Rigidity, Triangles and Minors

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Tensegrity

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Stress

Given a embedding $\rho : V \longmapsto \mathbb{R}^d$ of a graph $G = (V, E)$. A **stress** on ρ is a function $\omega : V \times V \rightarrow \mathbb{R}$ such that for all $u \in V$:

$$\sum_{\{u,v\} \in E} \omega(\{u, v\})(\rho(v) - \rho(u)) = 0.$$

Definition

Let $G = (V, E)$ a graph. An embedding $\rho : V \longmapsto \mathbb{R}^d$ of G is d-stress free if every stress is trivial ($\omega = 0$).

Definition

G is generically d-stress free if the set of all d-stress free embeddings of G is open and dense in the set of all embeddings of G ($\simeq \mathbb{R}^{dn}$).

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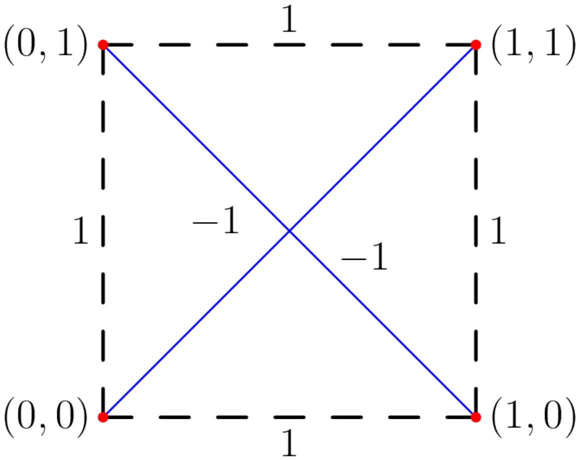
Example: A non-trivial 2-stress on K_4



History

Theorem (Cauchy, 1813)

Every convex polyhedron is 3-stress free.

Theorem (Maxwell, 1864)

Every polyhedron admits a non-trivial 2-stress.

Corollary

Every 3-connected planar graph admits a non-trivial 2-stress.

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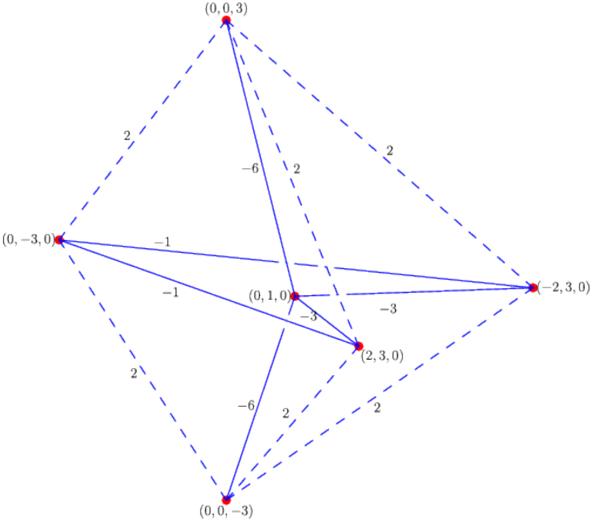
Example: A non-trivial 3-stress on a non-convex polyhedron



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Theorem (Gluck, 1974)

Almost all polyhedra are 3-stress free.

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Planar graphs are generically 3-stress free.

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Rigidity and Minors of Graphs

Theorem (Nevo, 2007)

For $3 \leq r \leq 6$, every K_r-minor free graph is generically $r-2$-stress free.

Theorem (Nevo, 2007)

For $3 \leq d \leq 5$, if each edge of G belongs to at least $d-2$ triangles then G contains a K_d minor.
If each edge of G belongs to at least 4 triangles then G contains a K_6 minor or is a clique-sum over K_l, $l \leq 4$.

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Proof for $d = 3, 4, 5$

We will use extensively the following theorem of Mader.

Theorem (Mader, 1968)

For $1 \leq r \leq 7$, every K_r minor-free graph has at most $(r-2)n - \binom{r-1}{2}$ edges.

- For $d = 3$, each edge belongs to at least one triangle and trivially contains a K_3 minor.
- For $d = 4$, by Mader's theorem, $|E| \leq 2n - 3$, so there is a vertex u such that $\deg(u) \leq 3$. And since each edge belongs to at least 2 triangles, $N(u)$ is isomorphic to K_3.

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Proof for $d = 3, 4, 5$

- For $d = 5$, by Mader's theorem, $|E| \leq 3n - 6$, so there is a vertex u such that $\deg(u) \leq 5$.

  If $\deg(u) = 4$ then $N(u)$ is isomorphic to K_4.

  Now suppose that $\deg(u) = 5$. Since each edge uv with $v \in N(u)$ belongs to at least 3 triangles, then for every $v \in N(u)$, $\deg(v) \geq 3$ in $N(u)$.

  Hence $|e(N(u))| \geq \lceil (3 \cdot 5)/2 \rceil = 8$. But $N(u)$ is K_4-minor free and so $|e(N(u))| \leq 2 \cdot 5 - 3 = 7$.

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Proof of the rigidity theorem

Theorem (Whiteley, 1989)

Let G' be obtained from a graph G by contracting an edge uv. If u and v have at most $d-1$ common neighbours and G' is generically d-stress free, then G is generically d-stress free.

Suppose that G is K_{d+2}-minor free for $3 \leq d \leq 6$.

Contract edges that belong to at most d triangles as long as it is possible and denote G' the graph obtained. By the previous theorem, G is d-stress free if G' is d-stress free.

If G' has no edge then it is trivially d-stress free.

Otherwise every edge belongs to at least d triangles and by the previous theorem G contains a K_{d+2}-minor, a contradiction.

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The case of 4 triangles

Theorem (A. & Gonçalves, 2012)

If G is a K_6 minor-free graph then G has a vertex u and an edge uv such that $\deg(u) \leq 7$ and uv belongs to at most 3 triangles.

Corollary

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Proof

By Mader's theorem $|E| \leq 4n - 10$, so there is a vertex u such that $\deg(u) \leq 7$.

If $\deg(u) \leq 4$, we have a contradiction, each edge uv with $v \in N(u)$ can't belong to 4 triangles.

If $\deg(u) = 5$, since each edge uv with $v \in N(u)$ belongs to 4 triangles then $N(u)$ is isomorphic to K_5, a contradiction.

Lemma

$N(u)$ is planar and 4-connected.

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Theorem (Chen & Kanevsky, 1993)

Every 4-connected graph can be assembled from either the complete graph K_5 or the double-axle wheel W_4^2 on four vertices using operations involving only vertex splitting and edge addition.

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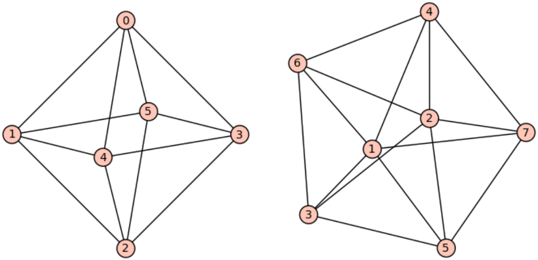

Figure: Double-axle wheel on 4 and 5 vertices

Proof

- We assume that (A, B) is a (≤ 3)-separation of G such that $A \cap B$ is a clique and B is minimal for this property and $u \in A$.
- We can find a vertex u' of small degree in $B \backslash (A \cap B)$.
- $N(u')$ is isomorphic to one of the two double-axle wheels.
- We can find a (≤ 3)-separation (A', B') with $B' \subsetneq B$ where $A' \cap B'$ is a clique and $\{u, u'\} \in A'$, contradicting the minimality of B.

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The case of 5 triangles

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G is K_7-minor free, so by Mader's theorem $|E| \leq 5n - 15$ and there is a vertex u of degree at most 9.

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We use a computer to generate all 5-connected K_6-minor free graphs with at most 9 vertices. We ended up with 22 possible graphs for $N(u)$.

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The case of 6 triangles

Definition

Let G a graph. A (G,k)-cockade is a graph constructed recursively as follows:

- *G is a (G,k)-cockade.*
- *If G_1 and G_2 are (G,k)-cockades and H_1 and H_2 are cliques of size k in respectively G_1 and G_2, then the graph obtained by taking the disjoint union of G_1 and G_2 and identifying H_1 with H_2 is a (G,k)-cockade.*

Theorem (Jørgensen, 1994)

Every graph on $n \geq 8$ vertices and at least $6n-20$ edges either has a K_8-minor, or is a $(K_{2,2,2,2,2},5)$-cockade.

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If each edge of G belongs to at least 6 triangles then G contains a K_8 minor or is a $(K_{2,2,2,2,2}, 5)$-cockade.

The idea is the same. We can still find a vertex u of degree at most 11 but some differences occur compared to the previous cases :

- $N(u)$ is not 6-connected! Some graphs are just 5-connected.
- If we assume that only edges incident to vertices of small degree belong to 6 triangles then some counting arguments fail. This can be fixed by relaxing the assumptions and assuming that all edges belong to 6 triangles.

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Back to rigidity

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Conjectures and Open Problems

Theorem (Song & Thomas, 2005)

Every graph on $n \geq 9$ vertices and at least $7n - 27$ edges either has a K_8-minor, or is a $(K_{2,2,2,2,2,1}, 6)$-cockade, or is isomorphic to $K_{2,2,2,3,3}$.

Conjecture

Every K_9-minor free graph is generically 7-stress free or is a $(K_{2,2,2,2,2,1}, 6)$-cockade, or is isomorphic to $K_{2,2,2,3,3}$.

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Conjectures and Open Problems (2)

- Can we prove smaller degeneracy for $K_{\leq 9}$-minor free graphs?

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Thank you!