Limits of near-coloring of sparse graphs

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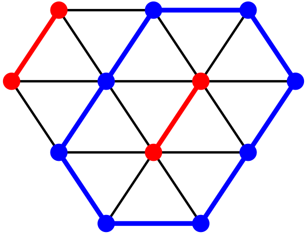
Near-coloring?

Definition - Near-Coloring

A graph G is $(d_1, \ldots, d_k)$-colorable, if and only if,

- $V(G) = V_1 \dot{\cup} \ldots \dot{\cup} V_k$
- $\forall i \in [1, k], \Delta(G[V_i]) \leq d_i$



$(2, 1)$-coloring

$$\underbrace{(0, \ldots, 0)}_{k}\text{-coloring} \Leftrightarrow \text{proper } k\text{-coloring}$$

- $V(G) = V_1 \dot{\cup} \ldots \dot{\cup} V_k$
- $\forall i \in [1, k], \Delta(G[V_i]) = 0$

$$\underbrace{(d, \ldots, d)}_{k}\text{-coloring} \Leftrightarrow d\text{-improper } k\text{-coloring}$$

- $V(G) = V_1 \dot{\cup} \ldots \dot{\cup} V_k$
- $\forall i \in [1, k], \Delta(G[V_i]) \leq d$

Some history

4CT - Appel and Haken '76

Every planar graph is $(0, 0, 0, 0)$-colorable.

Theorem - Cowen, Cowen, and Woodall '86

Every planar graph is $(2, 2, 2)$-colorable.

[list version: Eaton and Hull '99, Škrekovski '99]

Sparse graphs?

$\Rightarrow$ graph with small maximum average degree

$$\mathrm{mad}(G) = \max\left\{\frac{2|E(H)|}{|V(H)|}, H \subseteq G\right\}$$

Theorem - Havet and Sereni '06

Every graph G with $\mathrm{mad}(G) < k + \frac{kd}{k+d}$ is d-improperly k-colorable (in fact d-improperly k-choosable), i.e. $(\underbrace{d, \ldots, d}_{k})$-coloring.

Asymptotically sharp:

Theorem - Havet and Sereni '06

There exists a non-d-improperly k-colorable graph whose maximum average degree tends to $2k$ when d goes to infinity.

What about $(d, 0)$-coloring?

Bipartition V_1, V_2 of $V(G)$ such that:

$$\Delta(G[V_1]) \leq d \text{ and } G[V_2] \text{ is a stable set}$$

$(1,0)$-coloring

Theorem - Glebov and Zambalaeva '07

Every planar graph G with $g(G) \geq 16$ is $(1,0)$-colorable.

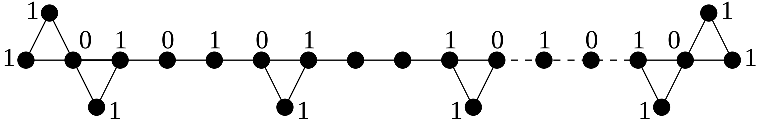
Theorem - Borodin and Ivanova '09

Every graph G with $\mathrm{mad}(G) < \frac{7}{3}$ is $(1,0)$-colorable.

$\Rightarrow$ Every planar graph G with $g(G) \geq 14$ is $(1,0)$-colorable.

Theorem - Borodin and Kostochka '11

Every graph G with $\mathrm{mad}(G) \leq \frac{12}{5}$ is $(1,0)$-colorable.
Moreover $\frac{12}{5}$ is **sharp**.



$\Rightarrow$ Every planar graph G with $g(G) \geq 12$ is $(1,0)$-colorable.

Question

Smallest g such that all planar graphs G with $g(G) \geq g$ are $(1,0)$-colorable?

[Esperet, M. , Ochem, and Pinlou '12: $g \geq 10$
(there exist non-(1,0)-colorable planar graphs with girth 9).]

$(d, 0)$-coloring

Theorem - Borodin, Ivanova, M., Ochem, and Raspaud '10

- Let $d \geq 2$. Every graph G with $\mathrm{mad}(G) < 3 - \frac{2}{d+2}$ is $(d, 0)$-colorable.
- There exist non-$(d, 0)$-colorable graphs G with $\mathrm{mad}(G) = (3 - \frac{2}{d+2}) + \frac{1}{d+3}$.

Asymptotically sharp.

Theorem - Borodin and Kostochka '11

Let $d \geq 2$. Every graph G with $\mathrm{mad}(G) \leq 3 - \frac{1}{d+1}$ is $(d, 0)$-colorable.
Moreover $3 - \frac{1}{d+1}$ is **sharp**.

Problem

- $(\underbrace{d, \ldots, d}_{k})$-coloring
- $(d, 0)$-coloring

$$(\underbrace{d, \ldots, d}_{a}, \underbrace{0, \ldots, 0}_{b})\text{-coloring?}$$

Partition of V in $a + b$ sets:

"a" subgraphs with maximum degree at most d

"b" stable sets

What happens when $d \to \infty$?

Observation

[Havet and Sereni '06]

$$G : \mathrm{mad}(G) < \underbrace{k + \frac{kd}{k+d}}_{\to 2k} \Rightarrow (\underbrace{d, \dots, d}_{k})\text{-coloring}$$

sharp

[Borodin and Kostochka '11]

$$G : \mathrm{mad}(G) < \underbrace{3 - \frac{1}{d+1}}_{\to 3} \Rightarrow (d, 0)\text{-coloring}$$

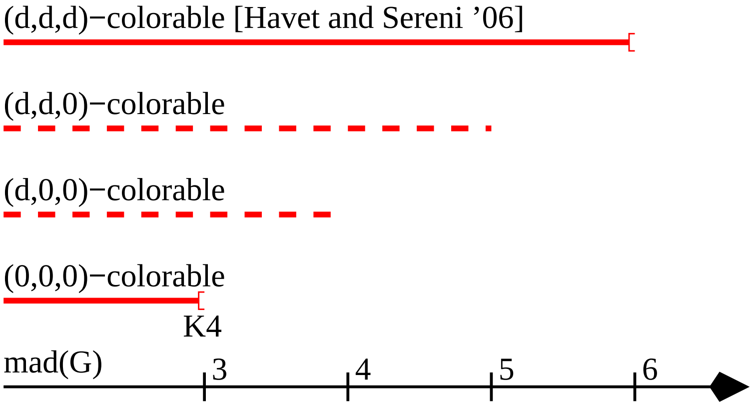
sharp

Question

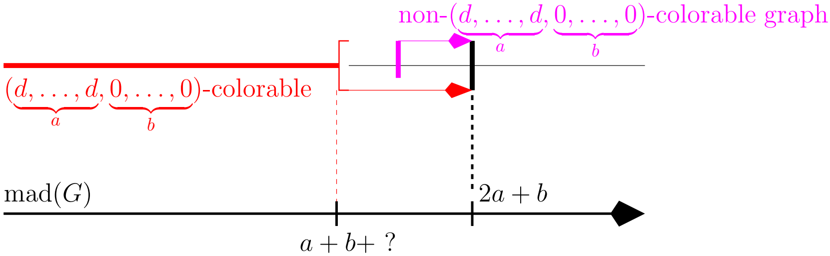
$$G : \mathrm{mad}(G) \to ? \Rightarrow (\underbrace{d, \dots, d}_{a}, \underbrace{0, \dots, 0}_{b})\text{-coloring}$$

Largest value m such that every graph with $\mathrm{mad} < m$ is $(\underbrace{d, \dots, d}_{a}, \underbrace{0, \dots, 0}_{b})$-colorable ($d \to \infty$)?

Case: (?,?,?)-coloring



Idea



non-$(\underbrace{d, \ldots, d}_{a}, \underbrace{0, \ldots, 0}_{b})$-colorable graph
$(\underbrace{d, \ldots, d}_{a}, \underbrace{0, \ldots, 0}_{b})$-colorable
mad(G)
$2a + b$
$a + b +$?

Limits

Notation: $(\underbrace{d,\dots,d}_{a},\underbrace{0,\dots,0}_{b})$-coloring $\Leftrightarrow$ $(d,a,b)^*$-coloring

Theorem - Dorbec, Kaiser, M. and Raspaud '12

Let $a+b>0$ and $d>0$.

- Every graph G with $\mathrm{mad}(G) < a+b+\frac{da(a+1)}{(a+d+1)(a+1)+ab}$ is $(d,a,b)^*$-colorable.
- There exist non-$(d,a,b)^*$-colorable graphs G with $\mathrm{mad}(G) = 2a+b-\frac{2}{(d+1)(b+1)-1}+\frac{2a+2}{(d+1)^{a+1}(b+1)^{a+1}-1}$.

Asymptotically sharp.

Answer

Largest value m such that every graph with $\mathrm{mad} < m$ is $(\underbrace{d,\dots,d}_{a},\underbrace{0,\dots,0}_{b})$-colorable?

When $d\to\infty$: $2a+b$.

Sketch of the proof

[1] reducible configurations $+$ discharging procedure

[2] exhibit a non-$(d, a, b)^*$-colorable graph G $+$ compute $\mathrm{mad}(G)$

[1] - $(d, a, b)^*$-coloring

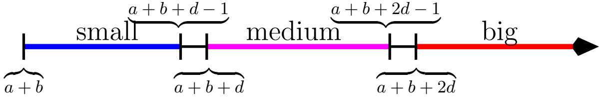
Let G be a counterexample with the minimum order.

Claim 0

$\delta(G) \geq a + b$

Define 3 objects:

Small vertex	$v : d_G(v) \leq a + b + d - 1$
Medium vertex	$v : a + b + d \leq d_G(v) \leq a + b + 2d - 1$
Big vertex	$v : d_G(v) \geq a + b + 2d$



Reducible configurations

Claim 1

A small vertex is adjacent to at least "a" non-small vertices.

Light small vertex:
A small vertex adjacent to exactly "a" non-small vertices.

Claim 2

A medium vertex is adjacent to at least "$a-1$" non-small vertices and to at least "$a-1+b$" non-(light small) vertices

Claim 3

A big vertex is adjacent to at least "b" non-(light small) vertices

Discharging procedure - Aim

Set $m = \mathrm{mad}(G)$

Step 1 Assign to each vertex a charge equal to its degree:

$$\forall v \in V(G), \omega(v) = d_G(v)$$

Observe that:

$$\sum_{v \in V(G)} \omega(v) < |V(G)| \cdot m$$

Step 2 Move charges in order to have on each vertex v a new charge $\omega^*(v)$ such that :

$$\forall v \in V(G), \omega^*(v) \geq m$$

Step 3 The contradiction completes the proof:

$$|V(G)| \cdot m \leq \sum_{v \in V(G)} \omega^*(v) = \sum_{v \in V(G)} \omega(v) < |V(G)| \cdot m$$

Observation

- Small vertices need charge.
- Medium vertices have enough charge but not too much.
- Big vertices have charge.

Idea

R1. A medium/big vertex gives r_1 to each adjacent light small vertex.

R2. A medium/big vertex gives r_2 to each adjacent small vertex that is not light.

(With $r_1 \geq r_2$)

Let v be a vertex of degree k.

[light small] $v : a + b \leq d_G(v) \leq a + b + d - 1$

$$\begin{aligned} \omega^*(v) &\geq k + a \times r_1 \text{ by R1.} \\ &\geq a + b + a \times r_1 \geq m \end{aligned}$$

[non-light small] $v : a + b \leq d_G(v) \leq a + b + d - 1$

$$\begin{aligned} \omega^*(v) &\geq k + (a + 1) \times r_2 \text{ by R2.} \\ &\geq a + b + (a + 1) \times r_2 \geq m \end{aligned}$$

[medium] $v : a + b + d \leq d_G(v) \leq a + b + 2d - 1$

$$\begin{aligned} \omega^*(v) &\geq k - (k - a - b + 1)r_1 - (a + b - 1 - (a - 1))r_2 \text{ by R1, R2, C2} \\ &\geq a + b + d - (d + 1)r_1 - br_2 \geq m \end{aligned}$$

[big] $v : d_G(v) \geq a + b + 2d$

$$\begin{aligned} \omega^*(v) &\geq k - (k - b)r_1 - br_2 \text{ by R1, R2, C3} \\ &\geq a + b + 2d - (a + 2d)r_1 - br_2 \geq m \end{aligned}$$

$$\begin{array}{rl}
\text{[light small]} & a + b + a \times r_1 \geq m \\
\text{[non-light small]} & a + b + (a+1) \times r_2 \geq m \\
\text{[medium]} & a + b + d - (d+1)r_1 - br_2 \geq m \\
\text{[big]} & a + b + 2d - (a+2d)r_1 - br_2 \geq m \\
\hline
\end{array}$$

Find r_1, r_2, m maximizing m

$$r_1 = \frac{d(a+1)}{(a+d+1)(a+1)+ab}$$

$$r_2 = \frac{da}{(a+d+1)(a+1)+ab}$$

$$m = a + b + \frac{da(a+1)}{(a+d+1)(a+1)+ab}$$

$$\forall v \in V(G), \omega^*(v) \geq m$$

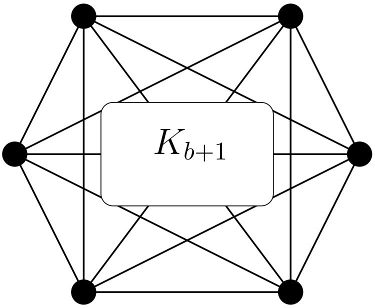
$$|V(G)| \cdot m \leq \sum_{v \in V(G)} \omega^*(v) = \sum_{v \in V(G)} \omega(v) < |V(G)| \cdot m$$

$\square$

[2] a non-$(d, a, b)^*$-colorable graph: $G_{d,a,b}$

By induction on a.

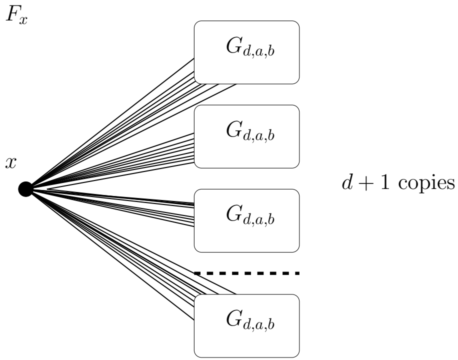
Case $a = 0$: $G_{d,0,b} = K_{b+1}$.



not $(d, 0, b)^*$-colorable

[2] a non-$(d, a, b)^*$-colorable graph: $G_{d,a,b}$

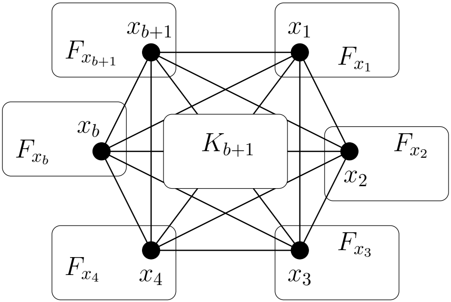
From a to $a + 1$.



$G_{d,a,b}$ not $(d, a, b)^*$-colorable $\Rightarrow$
in any $(d, a + 1, b)^*$-coloring, each copy contains a vertex of each color I_i for $1 \leq i \leq a + 1$ $\Rightarrow$
x must be colored with color 0_i for some $i \in \{1, ..., b\}$

[2] a non-$(d, a, b)^*$-colorable graph: $G_{d,a,b}$

From a to $a+1$: $G_{d,a+1,b}$



not $(d, a+1, b)^*$-colorable

[2] $\mathrm{mad}(G_{d,a,b})$?

Not so easy...

$$n = (b+1)\frac{(d+1)^{a+1}(b+1)^{a+1} - 1}{(d+1)(b+1) - 1}$$

$$e = (b+1)\frac{(d+1)^{a+1}(b+1)^{a+1}((a+\frac{b}{2})((d+1)(b+1)-1)-1) - \left(\frac{b}{2}-1\right)(d+1)(b+1) + \frac{b}{2}}{((d+1)(b+1)-1)^2}$$

$$mad(G) = ad(G) = \frac{2e}{n}$$

Conclusion

$$\mathfrak{f}(d,a,b) = a + b + \frac{da(a+1)}{(a+d+1)(a+1)+ab}$$

$$\mathfrak{g}(d,a,b) = 2a+b-\frac{2}{(d+1)(b+1)-1}+\frac{2a+2}{(d+1)^{a+1}(b+1)^{a+1}-1}$$

asymptotically sharp when $d \to \infty$

Largest value m such that every graph with $\mathrm{mad} < m$ is $(d,a,b)^*$-colorable?

$$\mathfrak{f}(d,a,b) \leq m < \mathfrak{g}(d,a,b)$$