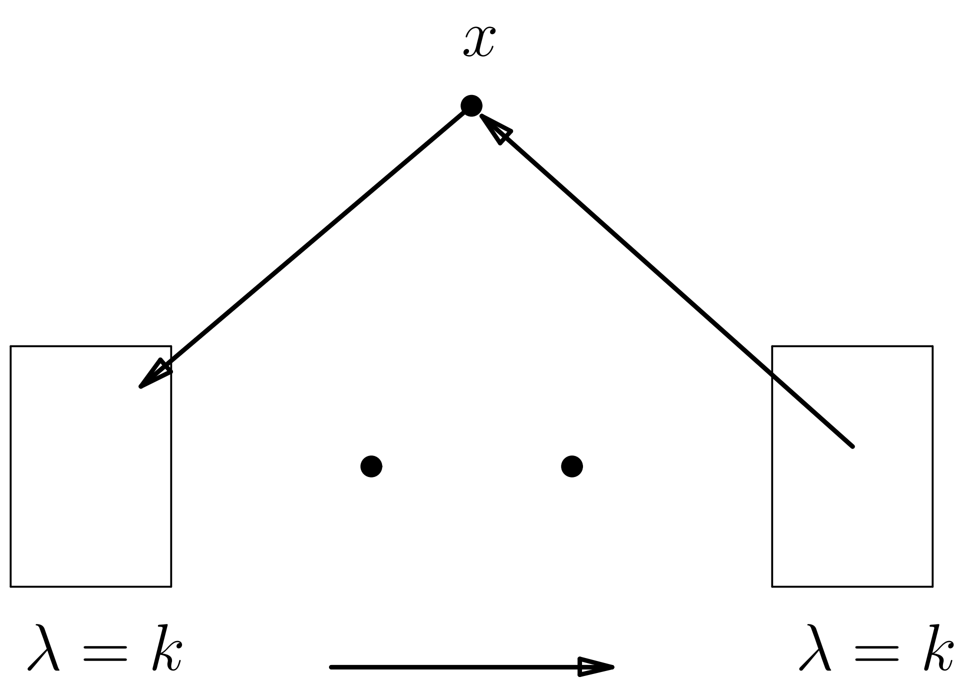
- A tournament is **almost transitive** if it can be obtained from a transitive tournament by reversing the arc from the vertex of maximum out-degree to the vertex of maximum in-degree.
- Thomassen 1989: a tournament T contains two arc-disjoint hamiltonian paths unless it has a strong component which is an almost transitive tournament of odd order or has two consecutive strong components of size 1.





Problem 4 *Characterize those tournaments which contain two arc-disjoint hamiltonian paths with prescribed start vertices.*

- By inspection of the example below we see that no arc-strong connectivity suffices.

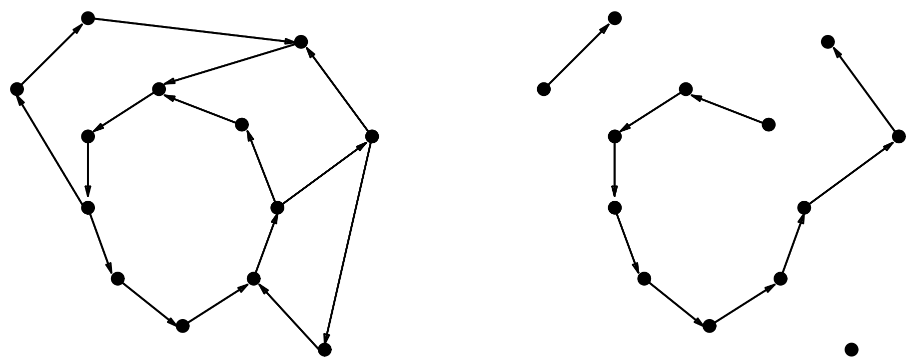


Small certificates for strong connectivity

Problem 5 (The MSSS problem) *: Given a strong digraph* $D = (V, A)$
find a spanning strong subdigraph $D' = (V, A')$ *of* D *such that* $|A'|$ *is minimum.*

- The MSSS problem is $\mathcal{NP}$-hard.
- Let $pc(D)$ denote the path covering number of D and let $pc^*(D) = 0$ if D is hamiltonian and $pc^*(D) = pc(D)$ otherwise.

- Lowerbound: For every strongly connected digraph D, every spanning strong subdigraph of D has at least $n + pc^*(D)$ arcs.



Quasi-transitive digraphs

- A digraph $D = (V, A)$ is **quasi-transitive** if $xy, yz \in A$ implies that either $yz \in A$ or $zy \in A$ (possibly both).
- Bang-Jensen, Huang and Yeo, 1999: The MSSS problem is solvable in polynomial time for quasi-transitive digraphs. Furthermore, if D is a strong quasi-transitive digraph on n vertices, then the number of arcs in a minimum spanning strong subdigraph of D is $n + pc^*(D)$.
- Bang-Jensen and Yeo, 1999: The same statement as above holds for digraphs that are either extended semicomplete of semicomplete bipartite.

- Bang-Jensen and Yeo, 1999: Let D be a strong digraph which is either extended semicomplete or semicomplete bipartite on n vertices, then there exists a minimum spanning strong subdigraph D' of D which contains a longest cycle of D.
- **Conjecture 7** *The MSSS problem is polynomially solvable for semicomplete multipartite digraphs.*
- **Problem 6** *Does every strong semicomplete multipartite digraph D contain a minimum spanning strong subdigraph D' of D which contains a longest cycle of D?*

A $\frac{7}{4}$-approximation algorithm for MSSS

Theorem 0.6 *In time $O(n^2)$ we can find a $\frac{7}{4}$-approximation for the MSSS problem in a strong digraph with no cut vertex.*

Proof:

- If $n = 2$, we return D as the optimal solution so we may assume $n \geq 3$.
- Since D has no cut vertex it contains a cycle of length at least 3. Let P_0 be such a cycle and add to it a maximal sequence $P_1, P_2, \ldots, P_s$ of ears of size at least three.
- Let $X = V(P_0) \cup \ldots \cup V(P_s)$.
- We can show that $Y = V(D) - X$ is an independent set.

- Let D'_X be the strong spanning subdigraph of $D\langle X\rangle$ consisting of all arcs from $P_0, P_1, \dots, P_s$.
- Since each ear P_i, $i \leq s$, adds at least two new vertices to X, we have $s \leq \frac{|X|-3}{2}$.
- Hence we get

$$|A(D'_X)| = |X| + s \leq |X| + \frac{|X|-3}{2} < \frac{3}{2}|X|.$$

- By adding each vertex of Y to D'_X as an ear of size two we obtain a strong spanning subdigraph D' of D with $m' = |A(D'_X)| + 2(n - |X|)$ arcs.

- By the calculation above,

$$m' < \frac{3}{2}|X| + 2(n - |X|) \leq 2n - \frac{|X|}{2}. \tag{1}$$

- If $|X| \geq \frac{n}{2}$, then we get from (1) that $m' \leq \frac{7}{4}n$ and we are done.
- So assume $|X| < \frac{n}{2}$. It follows from the fact that Y is independent that every strong spanning subdigraph of D must contain at least $2(n - |X|)$ arcs (one in and out from every vertex of Y).

- Thus the approximation ratio α which we obtain by returning D' as our solution is no worse than $\frac{m'}{2(n-|X|)} \leq \frac{2n-\frac{|X|}{2}}{2(n-|X|)}$.
- This number is strictly increasing in $|X|$ and as $|X| < \frac{n}{2}$ we have $\alpha \leq \frac{7}{4}$.
- Best know polynomial approximation guarantee is $\frac{3}{2}$ (Vetta 2001).

Certificates for Higher Connectivities

- Every k-arc-strong digraph on n vertices contains a spanning k-arc-strong digraph with at most $2k(n-1)$ arcs.
- (Bang-Jensen, Huang and Yeo, 2000): For any $n \geq 3$ and $k \geq 1$, every k-arc-strong tournament T on n vertices contains a spanning k-arc-strong subdigraph D' with at most $nk + 136k^2$ arcs. Furthermore, such a spanning subdigraph can be found in polynomial time.
- For any tournament T we denote by $\delta_{\geq k}(T)$ the minimum number of arcs in a spanning subdigraph D of T in which has $\delta(D) \geq k$. If $\delta(T) < k$ we let $\delta_{\geq k}(T) = \infty$.
- For every tournament with $\delta(T) \geq k$ we have $\delta_{\geq k}(T) \leq nk + k(k+1)/2$ and this is sharp. Furthermore, if T is k-arc-strong $\delta_{\geq k}(T) \leq nk + k(k-1)/2$.

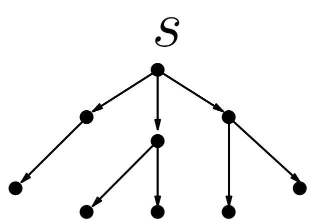
- For any k-arc-strong tournament T we denote by $i(k,T)$ the minimum number of arcs in a spanning k-arc-strong subdigraph D of T.

Conjecture 8 (Bang-Jensen, Huang and Yeo, 2000) *For every natural number k and every k-arc-strong tournament T we have $i(k,T) = \delta_{\geq k}(T)$.*

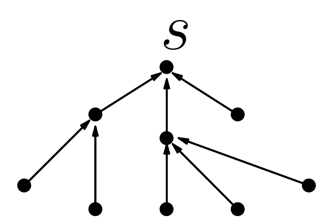
Problem 7 *Does there exist a function $g = g(k)$ such that every k-strong tournament contain a spanning k-strong subdigraph with at most $kn + g(k)$ arcs?*

Arc-disjoint branchings

An **out-branching** (**in-branching**) rooted at s in a directed multigraph D is a spanning tree in $UG(D)$ which is oriented in such a way that every vertex except s has precisely one arc entering (leaving):



F_s^+



F_s^-

Edmonds 1973: A digraph D contains k arc-disjoint out-branchings rooted at the vertex s if and only if

(2) $$d^-(X) \geq k \quad \text{for every } X \subseteq V - s.$$

or, equivalently (by Menger's theorem), s has k arc-disjoint paths to every other vertex.

Out-branchings with few leaves

- A **leaf** of an out-branching F_s^+ is a vertex of out-degree zero in F_s^+.
- By a slight modification of the Gallai-Millgram theorem and Edmond's branching theorem, every 2-arc-strong tournament contains arc disjoint out-branchings $F_{s,1}^+, F_{s,2}^+$ such that each has at most two leaves.

Problem 8 *When does a tournament contain two arc-disjoint out-branchings $F_{s,1}^+, F_{s,2}^+$ such that one of these is a hamiltonian path from s?*

The example in Figure 1 shows that 2-strong connectivity is not sufficient to guarantee arc-disjoint out-branchings $F^+_{s,1}, F^+_{s,2}$ such that one of these is a hamiltonian path from s.

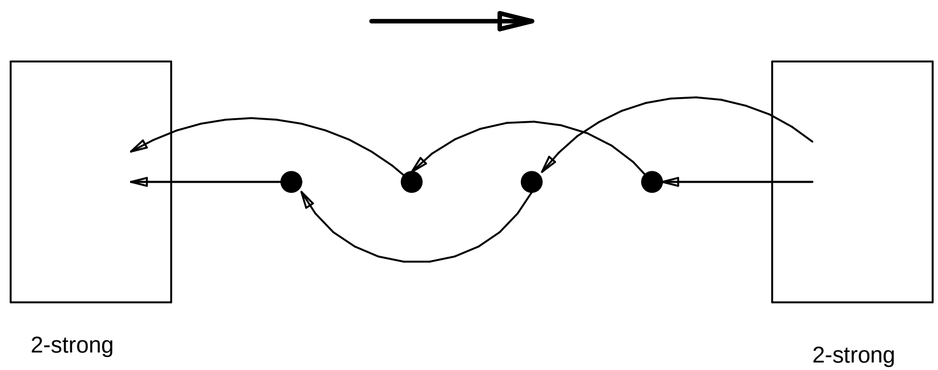


Figure 1: A 2-strong tournament

Arc-disjoint in- and out-branchings

- Thomassen 1986: deciding whether a digraph D has arc-disjoint in- and out-branchings F_v^-, F_v^+ with the same root is NP-complete
- Bang-Jensen 1986: every 2-arc-strong tournament T has arc-disjoint in- and out-branchings F_v^-, F_v^+ for every choice of $v \in V(T)$
- There is a polynomial algorithm for deciding whether a given tournament T has arc-disjoint in- and out-branchings F_u^-, F_v^+ for given $u, v \in V(T)$.
- Bang-Jensen and Yeo 2001: Every $74k$-arc-strong tournament contains $2k$ arc-disjoint branchings $F_{u_1}^-, \ldots, F_{u_k}^-, F_{v_1}^+, \ldots, F_{v_k}^+$.

Conjecture 9 (Bang-Jensen and Yeo, 2001) *Every $2k$-arc-strong tournament contains $2k$ arc-disjoint branchings $F^-_{u_1}, \ldots, F^-_{u_k}, F^+_{v_1}, \ldots, F^+_{v_k}$.*

Conjecture 10 (Bang-Jensen and Gutin, 1998) *There is a polynomial algorithm for deciding whether a given digraph D which is either locally semicomplete or quasi-transitive has arc-disjoint in- and out-branchings F^-_u, F^+_v for given $u, v \in V(D)$.*

Conjecture 11 (Thomassen) *Every 10^{10}-arc-strong digraph D has arc-disjoint in- and out-branchings F_v^-, F_v^+ for every choice of $v \in V(D)$.*

- Very little is known about this problem for general digraphs.
- There exist 2-strong directed multigraph with no arc-disjoint in- and out branchings rooted at the vertex s.

Problem 9 *Try to find a 3-arc-strong digraph D which does not have arc-disjoint in- and out-branchings F_v^-, F_v^+ for some choice of $v \in V(D)$.*

Edge-disjoint spanning trees in graphs

Theorem 0.7 (Tutte 1961) *An undirected graph $G = (V, E)$ has k edge-disjoint spanning trees if and only if*

(3) $$\sum_{1 \le i < j \le p} |(V_i, V_j)| \ge k(p-1),$$

holds for every partition $V_1, V_2, \ldots, V_p$ of V. Here $|(V_i, V_j)|$ denotes the number of edges with one end in V_i and the other in V_j.

Deciding whether a graph has k edge-disjoint trees and finding the desired trees if they exist can be done in polynomial time via matroid partition algorithms.

The characterization also holds for multigraphs.

Arc-disjoint out-branchings in digraphs

An **out-branching** rooted at s in a digraph $D = (V, A)$ is a connected subdigraph B_s^+ of D where each vertex distinct from s has in-degree 1 and s has in-degree 0. I.e it is an orientation of a spanning tree of $UG(D)$ such that s can reach every other vertex by a directed path.

Theorem 0.8 (Edmonds 1973) *A digraph $D = (V, A)$ has k arc-disjoint out-branchings rooted at $s \in V$ if and only if*

$$d^-(X) \geq k \quad \forall \quad \emptyset \neq X \subseteq V - s. \tag{4}$$

Deciding whether a digraph has k arc-disjoint out-branchings rooted at a given vertex s and finding the desired branchings if they exist can be done in polynomial time via flows

Thomassé's question

Problem 10 (Thomassé 2007?) *Find a good characterization of directed graphs having two disjoint directed spanning trees such that one of the spanning trees is an out-branching rooted at a given vertex.*

Equivalently: Characterize those digraphs which have an out-branching B_s^+ rooted at a given vertex s such that $UG(D - A(B_s^+))$ is connected.

Problem 11 *Is there a polynomial algorithm to decide the existence of such an out-branching?*

(arc-)disjoint (s,t)-paths in (di)graphs

- By Menger's theorem, a (di)graph H has two edge-disjoint (arc-disjoint) (s,t)-paths if and only if there there is no edge (arc) whose removal destroys all (s,t)-paths.
- Easy to find the desired paths in linear time if they exist.
- What about the mixed version: Given a digraph D and vertices s,t. Does $UG(D)$ contain two edge-disjoint (s,t)-paths P,Q such that P is also a directed (s,t)-path in D?
- If a digraph D contains a branching B_s^+ such that $UG(D - A(B_s^+))$ is connected, then for every other vertex t, $UG(D)$ contains two edge-disjoint st-paths P,Q so that P is a directed (s,t)-path in D.

Linkage problems

FORTUNE, HOPCROFT, and WYLLIE proved that most directed linkage problems are $\mathcal{NP}$-complete. To be more precise let us consider the following linkage problem. We fix a digraph P.

Problem 12 (Directed linkage with demand digraph P) *Given a digraph D and an injection $h : V(P) \to V(D)$, decide if h extends to an injection on $V(P) \cup A(P)$ such that, for every arc $a = st$ of P, $h(a)$ is an $(h(s), h(t))$-path in D if $s \neq t$ and a cycle in D containing $h(s)$ if $s = t$, and, for each $b \in A(P) - \{a\}$, $V(h(a)) \cap V(h(b)) \subseteq \{h(s), h(t)\}$.*

FORTUNE et al. call this the *Fixed directed subgraph homeomorphism problem*.

Theorem 0.9 (Fortune, Hopcroft and Wyllie 1980) *Assuming $\mathcal{P} \neq \mathcal{NP}$, the linkage problem with demand digraph P is polynomially solvable precisely when all arcs of P have the same head or they all have the same tail.*

It is an easy consequence of Graph Minors XIII that the undirected analogue is polynomially solvable for any fixed demand graph.

Theorem 0.10 (Fortune, Hopcroft and Wyllie, 1980) *For every demand digraph P, there is a polynomial time algorithm to decide if a given directed acyclic digraph D and an injection $h : V(P) \to V(D)$ admits an extension of h to a homeomorphism from P to D.*

Mixed linkages

Let us fix a mixed graph $P = (V', E' \cup A')$ with edges E' and arcs A', and consider a digraph D. We call an injection h on $V' \cup A' \cup E'$ a *mixed homeomorphism from* P *to* D, if

(H1) for every vertex x of P, $h(x)$ is a vertex of D,

(H2) for every arc $a = st$ of P, $h(a)$ is an $(h(s), h(t))$-path in D if $s \neq t$ and a cycle in D containing $h(s)$ if $s = t$,

(H3) for every edge $e = st$ of P, $h(e)$ is an $h(s)h(t)$-path in $UG(D)$ if $s \neq t$ and a cycle in $UG(D)$ containing $h(s)$ if $s = t$, and

(H4) for distinct $x, y \in A' \cup E'$, $V(h(x)) \cap V(h(y)) \subseteq h(V(x))$,

where $V(x)$ denotes the set of endvertices of an arc or edge from P

Problem 13 (Directed linkage with mixed demand graph $\boldsymbol{P}$) *Given a directed graph D and an injection $h : V(P) \to V(D)$, decide whether h extends to a mixed homeomorphism from P to D.*

Theorem 0.11 *The directed linkage problem with mixed demand graph P is polynomially solvable in the following cases:*

(a) *P has no arcs, or*

(b) *P has no edges and there is some vertex s in $V(P)$ that is either the head of all arcs in P or the tail of all arcs in P.*

The problem is $\mathcal{NP}$-complete for all other mixed demand graphs P.

Corollary 0.12 *The following problems are all $\mathcal{NP}$-complete for digraphs. Decide whether for a given input digraph D and vertices $s \neq t, p \neq q$, there exist*

(a) a cycle B in D containing s and a cycle C in $UG(D)$ containing p with $V(B) \cap V(C) \subseteq \{s\} \cap \{p\}$;

(b) an (s,t)-path P in D and a cycle C in $UG(D)$ containing p with $V(P) \cap V(C) \subseteq \{s,t\} \cap \{p\}$;

(c) a cycle B in D containing s and a pq-path Q in $UG(D)$ with $V(B) \cap V(Q) \subseteq \{s\} \cap \{p,q\}$;

(d) an (s,t)-path P in D and a pq-path Q in $UG(D)$ with $V(P) \cap V(Q) \subseteq \{s,t\} \cap \{p,q\}$. $\diamond$

Note that, by Corollary 0.12(d), it is already $\mathcal{NP}$-complete to decide whether the underlying graph of a given digraph D contains two internally disjoint (s,t)-paths P_1, P_2 so that P_1 is also a path in D.

The acyclic case

Theorem 0.13 *The directed linkage problem with mixed demand graph P is polynomially solvable for acyclic digraphs in the following cases:*

(a) P has no arcs, or

(b) P has no edges, or

(c) P contains a directed cycle.

The problem is $\mathcal{NP}$-complete for all other mixed demand graphs.

Corollary 0.14 *It is NP-complete to decide for a given acyclic digraph D and vertices s, t whether $UG(D)$ contains contains two edge-disjoint st-paths P, Q so that P is a directed (s, t)-path in D.*

Disjoint (un)directed cycles in a digraph

- Lovász (1965) characterized those undirected graphs that do not have two disjoint cycles. The characterization leads to a polynomial algorithm to decide the existence of such cycles.
- A digraph is **inter-cyclic** if it does not contain two disjoint cycles.
- McQuaig characterized inter-cyclic digraphs with minimum in- and out-degree at least 2.
- His characterization leads to a polynomial algorithm for deciding whether a given digraph is inter-cyclic.

Problem 14 *What is the complexity of deciding for a given digraph D whether $UG(D)$ contains two disjoint cycles B, C such that B is also a cycle in D?*

Theorem 0.15 *There exists a polynomial algorithm for deciding whether the underlying graph of a given strongly connected digraph D contains two disjoint cycles B, C so that B is also a cycle of D.*

Seems non-trivial to prove. Our proof uses

- McCuaig's characterization of inter-cyclic digraphs,
- Thomassen's solution of the 2-path-problem for acyclic digraphs and
- a non-trivial algorithm for the case of cycle transversal number one.

Why so difficult?

In the mixed version we can not, like in the directed case, employ three important concepts:

(i) Symmetry of the objects we are looking for,

(ii) strongly connectedness — the general directed version reduces immediately to this case, whereas here we needed to add it as a condition to the input digraph —, and, finally,

(iii) reduction by contracting an arc which is either the unique out-arc at their tail or the unique in-arc at their tail — in the directed case, this does not change the answer, so that we can immediately assume that all vertices have in- and out-degree at least 2.

- Every 6-connected graph is 2-linked [Seymour 1980, Thomassen 1980].
- For every natural number k there exists and k-strong digraph which is not 2-linked [Thomassen 1991].

Let us call digraph D **2-mixed-linkable** if, for every choice of vertices $s_1, s_2, t_1, t_2 \in V(D)$, the underlying graph of D contains a pair of internally disjoint paths P_1, P_2 such that P_1 is an $s_1 t_1$ path in the underlying graph of D and P_2 is a directed (s_2, t_2)-path.

Problem 15 *Does there exist an integer N so that every N-strong digraph is 2-mixed-linkable?*

Conjecture 12 (Bang-Jensen and Yeo, 2002) *There exists an integer N such that every N-arc-strong digraph D contains arc-disjoint spanning strong subdigraphs D_1, D_2.*

Conjecture 13 *There exists an integer N such that every N-arc-strong digraph D contains a spanning strong subdigraph D' such that the underlying graph of $D - A(D')$ is (2-edge-)connected.*

Theorem 0.16 (BJ+Yeo 09) *It is NP-complete to decide whether a 2-regular digraph D contains a spanning strong subdigraph D' such that the underlying graph of $D - A(D')$ is connected.*