Lower bounds on odd perfect numbers

Pascal Ochem, Michael Rao

28 octobre 2012

Perfect numbers

- A number equal to the sum of its proper divisors.
- Examples : 6=1+2+3, 28=1+2+4+7+14

$\sigma_i(N)$

- $\sigma_i(N) = \Sigma_{d|N} d^i$.
- N is perfect : $\sigma_1(N) = 2N$.
- N is perfect : $\sigma_{-1}(N) = \sigma_1(N)/N = 2$. (abundancy)
- $GCD(a, b) = 1$ implies $\sigma_i(ab) = \sigma_i(a)\sigma_i(b)$.
- $\sigma_i(p_1^{e_1} p_2^{e_2} p_3^{e_3} \dots) = \sigma_i(p_1^{e_1})\sigma_i(p_2^{e_2})\sigma_i(p_3^{e_3}) \dots$.
- $\sigma_1(p^e) = 1 + p + p^2 + \cdots + p^e = \frac{p^{e+1}-1}{p-1}$.
- $1 + \frac{1}{p} \leq \sigma_{-1}(p^e) < 1 + \frac{1}{p-1}$.

Even perfect numbers

- Suppose $2^k \| N$, $k \geq 1$, $\sigma_1(N) = 2N$.
- $\sigma_1(2^k) | \sigma_1(N)$, so $2^{k+1} - 1 | 2N$, so $2^{k+1} - 1 | N$.
- $N = 2^k \times (2^{k+1} - 1) \times i$ with i odd.

$$
\begin{aligned}
\sigma_{-1}(N) &= \sigma_{-1}(2^k \times (2^{k+1} - 1) \times i) \\
&\geq \sigma_{-1}(2^k \times (2^{k+1} - 1)) \\
&= (2^{k+1} - 1)/2^k \times \sigma_1(2^{k+1} - 1)/2^{k+1} - 1 \\
&= 2^{-k} \times \sigma_1(2^{k+1} - 1) \\
&\geq 2^{-k} \times (1 + (2^{k+1} - 1)) = 2
\end{aligned}
$$

So, N is an even perfect number iff $N = 2^k \times (2^{k+1} - 1)$ and $2^{k+1} - 1$ is prime.

Odd perfect numbers

- Suppose N is odd and $\sigma_1(N) = 2N$.
- [Euler]
  $N = p^e m^2$ with p prime, $p \nmid m$, and $p \equiv e \equiv 1 \pmod 4$.
- Because $2 \| \sigma_1(N)$, so $2 \| \sigma_1(p^e)$ and $\sigma_1(m^2)$ is odd.
- p is the *special prime*. p^e is the *special component*.

Odd perfect numbers

- [Brent, Cohen, Riele 1991] $N > 10^{300}$.
  [O., Rao 2012] $N > 10^{1500}$ $\quad$ ($N > 10^{1600}$, unpublished).
- [Nielsen 2007] (distinct prime factors) $\omega(N) \geq 9$.
- [O., Rao 2012] (prime factors with multiplicity)
  $\Omega(N) \geq \max(101,\ 2\omega(N) + 51,\ (18\omega(N) - 31)/7)$.
- [Nielsen 2003] $N < 2^{4^{\omega(N)}}$.
- [Goto, Ohno 2008] One prime factor $> 10^8$.
- [Iannucci 1999] Two prime factors $> 10^4$.
- [Iannucci 2000] Three prime factors $> 10^2$.
- [O., Rao 2012] One component $> 10^{62}$.

Factor chains

Suppose $3^2 \| N$. Then $\sigma_1(3^2) | \sigma_1(N)$, i.e., $13 | N$.
Suppose $13^1 \| N$. Then $2 \cdot 7 | \sigma_1(N)$, i.e., $7 | N$. (Special prime)

$3^2 \Longrightarrow 13$
$13^1 \Longrightarrow 2 \cdot 7$
$7^2 \Longrightarrow 3 \cdot 19 \quad [3^2 \cdot 7^2 \cdot 13 \cdot 19^2 > 10^6]$
$7^4 \Longrightarrow 2801 \quad [3^2 \cdot 7^4 \cdot 13 \cdot 2801^2 > 10^6]$
7^e, $e \geq 6 \quad [3^2 \cdot 7^6 \cdot 13 > 10^6]$
$13^2 \Longrightarrow 3 \cdot 61$
$61^1 \Longrightarrow 2 \cdot 31 \quad [3^2 \cdot 13^2 \cdot 61 \cdot 32^2 > 10^6]$
61^e, $e \geq 2 \quad [3^2 \cdot 13^2 \cdot 61^2 > 10^6]$
$13^4 \Longrightarrow 30941 \quad [13^4 \cdot 3094 > 10^6]$
13^e, $e \geq 5 \quad [3^2 \cdot 13^5 > 10^6]$

Factor chains

$3^4 \Longrightarrow 11^2$

$\quad 11^2 \Longrightarrow 7 \cdot 19 \quad [\sigma_{-1}(3^4 \cdot 7^2 \cdot 11^2 \cdot 19^2) > 2]\ [> 10^6]$

$\quad 11^e,\ e \geq 4 \quad [3^4 \cdot 11^4 > 10^6]$

$3^6 \Longrightarrow 1093$

$\quad 1093^1 \Longrightarrow 2 \cdot 547 \quad [> 10^6]$

$\quad 1093^e,\ e \geq 2 \quad [3^6 \cdot 1093^2 > 10^6]$

$3^8 \Longrightarrow 13 \cdot 757 \quad [3^8 \cdot 13^2 \cdot 757 > 10^6]$

$3^{10} \Longrightarrow 23 \cdot 3851 \quad [3^{10} \cdot 23^2 \cdot 3851^2 > 10^6]$

$3^{12} \Longrightarrow 797161 \quad [3^{12} \cdot 797161 > 10^6]$

$3^e,\ e \geq 14 \quad [3^{14} > 10^6]$

Factor chains

$5^1 \Longrightarrow 2 \cdot 3$ [3 is forbidden]

$5^2 \Longrightarrow 31$

$31^2 \Longrightarrow 3 \cdot 331$ [3 is forbidden]

31^e, $e \geq 4$ $[5^2 \cdot 31^4 > 10^6]$

$5^4 \Longrightarrow 11 \cdot 71$ $[5^4 \cdot 11^2 \cdot 71^2 > 10^6]$

$5^5 \Longrightarrow 2 \cdot 3^2 \cdot 7 \cdot 13$ [3 is forbidden]

$5^6 \Longrightarrow 19531$ $[5^6 \cdot 19531^2 > 10^6]$

$5^8 \Longrightarrow 19 \cdot 31 \cdot 8291$ $[5^8 \cdot 19^2 \cdot 31^2 \cdot 8291^2 > 10^6]$

5^e, $e \geq 10$ $[5^{10} > 10^6]$

Final argument

- Suppose N is an odd perfect number such that $GCD(N, 3 \cdot 5) = 1$.
- Suppose N has at most 5 distinct prime factors.
- $\sigma_{-1}(N) < \sigma_{-1}(7^{\infty} \cdot 11^{\infty} \cdot 13^{\infty} \cdot 17^{\infty} \cdot 19^{\infty}) = 1.5592 \cdots < 2$.
- Contradiction : N has at least 6 distinct prime factors.
- So $N \geq 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 > 10^6$.

$N > 10^{1500}$

- We forbid $\{127, 19, 7, 11, 331, 31, 97, 61, 13, 398581, 1093, 3, 5, 307, 17\}$.
- Improved final argument.
- We circumvent roadblocks.

Example of roadblock :

$7^4 \Longrightarrow 2801$

$2801^{82} \Longrightarrow C283$

Circumventing roadblocks

Example of roadblock :
$7^4 \Longrightarrow 2801$
$2801^{82} \Longrightarrow C283$

So $N = 7^4 \cdot 2801^{82} \cdot i$.
Let p be the smallest prime of i. Suppose $p \geq 641$.
Then i has ≥ 341 prime factors, otherwise
$\sigma_{-1}(N) < \sigma_{-1}(7^4 \cdot 2801^{82}) \times (1 + 1/640)^{340} < 2$.
And then $N > 7^4 \cdot 2801^{82} \cdot 641^{341} > 10^{1500}$.

So $p < 641$.
We branch on every prime < 641 to rule them out.

Circumventing roadblocks recursively

Example of roadblock :

$7^4 \Longrightarrow 2801$

$\quad 2801^{82} \Longrightarrow C283$

We branch on every prime < 641 to rule them out.

So we have to branch on the prime 97 and on the component 97^1, and we hit a new roadblock :

$7^4 \Longrightarrow 2801$

$\quad 2801^{82} \Longrightarrow C283$

$\quad\quad 97^1 \Longrightarrow 2 \cdot 7^2$

$\Omega(N) \geq (18\omega(N) - 31)/7$ - variables

- $p = \omega(N)$: number of distinct prime factors
- $f = \Omega(N)$: total number of prime factors
- p_2 : number of distinct prime factors with exponent 2, distinct from 3
- $p_{2,1}$: number of distinct prime factors with exponent 2 congruent to 1 mod 3
- p_4 : number of distinct prime factors with exponent at least 4, distinct from 3 and the special prime
- f_4 : total number of prime factors with exponent at least 4, distinct from 3 and the special prime
- e : exponent of the special prime
- f_3 : exponent of the prime 3

$\Omega(N) \geq (18\omega(N) - 31)/7$ - inequalities

(1) $1 \leq e$

(2) $e + f_3 + 2p_2 + f_4 = f$

(3) $4p_4 \leq f_4$

(4) $p_{2,1} \leq f_3$

(5) $p \leq f_3/2 + 1 + p_2 + p_4$

(6) $p \leq 2 + p_2 + p_4$

(7) $f \leq (18p - 32)/7$

(8) $2p_2 \leq 1 + e + 3p_{2,1} + p_4 + f_4$

The combination
$5\times(1)+7\times(2)+5\times(3)+6\times(4)+2\times(5)+16\times(6)+7\times(7)+2\times(8)$
gives $1 \leq 0$, a contradiction.
So (7) is false, thus $\Omega(N) \geq (18\omega(N) - 31)/7$.