Edge coloring: not so simple on multigraphs?

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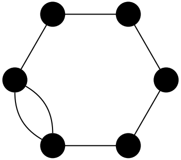
Edge coloring



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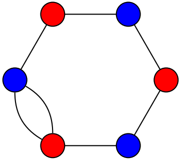

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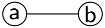

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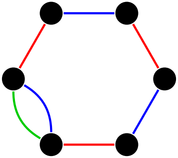

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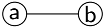

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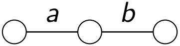

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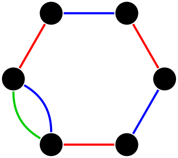

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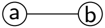

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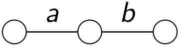

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Δ: Maximum degree of the graph.

$$\Delta \leq \chi'$$

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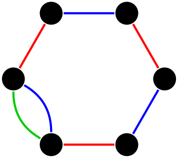

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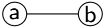

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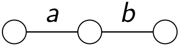

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$$\Delta \leq \chi' \leq 2\Delta - 1.$$

Simple graphs

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For any simple graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

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Theorem (König '16, Sanders Zhao '01)

For any simple graph G, if G is bipartite, or G is planar with $\Delta(G) \geq 7$*, then* $\chi'(G) = \Delta(G)$.

Theorem (Erdös Wilson '77)

Almost every simple graph G verifies $\chi'(G) = \Delta(G)$.

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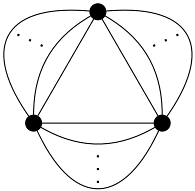
Theorem (Misra Gries '92 (Inspired from the proof of Vizing's theorem))

For any simple graph $G = (V, E)$*, a* $(\Delta + 1)$*-edge-coloring can be found in* $\mathcal{O}(|V| \times |E|)$.

Multigraphs



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$\chi' = 3p$.
$\mu = p$.

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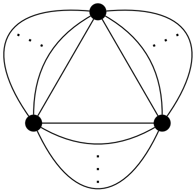

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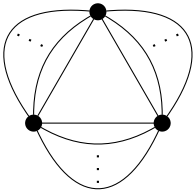

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Both theorems are optimal!

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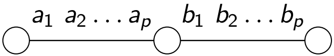

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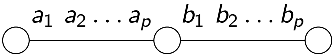

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Property

$\chi'_f(G) = inf_p \frac{\chi'_p(G)}{p}$.

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Which would be optimal. And is computable in polynomial time.

Goldberg's conjecture

Theorem (Yu '08)

For any multigraph G, $\chi'(G) \leq max(\Delta(G) + \sqrt{\frac{\Delta(G)}{2}}, w(G))$.

Theorem (Kurt '09)

For any multigraph G, $\chi'(G) \leq max(\Delta(G) + \frac{\Delta(G)+22}{24}, w(G))$.

Seymour's conjecture

Conjecture (Seymour)

For any k-regular planar multigraph G s.t. every odd subset H has $d(H) \geq k$ verifies $\chi'(G) = k$.

- $k = 3 \Leftrightarrow$ 4CT. (Tait)
- $k = 4, 5$ (Guenin)
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Sketch of the proof.

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Thanks for your attention.
Any questions?