Algorithmic Graph Minors:

turning Combinatorics to Algorithms

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Examples:

- $2^{\mathbb{N}}$ is not W.Q.O. by set inclusion.
- $\mathbb{N}$ is not W.Q.O. by divisibility.
- $\mathbb{N}^k$ is W.Q.O. by by component-wise ordering.

General question: Given a set $\mathcal{X}$ and an ordering relation $\leq$ on it,

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Remind: This talk is about graphs and algorithms!

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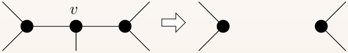
1. $\backslash v$: vertex removal: 

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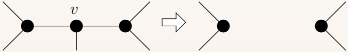

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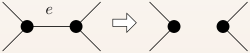

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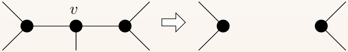

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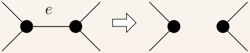

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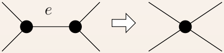

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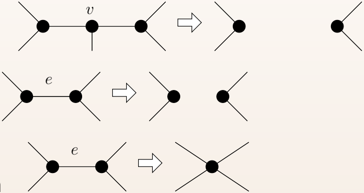

Minor Relation:

$H \leq G$ if H can be obtained from G after a sequence of the above operations

Wagner's Conjecture:

- The set of all graphs is W.Q.O. by the minor relation

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10/11 Fulkerson Prize (2006) (2/3 for N.R. & 4/5 for P.S.)

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$$O(f(k) \cdot n^{O(1)}) \text{ steps}$$

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- Not all parameterized problems admit FPT-algorithms.

There are parameterized complexity classes like W[1], W[2], or W[P] and adequate reductions such that when a problem is hard for them is not expected to have an FPT-algorithm.

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A **meta**-problem:

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Instance: a graph G and an integer $k \geq 0$.
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Question: $\mathbf{p}(G) \leq k$?

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- Holy grail (meta)-question

For which functions **p** it holds that k-PARAMETER **p**-CHECKING∈FPT?

(i.e., there is an $f(k) \cdot n^{O(1)}$-step algorithm checking whether **p**$(G) \leq k$?)

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Minor-closed parameters:

- vertex cover, $\mathbf{vc}(G)$
- feedback vertex set, $\mathbf{fvs}(G)$
- branchwidth/treewidth/pathwidth/tree-depth,

  $\mathbf{bw}(G)/\mathbf{tw}(G)/\mathbf{pw}(G)/\mathbf{td}(G)$
- minimum maximal matching, $\mathbf{mmm}(G)$
- $\mathbf{p}(G) = |V(G)| - \alpha(G)$ ($\alpha(G)$ is the max independent set size)
- the genus of a graph, $\gamma(G)$
- the apex number of a graph, $\mathbf{apx}(G)$
- $\mathbf{p}(G) = \min\{k \mid P_k \not\leq G\}$ (Longest Path)

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In other words,

- **p**$(G) \leq k$ can be checked in $f(k) \cdot n^3$ steps.
- Every minor-closed graph class $\mathcal{G}$ can be recognized in $O(n^3)$.

  [Take **p**$(G) = 0$ if $G \in \mathcal{G}$ and **p**$(G) = 1$, otherwise]

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... go back to the proofs!

- For any minor closed parameter $\mathbf{p}$ and any k, we define $\mathbf{ob}_k(\mathbf{p})$ as the set of minor-minimal elements in

$$\{G \mid \mathbf{p}(G) > k\}$$

- we call $\mathbf{ob}_k(\mathbf{p})$ obstruction family of $\mathbf{p}$.
- Observe: $\mathbf{p}(G) \leq k \Leftrightarrow \forall_{H \in \mathbf{ob}_k(\mathbf{p})}\ H \not\leq G$
- Observe: $\mathbf{ob}_k(\mathbf{p})$ is an antichain.
- GMT Consequence: $\mathbf{ob}_k(\mathbf{p})$ is finite!

An algorithm for the k-PARAMETER **p**-CHECKING problem

1. for all $H \in \mathbf{ob}_k(\mathbf{p})$
2. check (in $O(g(k) \cdot n^3)$ steps) whether $H \leq_{mn} G$
3. and if this holds, then output NO
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-news: $g(k)$ is **huge**!

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2. we know $\mathbf{ob}_k(\mathbf{p})$ for few parameters and for small values of k
3. when we have upper bounds for $|\mathbf{ob}_k(\mathbf{p})|$, they are immense.

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Theorem (GM-VI, GM-VII, GM-XIII, GM-XXI, GM-XII)

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k-DISJOINT PATHS

Instance: A graph G and a sequence of pairs of terminals

$$T=(s_1,t_1),\dots,(s_k,t_k)\in (V(G)\times V(G))^k.$$

Parameter: k.

Question: Are there k pairwise vertex disjoint paths $P_1,\dots,P_k$ in G such that for every $i\in\{1,\dots,k\}$, P_i has endpoints s_i and t_i?

Both k-MINOR CONTAINMENT and k-DISJOINT PATHS PROBLEM where solved using the

The irrelevant vertex Technique

introduced in [GM XIII]

Given an instance (G, T, k) of the k-Disjoint Paths problem,

a vertex $v \in V(G)$ is an *irrelevant* vertex of G if

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We give a outline of the idea for the case of k-Disjoint Paths.

The general scheme of the algorithm in [GM XIII] is the following:

Input: An instance (G, T, k) of k-Disjoint Paths

Output: An equivalent instance (G, T, k) k-Disjoint Paths

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Here $\mathcal{G}_k$ represents some structural condition for the problem input.

The algorithmic scheme depends on the parameterized class $\mathcal{G}_k$ and creates an equivalent instance that belongs in $\mathcal{G}_k$.

- It is applied first for

$$\mathcal{G}_k = \{G \mid G \text{ is a } K_{h(k)}\text{-minor free graph}\}$$

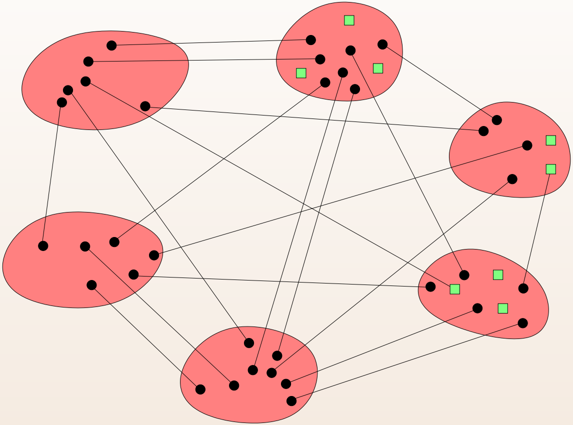
and then for

$$\mathcal{G}_k = \{G \mid G \text{ is a } \Gamma_{j(k)}\text{-minor free graph}\},$$

for some suitable choice of recursive functions $h, j : \mathbb{N} \to \mathbb{N}$.

- Γ_r is the $(r \times r)$-grid.

Phase 1: What to do with a "big" *clique minor*?



(technical) details omitted...

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- Weak Structure Theorem [GM XIII]
- Strong Structure Theorem [GM XVI]

Theorem (Weak Structure Theorem)

There exists recursive functions $g_1 : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $g_2 : \mathbb{N} \to \mathbb{N}$, such that for every graph G and every $r, q \in \mathbb{N}$, one of the following holds:

1. K_r *is a minor of* G,
2. $\Gamma_{g_1(r,q)}$ *is not a minor of* G,
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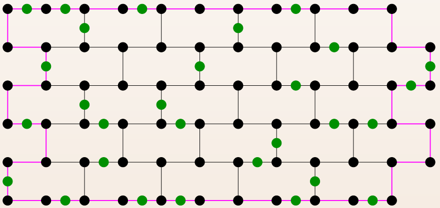
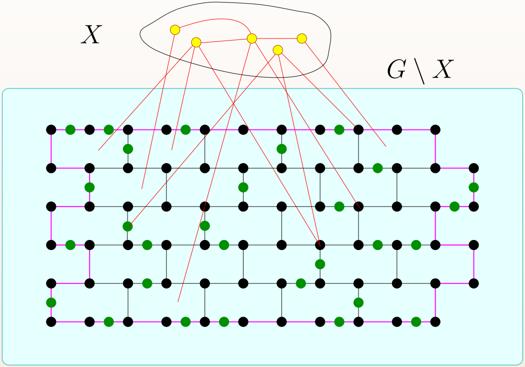
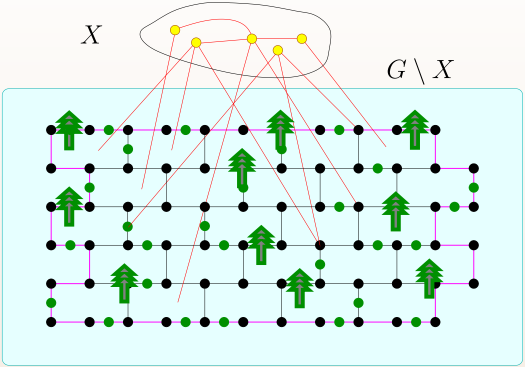
A subdivided Wall W of heigh 5:





The compass is the part of the $G \setminus X$ that is "inside" the perimeter of the subdivided wall W. The perimeter is as a separator between the internal compass vertices and the part of $G \setminus X$ that is outside the perimeter



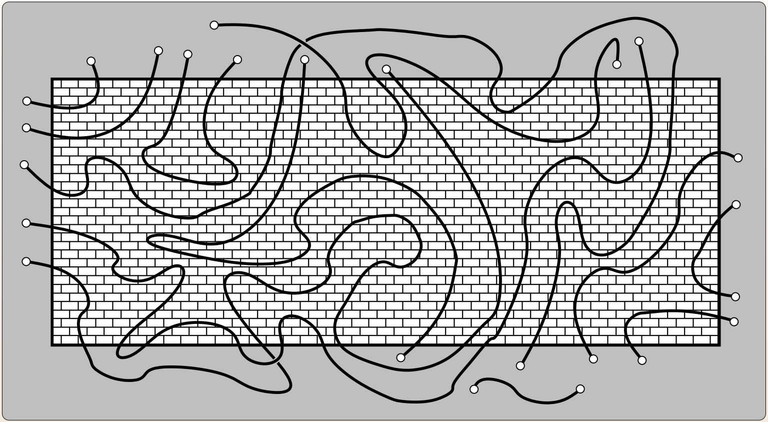
The compass can be decomposed to graphs of bounded treewidth (flaps) whose "roots" have size ≤ 3 and form a planar hypergraph inside the disk bounded by the perimeter

Weak structure theorem $\longrightarrow$ sunny forest theorem!

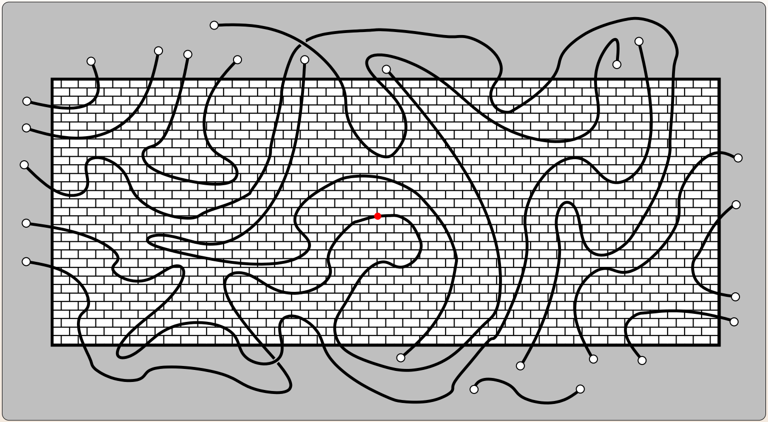
We examine only the simpler case where $X = \emptyset$.

[the forest is **dark**!]

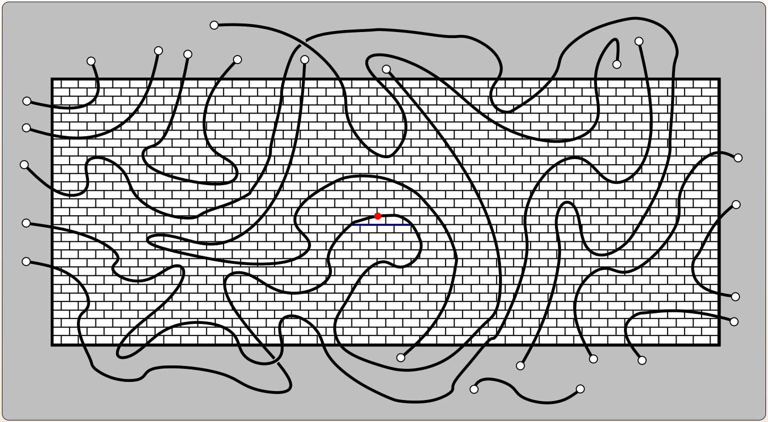
A solution to the k-Disjoint Paths Problem, for $k = 12$



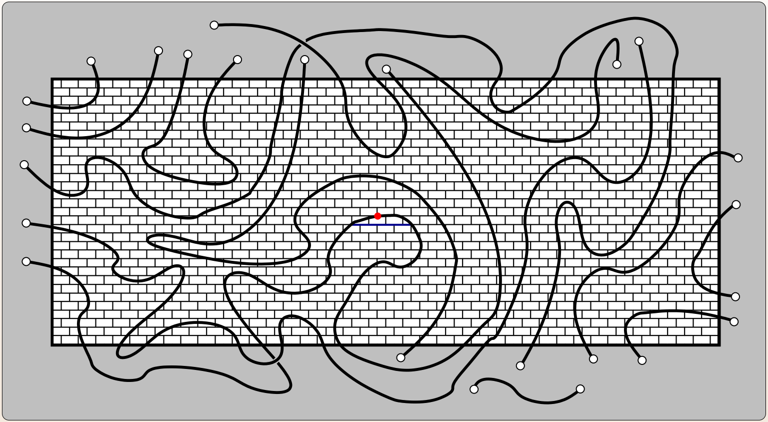
The middle vertex of the subdivided wall W



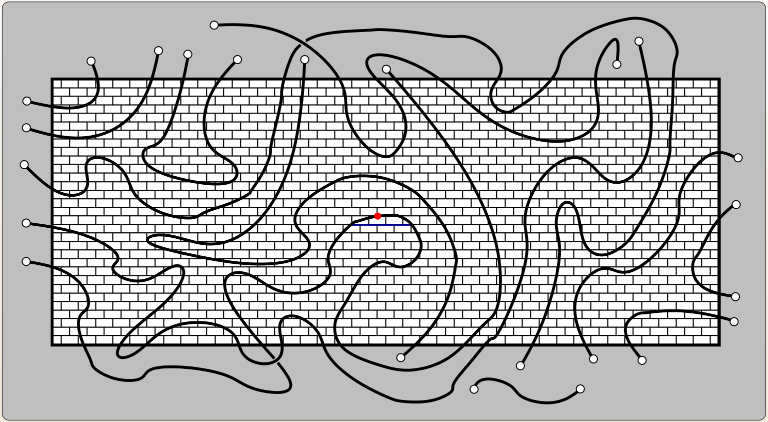
A way to avoid the middle vertex



Is it always possible to avoid the middle vertex?



The answer is YES given that the height of W is at least $\lambda(k)$!



Therefore, if the height of W is "big enough", then we can safely detect an irrelevant vertex (and remove it)!

Theorem (Weak Structure Theorem)

There exists recursive functions $g_1 : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $g_2 : \mathbb{N} \to \mathbb{N}$, such that for every graph G and every $r, q \in \mathbb{N}$, one of the following holds:

1. K_r *is a minor of* G, *(This is now excluded for $r = h(k)$)*
2. $\Gamma_{g_1(r,q)}$ *is not a minor of* G, *(This is what we want to achieve!)*
3. $\exists X \subseteq V(G)$ *with* $|X| \leq g_2(r)$ *such that* $G \setminus X$ *contains as a subgraph a flat subdivided wall* W *where* W *has height* q *z and the compass of* W *has a rural division* $\mathcal{D}$ *such that each internal flap of* $\mathcal{D}$ *has treewidth at most* $g_1(r,q)$. *(irrelevant vertex wanted...)*

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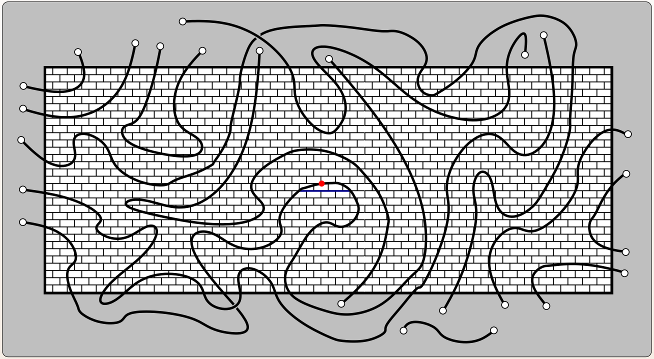
- After the second phase, we have an equivalent instance satisfying cond. 2.
- This means that G has treewidth bounded by some function of k: the problem can be solved using dynamic programming.

We can reroute the k disjoint path, given that the height of W is at least $\lambda(k)$!



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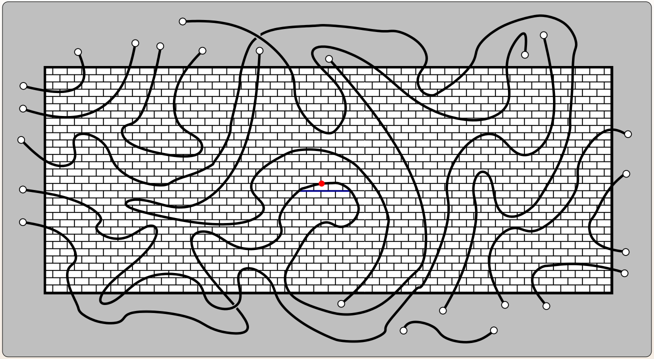

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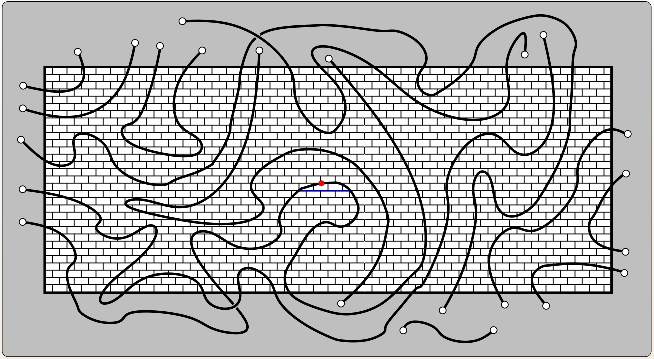

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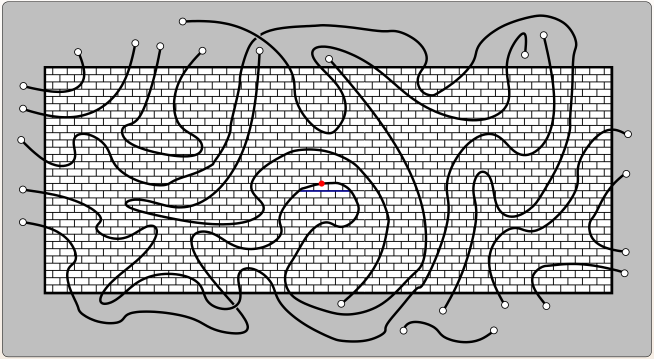

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☠ What is the estimation of $\lambda(k)$? Huge!

David Johnson mentioned in his ongoing guide on NP-completeness:

> *"for any instance $G = (V, E)$ that one could fit into the known universe, one would easily prefer $|V|^{70}$ to even constant time, if that constant had to be one of Robertson and Seymour's"*.

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David Johnson also estimated that just one constant in the parametric dependence of the strong structural Theorem to be roughly

$$2^{\uparrow 2^{2^{2^{2^{\uparrow 2^{\uparrow \Theta(r)}}}}}}$$

where $2^{\uparrow r}$ denotes a tower $2^{2^{2^{\cdot^{\cdot^{\cdot}}}}}$ involving r 2's.

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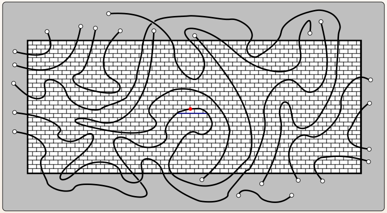

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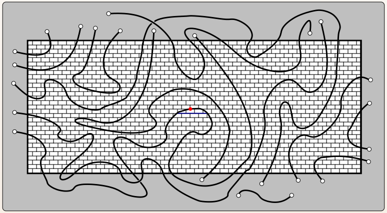

This, still, gives an algorithm of $2^{2^{2^{2^{\Omega(k)}}}}$ steps

for the k-Disjoint Paths Problem

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- A non-☠ algorithm for the k-Disjoint Paths Problem (and related problems) would require radically different techniques!

Last words on Algorithmic Graph Minors Theory...

Some **recent** FPT-Algorithms using the *irrelevant vertex technique* or variants of it:

- Bipartite Contraction, Partial Vertex Cover, Partial Dominating Set,
- Topological Minor Containment, Immersion Containment,
- Bounded Genus Contraction Containment, Odd Cycle Induced Packing,
- Odd Cycle Packing, Induced cycle, Optimal Embedding in a Surface

Piet Mondrian, Composition with Yellow, Blue, and Red, 1921