Les matroïdes orientés en tant que graphes signés

Oriented matroids as signed graphs

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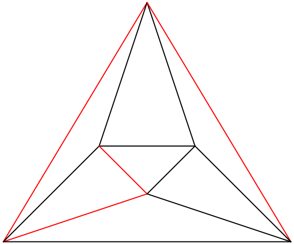
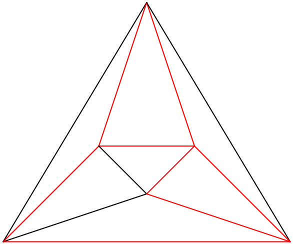
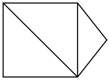




Let G be a connected graph :



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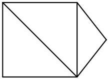
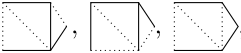
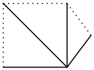
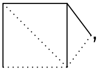


And let $\mathcal{F}$ be the set of its spanning trees :

$\mathcal{F} = \{$  $, \dots \}$

What is the key property of this set of spanning trees, for a given graph?

Let's take two of them:  and , say F_1 and F_2.

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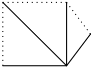
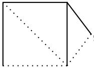
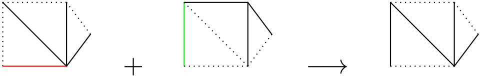
Let's take two of them:  and , say F_1 and F_2.

If I delete one edge in F_1 that is not in F_2, I can then add an edge of F_2 to F_1 to reobtain a spanning tree:



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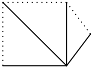
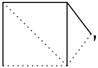
Let's take two of them:  and , say F_1 and F_2.

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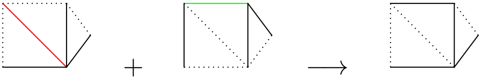

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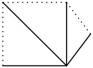
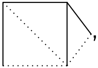
Let's take two of them:  and , say F_1 and F_2.

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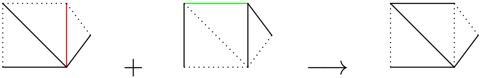

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1. the set $\mathcal{B}$ is nonempty
2. for all $B_1 \neq B_2$ in $\mathcal{B}$, for all e in $B_1 \setminus B_2$, there exists f in $B_2 \setminus B_1$ so that $(B_1 - e + f)$ is in $\mathcal{B}$ (exchange axiom)

We call those set systems matroids. E is the ground set and elements of $\mathcal{B}$ are the bases of the matroid. The number of elements in a base is the rank of the matroid.

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Matroids obtained from graphs are a proper subclass of the class of matroids. Finite families of vectors also give rise to matroids (take for $\mathcal{B}$ the set of maximal linearly independent subfamilies).

We define a distance between two bases :

$$d(B_1, B_2) = \frac{B_1 \Delta B_2}{2}$$

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We then define a graph $BG_{\mathcal{B}}$ whose vertices are the elements of $\mathcal{B}$, with edges linking two vertices if and only if their distance is one.

The axiomatic can be a bit simplified : a set system $(E, \mathcal{B})$ is a matroid if and only if

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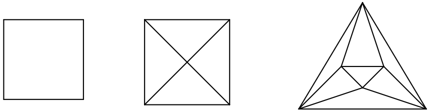
What happens locally for such bases in the graph?

If $(E, \mathcal{B})$ is a matroid, then the *closed common neighbourhood* of two bases at distance two is a C_4, a pyramid or an octahedron.



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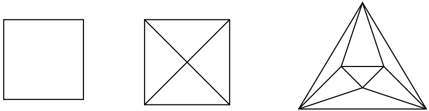

This is the easy part of the characterisation of matroid basis graphs. The other part is the characterisation of graphs that can be properly labelled by bases.

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- a nice homotopy property for paths (Maurer)
- what class of matroids do they represent

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We will only consider orientations of uniform matroids. The uniform matroid of rank r on n elements (we write $U(n, r)$ to denote it) is the matroid $(E = \{1, \ldots, n\}, \mathcal{P}_r(E))$

Axiomatics

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2. for all distincts $x_1, x_2, \ldots, x_{r-2}, a, b, c, d \in E$ the set formed by the three signs

$$\begin{aligned} \chi(x_1,\ldots,x_{r-2},a,b)\cdot\chi(x_1,\ldots,x_{r-2},c,d) \\ -\chi(x_1,\ldots,x_{r-2},a,c)\cdot\chi(x_1,\ldots,x_{r-2},b,d) \\ \chi(x_1,\ldots,x_{r-2},a,d)\cdot\chi(x_1,\ldots,x_{r-2},b,c) \end{aligned}$$

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Oriented matroids are the pairs $\{\chi, -\chi\}$.

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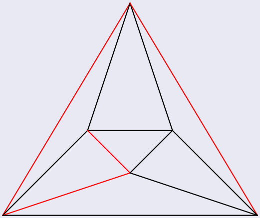
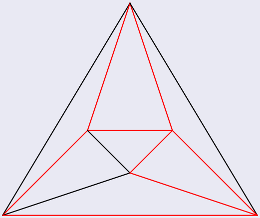
We can now give our principal theorem.

Our result

Theorem

Relabeling classes of oriented uniform matroids of rank r on n elements are in one-to-one correspondence with bicolorations of the edges of $J(n, r)$ such that each octahedron has one of the two coloration :





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It is conjectured that for all r and n, for every good bicoloration of $J(n, r)$, an authorized flip can be done for some vertex.

Work to be done

Generalize the characterisation to non-uniform oriented matroids.

The end

Thank you for your attention.