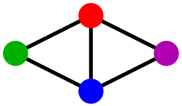
The Maximum Clique Problem in Multiple Interval Graphs

Mathew Francis Daniel Gonçalves Pascal Ochem

Interval graphs



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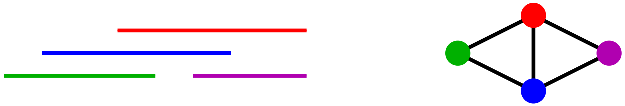

When all intervals are of the same length: “Unit” interval graph

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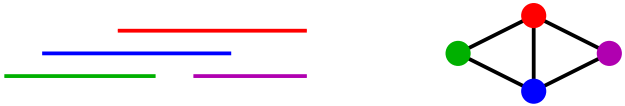

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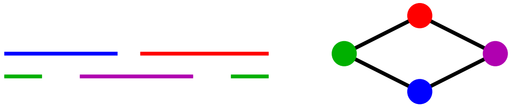

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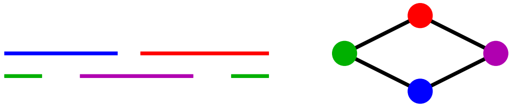
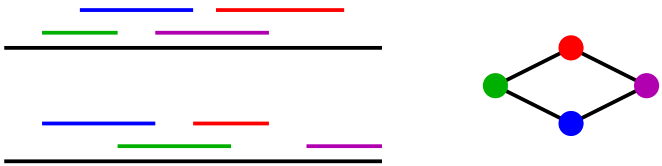


When all intervals are of the same length: “Unit” t-interval graph

t-track graphs



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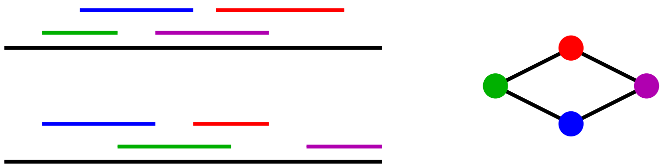

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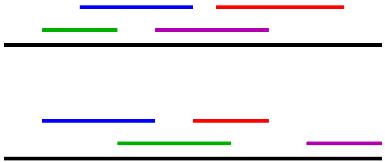

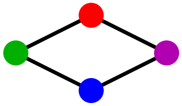

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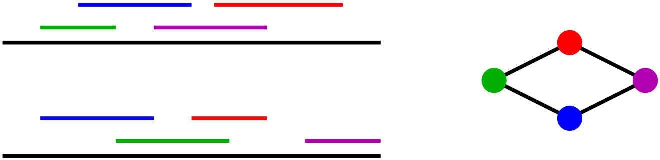

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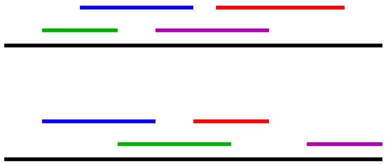

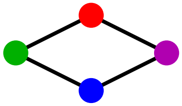

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- "Boxicity" $\leq t$ graph: Edge intersection of t interval graphs.

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- t-interval graphs: $\frac{t^2-t+1}{2}$-approximation algorithm [Butman et al. '07].
- t-track graphs: $\frac{t^2-t}{2}$-approximation algorithm [Koenig '09]. Therefore, polynomial-time solvable on 2-track graphs.

Our results

Butman et al. ask the following questions:

1. Is MAXIMUM CLIQUE NP-hard for 2-interval graphs?
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We show:

1. MAXIMUM CLIQUE is APX-complete for 2-interval graphs, 3-track graphs, unit 3-interval graphs and unit 4-track graphs.
2. MAXIMUM CLIQUE is NP-complete for unit 2-interval graphs and unit 3-track graphs.
3. There is a t-approximation algorithm for MAXIMUM CLIQUE on t-interval graphs.

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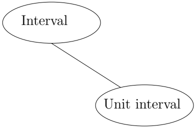

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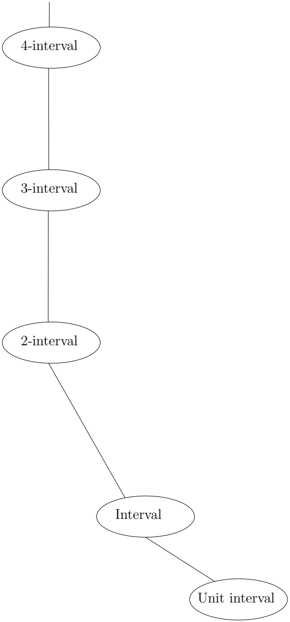

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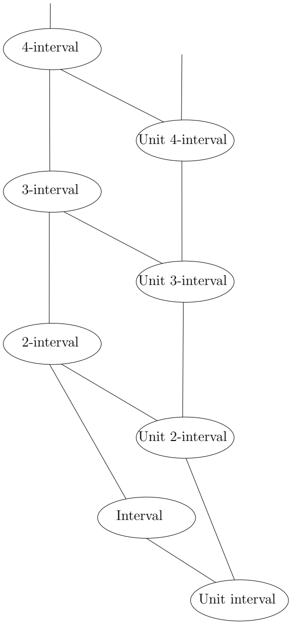

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Unit 4-interval
3-interval
Unit 3-interval
2-interval
Unit 2-interval
Interval
Unit interval

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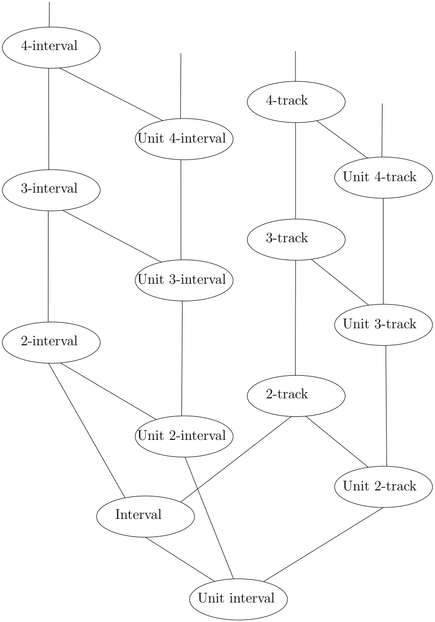
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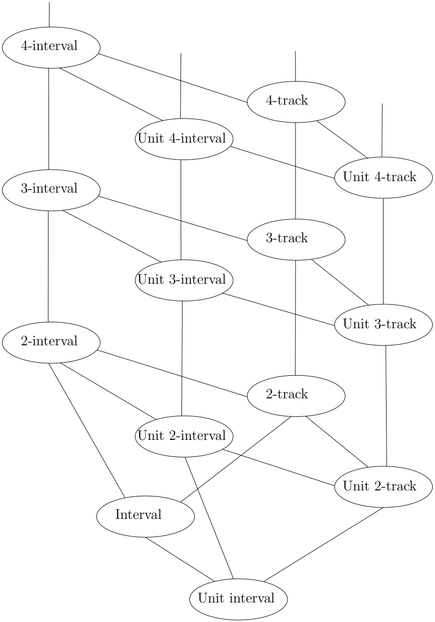
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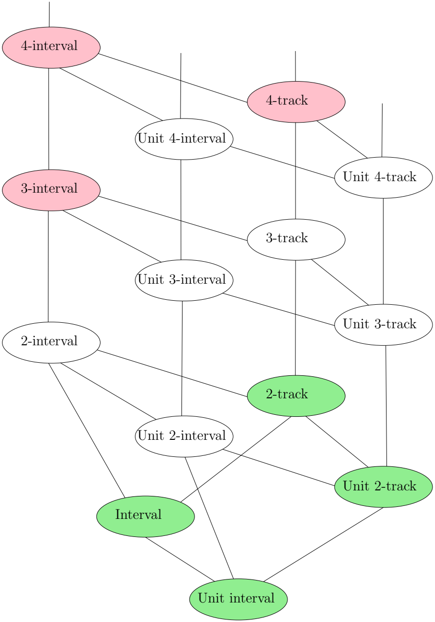
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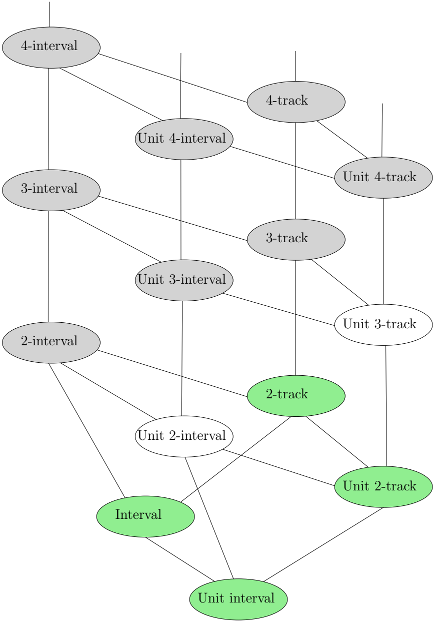
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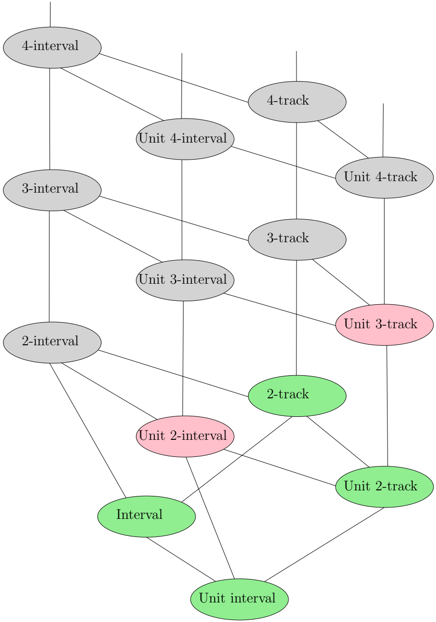
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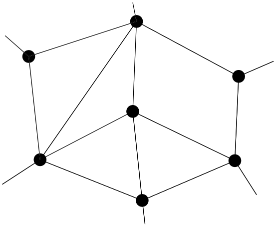
The "even subdivision" of a graph:

Given a graph G, construct G' by *even subdivision* of edges.



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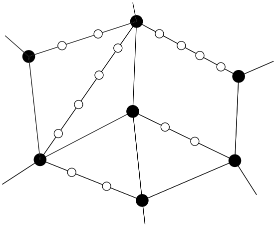

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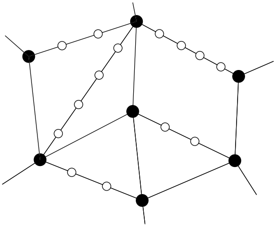

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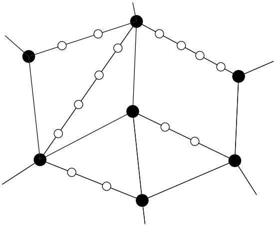

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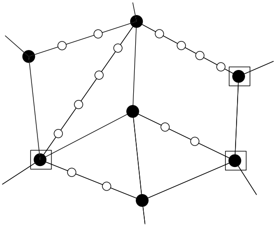

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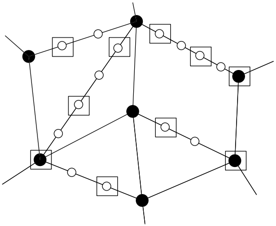

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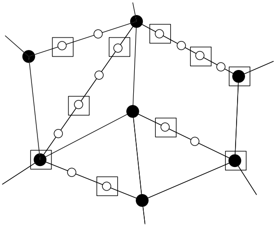

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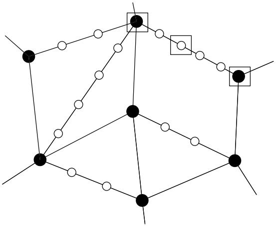

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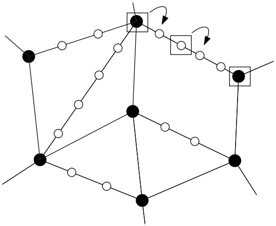

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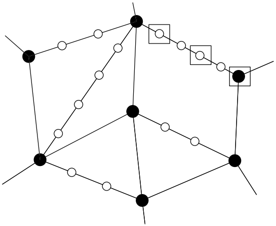

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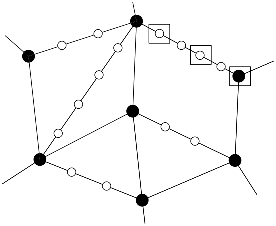

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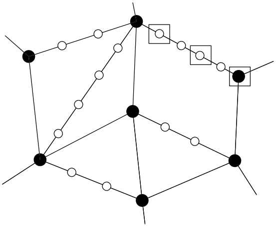

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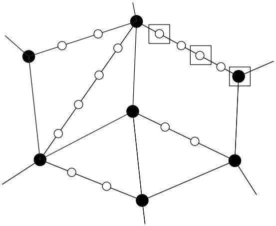

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An M.I.S. in G' $\xrightarrow{poly.time}$ an M.I.S. in G.

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Therefore, MAXIMUM INDEPENDENT SET is NP-hard in $\mathcal{C}$ as well.

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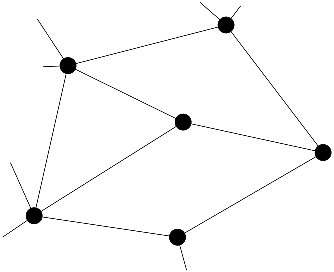

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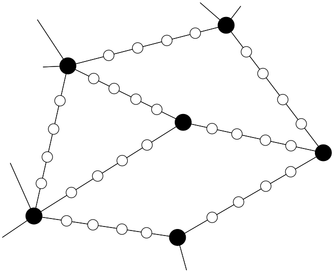

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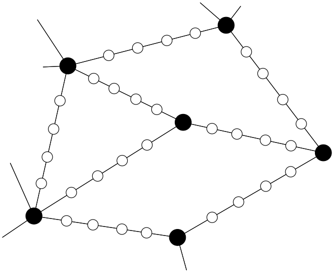

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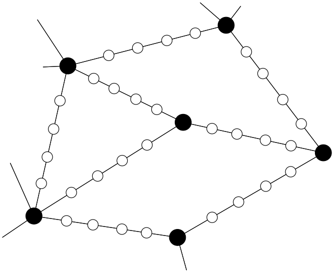

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X - set of original vertices of G.

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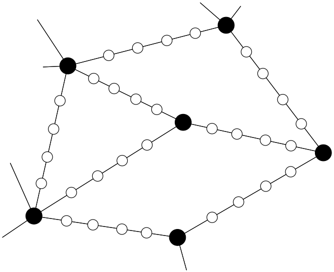
We show: The 4-subdivision of any graph is a co-2-interval graph.



Let G be any graph and G' its 4-subdivision.
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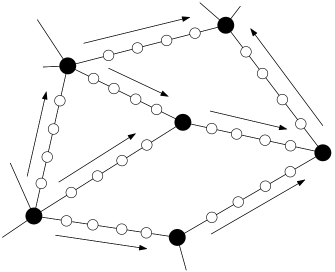

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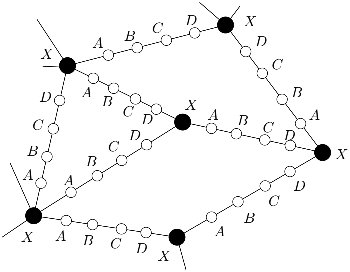

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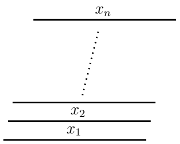
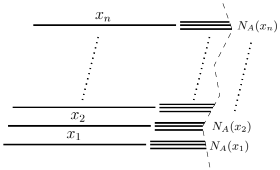
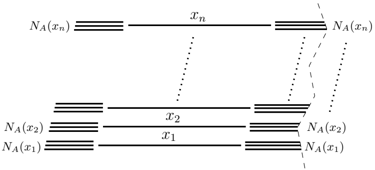
Remaining vertices are called “new” vertices.

Every path of length 5 between two vertices of X is given an arbitrary direction.

A new vertex is in A, B, C or D according as whether it occurs first, second, third or fourth in its path.


x_n
x_2
x_1


x_n
$N_A(x_n)$
x_2
$N_A(x_2)$
x_1
$N_A(x_1)$



$N_A(x_n)$
x_n
$N_A(x_n)$
x_2
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$N_A(x_2)$
x_1
$N_A(x_1)$
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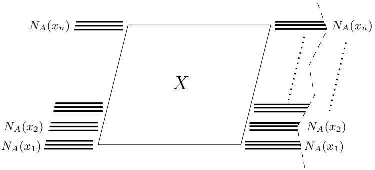
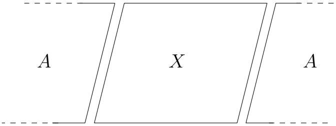
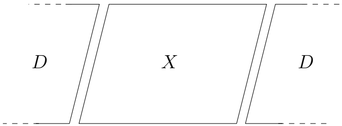
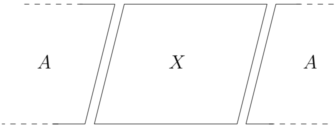
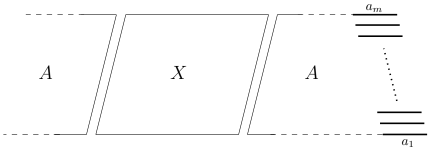
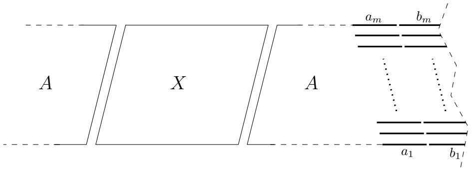

$N_A(x_n)$
X
$N_A(x_n)$
$N_A(x_2)$
$N_A(x_2)$
$N_A(x_1)$
$N_A(x_1)$

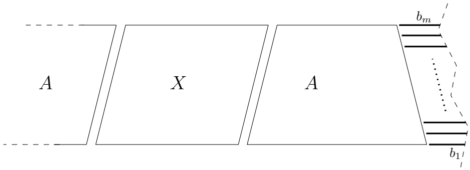

A
X
A


D
X
D

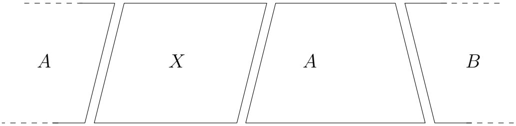

A
X
A


A
X
A
a_m
a_1

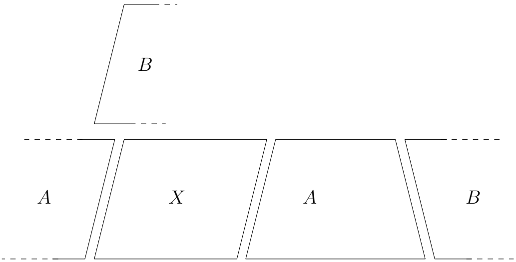

a_m
b_m
A
X
A
a_1
b_1

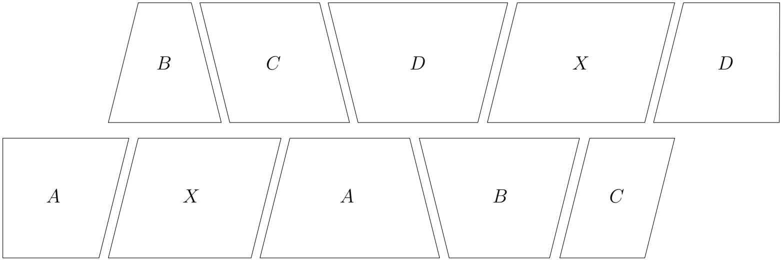


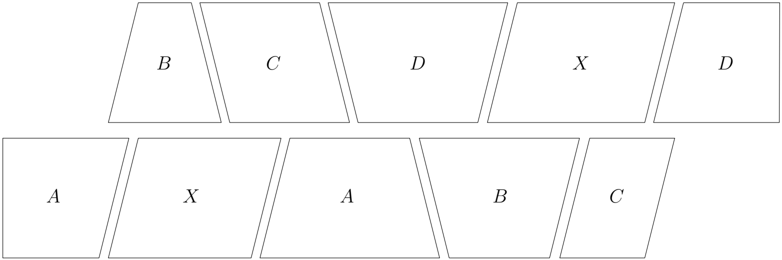
b_m
A
X
A
b_1



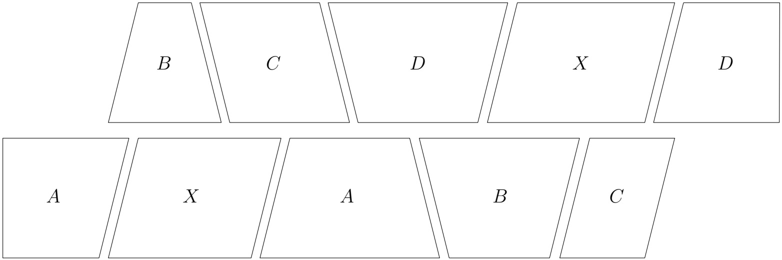
A
X
A
B


B
A
X
A
B

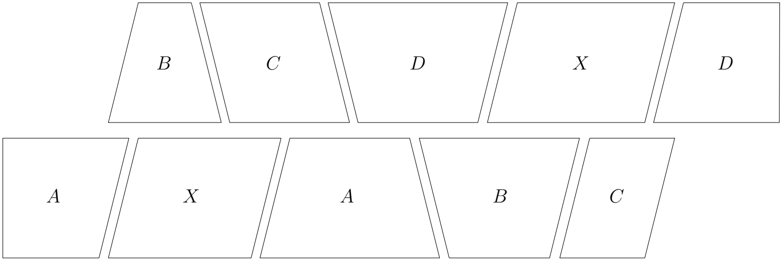

B
C
D
X
D
A
X
A
B
C



Given any graph G, its 4-subdivision G' is the complement a 2-interval graph.



Given any graph G, its 4-subdivision G' is the complement a 2-interval graph.
MAXIMUM INDEPENDENT SET is NP-hard in complements of 2-interval graphs.



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Approximation hardness

Theorem (Chlebík and Chlebíkova)

For any fixed even k, the MAXIMUM INDEPENDENT SET problem is APX-hard in k-subdivisions of 3-regular graphs.

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Algorithm $\mathcal{B}(\epsilon)$:
Input: G
Constructs G' and runs $\mathcal{A}(\epsilon)$ on G. Let I be the output of $\mathcal{A}(\epsilon)$.
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Output of $\mathcal{B}(\epsilon)$ is an independent set of G.

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As G is 3-regular, $\alpha \geq \frac{n}{4}$

$$\begin{aligned} |I| &\geq \frac{\alpha'}{(1+\epsilon)} - m \cdot \frac{k}{2} = \frac{\alpha + m \cdot \frac{k}{2}}{(1+\epsilon)} - m \cdot \frac{k}{2} = \frac{\alpha - \epsilon m \cdot \frac{k}{2}}{(1+\epsilon)} \\ &= \frac{\alpha - \epsilon \cdot \frac{3n}{2} \cdot \frac{k}{2}}{(1+\epsilon)} = \frac{\alpha - 3\epsilon k \cdot \frac{n}{4}}{(1+\epsilon)} \\ &\geq \frac{\alpha - 3\epsilon k \alpha}{(1+\epsilon)} \end{aligned}$$

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Approximation hardness

Theorem (Chlebík and Chlebíkova)

For any fixed even k, the MAXIMUM INDEPENDENT SET problem is APX-hard in k-subdivisions of 3-regular graphs.

Proof:

Thus, for every $\epsilon > 0$, $\mathcal{B}(\epsilon)$ is a $\left(\frac{1+\epsilon}{1-3\epsilon k}\right)$-approximation algorithm for MAXIMUM INDEPENDENT SET in 3-regular graphs.

But there can be no PTAS for MAXIMUM INDEPENDENT SET in 3-regular graphs unless P=NP, i.e., the problem is APX-hard [Alimonti and Kann '00].

Therefore, MAXIMUM INDEPENDENT SET in k-subdivisions of 3-regular graphs is also APX-hard.

Approximation hardness

Theorem (Chlebík and Chlebíkova)

For any fixed even k, the MAXIMUM INDEPENDENT SET problem is APX-hard in k-subdivisions of 3-regular graphs.

We have shown that given any graph, its 4-subdivision is the complement of a 2-interval graph, or a co-2-interval graph.

Approximation hardness

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We have shown that given any graph, its 4-subdivision is the complement of a 2-interval graph, or a co-2-interval graph. Therefore,

Theorem

MAXIMUM CLIQUE is APX-hard in 2-interval graphs.

Approximation hardness

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Theorem

MAXIMUM CLIQUE is APX-hard in 2-interval graphs.

Similar constructions show that:

- The 2-subdivision of any graph is co-3-track
- The 2-subdivision of any graph is co-unit-3-interval
- The 2-subdivision of any graph is co-unit-4-track

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Theorem

MAXIMUM CLIQUE is APX-hard in 3-track graphs.

Theorem

MAXIMUM CLIQUE is APX-hard in unit-3-interval graphs.

Theorem

MAXIMUM CLIQUE is APX-hard in unit-4-track graphs.

Unit 2-interval and unit 3-track graphs

Reduction from MAXIMUM INDEPENDENT SET for planar degree bounded graphs.
MAXIMUM INDEPENDENT SET remains NP-hard for planar graphs with degree at most 4.

Unit 2-interval and unit 3-track graphs

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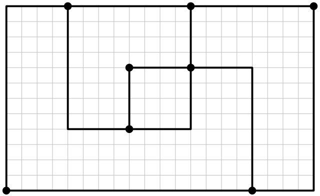

Every planar graph with $\Delta \leq 4$ can be "embedded" on a linear-sized rectangular grid [Valiant].
Vertices mapped to points with integer coordinates.
Edges are piecewise linear curves made up of horizontal and vertical segments whose end-points have integer coordinates.

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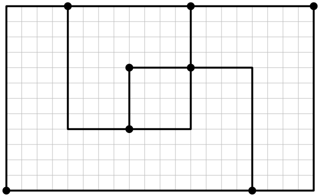

Given a planar graph G, take an embedding of it on such a grid.

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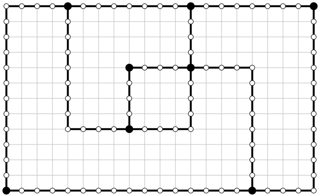

Given a planar graph G, take an embedding of it on such a grid.
Insert vertices at all integer points.

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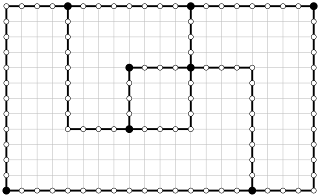

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We get a subdivision G' of G.

Unit 2-interval and unit 3-track graphs

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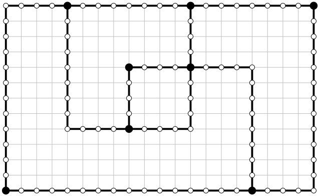

Given a planar graph G, take an embedding of it on such a grid.
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Not necessarily an even subdivision.

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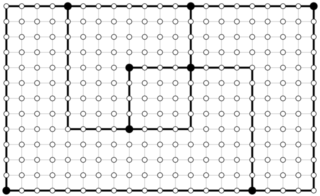

Given a planar graph G, take an embedding of it on such a grid.
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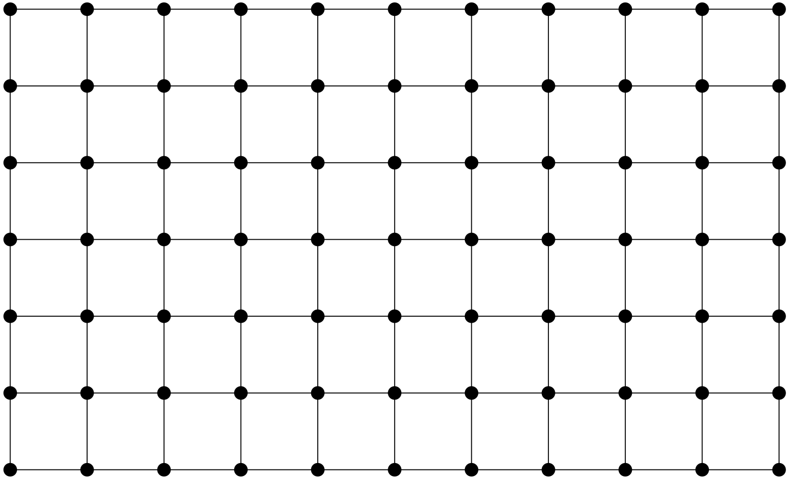
Not necessarily an even subdivision.
G' is an induced subgraph of the rectangular grid graph.

The weird grid



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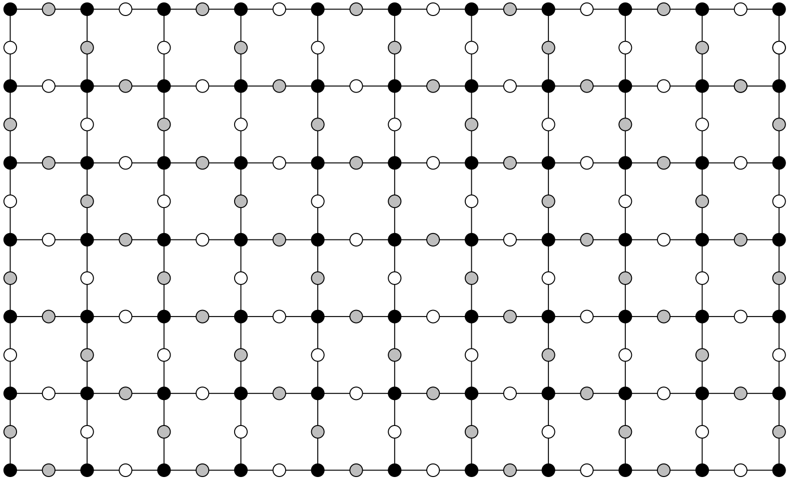
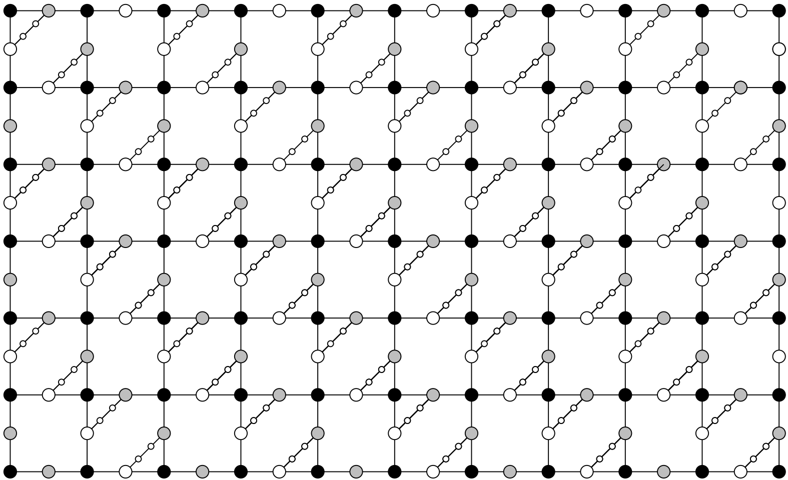


The weird grid



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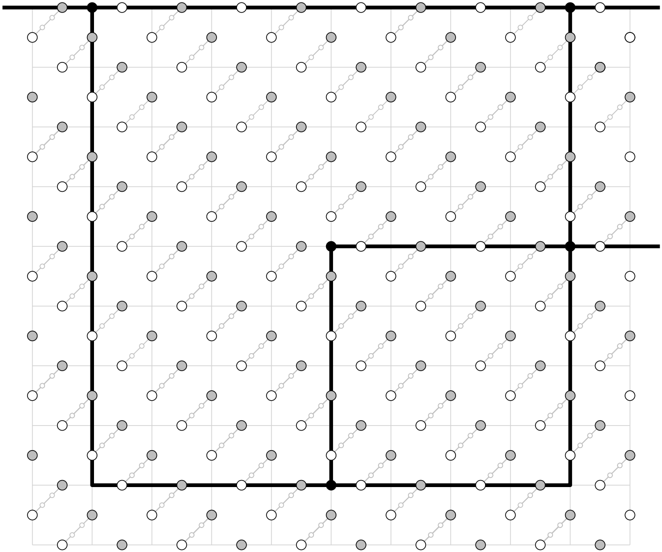

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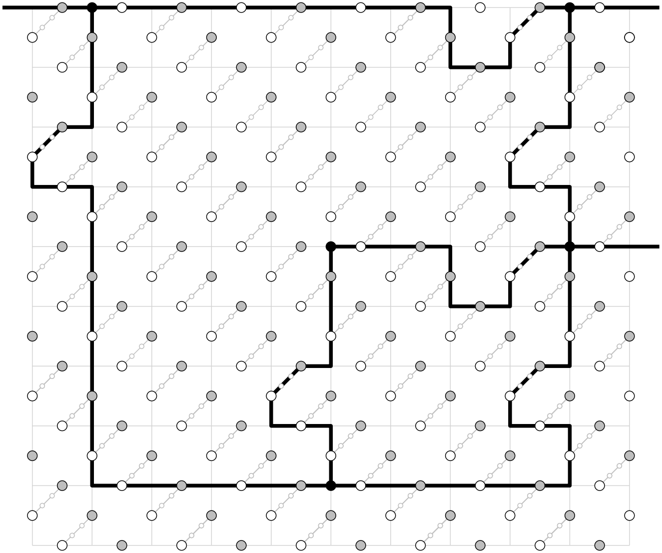

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We show:

The complement of the weird grid is both a unit 2-interval graph and a unit 3-track graph.

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We show:

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Therefore:

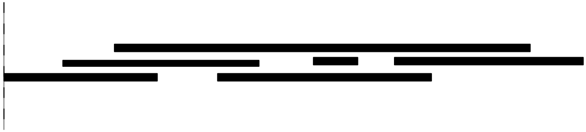
Theorem

MAXIMUM CLIQUE is NP-hard on unit 2-interval graphs.

Theorem

MAXIMUM CLIQUE is NP-hard on unit 3-track graphs.

Approximation algorithm



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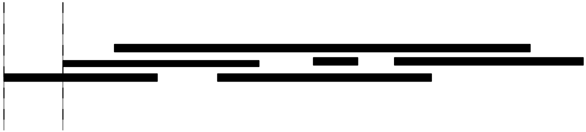

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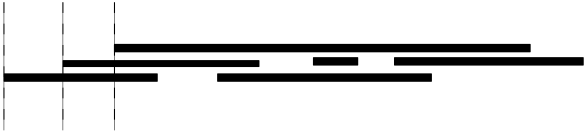

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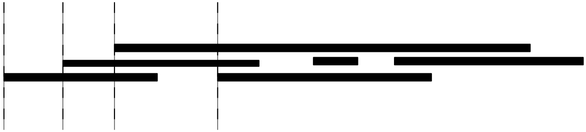

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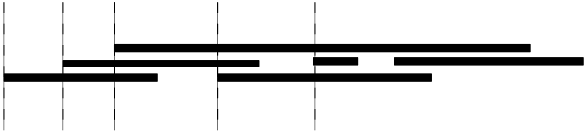

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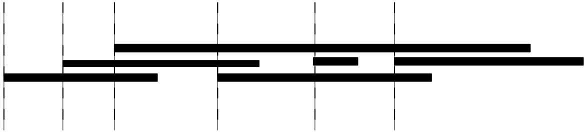

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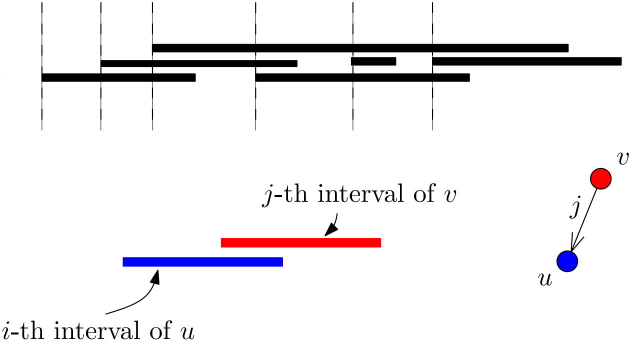

Some more results

MAXIMUM CLIQUE on circular analogues of t-interval and t-track graphs.

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The complement of the 2-subdivision of any graph is both a circular 2-interval graph and a circular 2-track graph.

Theorem

MAXIMUM CLIQUE is APX-hard in circular 2-interval graphs.

Theorem

MAXIMUM CLIQUE is APX-hard in circular 2-track graphs.

Corollary

MAXIMUM CLIQUE is NP-complete on unit circular 2-interval graphs.

Open problems

- Is there a PTAS for MAXIMUM CLIQUE in unit 2-interval graphs and unit 3-track graphs or are the problems APX-hard?
- Can the approximation ratio of t for MAXIMUM CLIQUE in t-interval graphs be improved? Not better than $O(t^{1-\epsilon})$.
- Is MAXIMUM CLIQUE NP-complete for unit circular 2-track graphs?
- What approximation ratio can be obtained if a representation of the graph is not known?