Torus Squarings

GK MDS
Monday lecture
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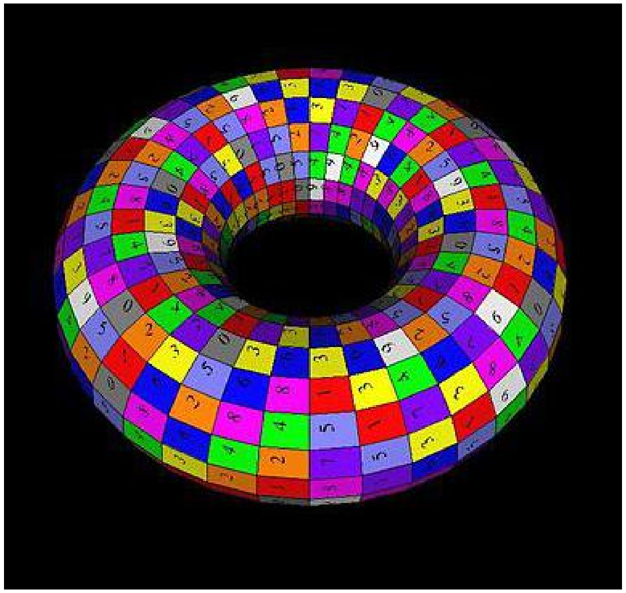
joint work with **Éric Fusy**



Overview

Rectangulations and Squarings

Segment Contact Representations

Segment Contacts on the Torus

Rectangular and Square Duals

Square Duals on the Torus

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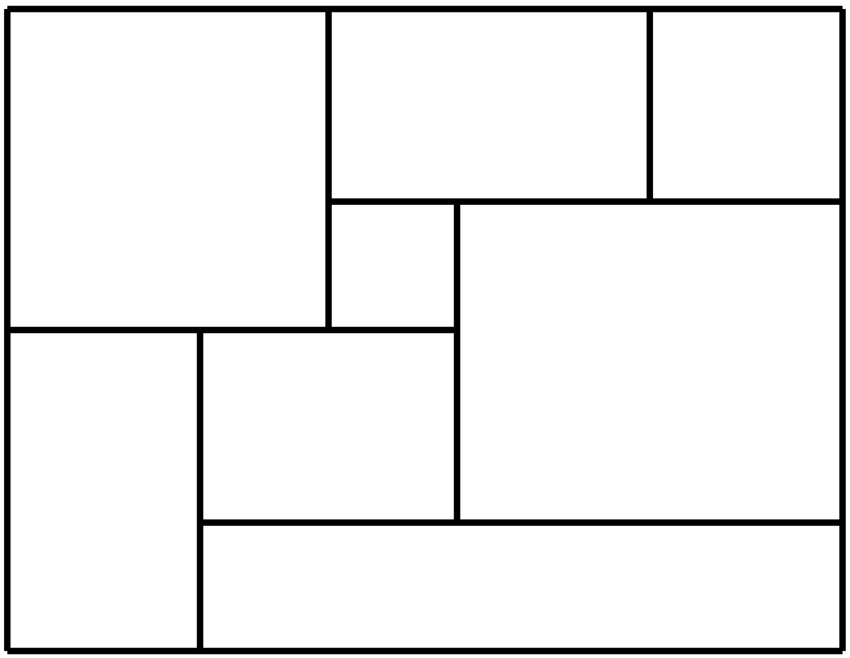
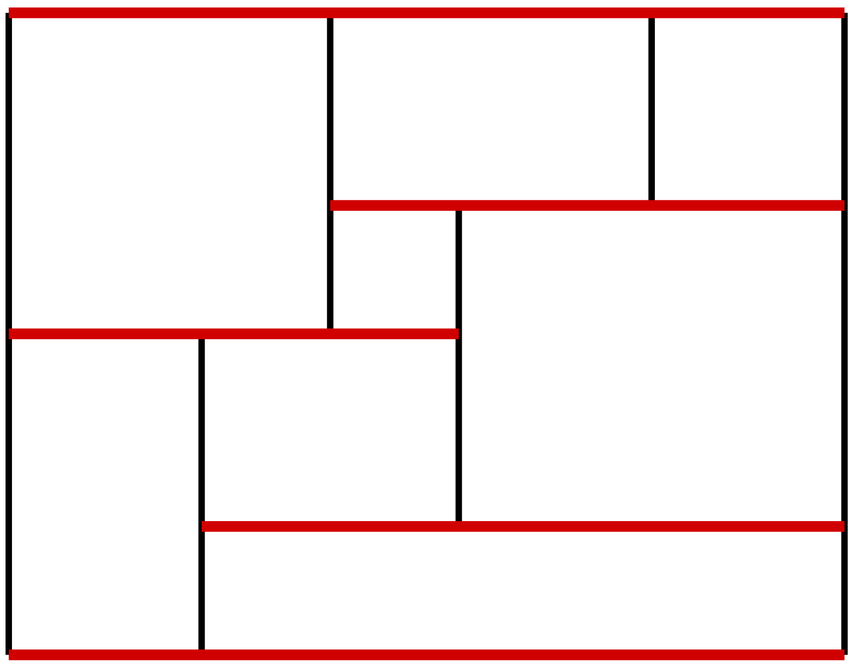
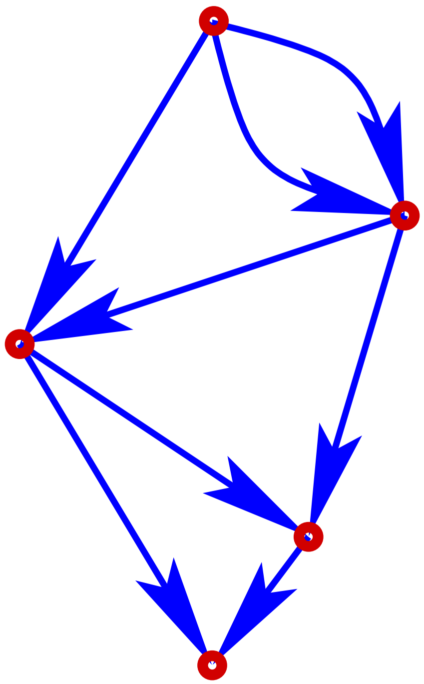
The Main Character



A rectangular dissection of a rectangle

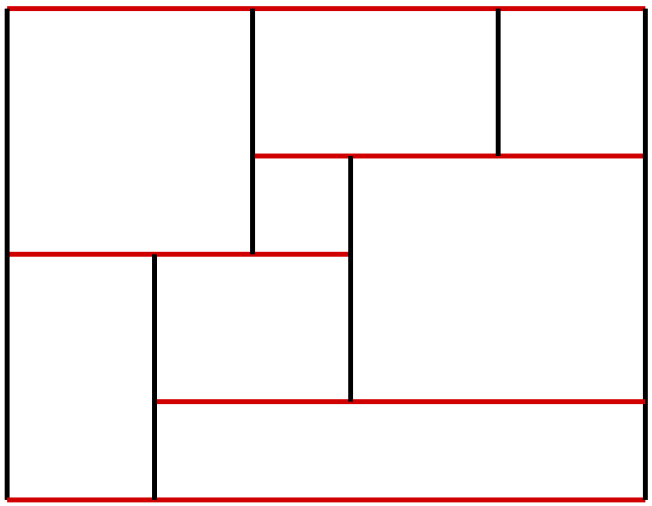
Rectangular Dissections and Graphs

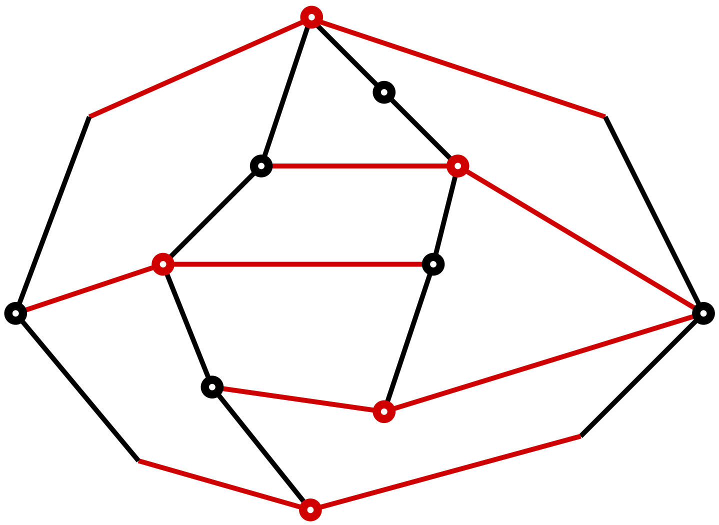




The bipolar graph induced by R.

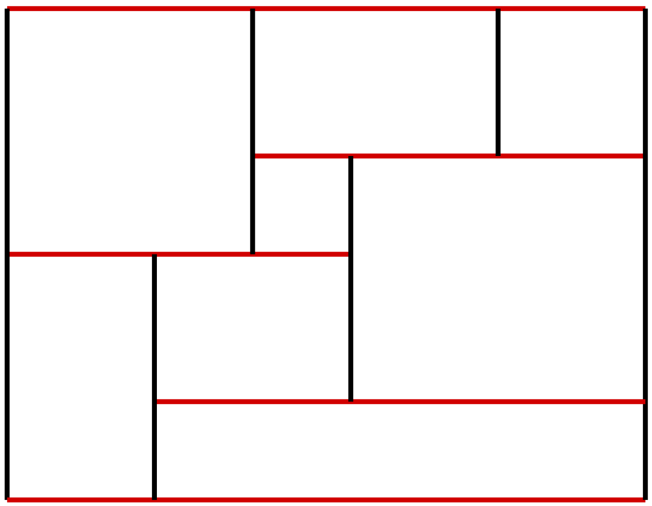
Rectangular Dissections and Graphs

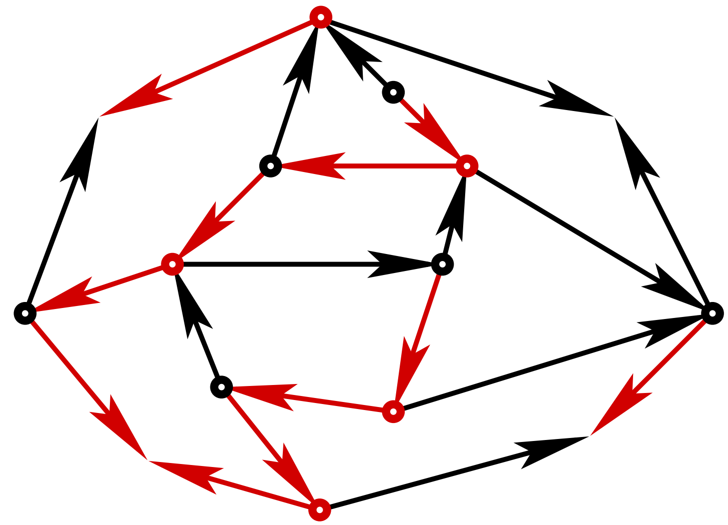




A quadrangulation induced by segment contacts.

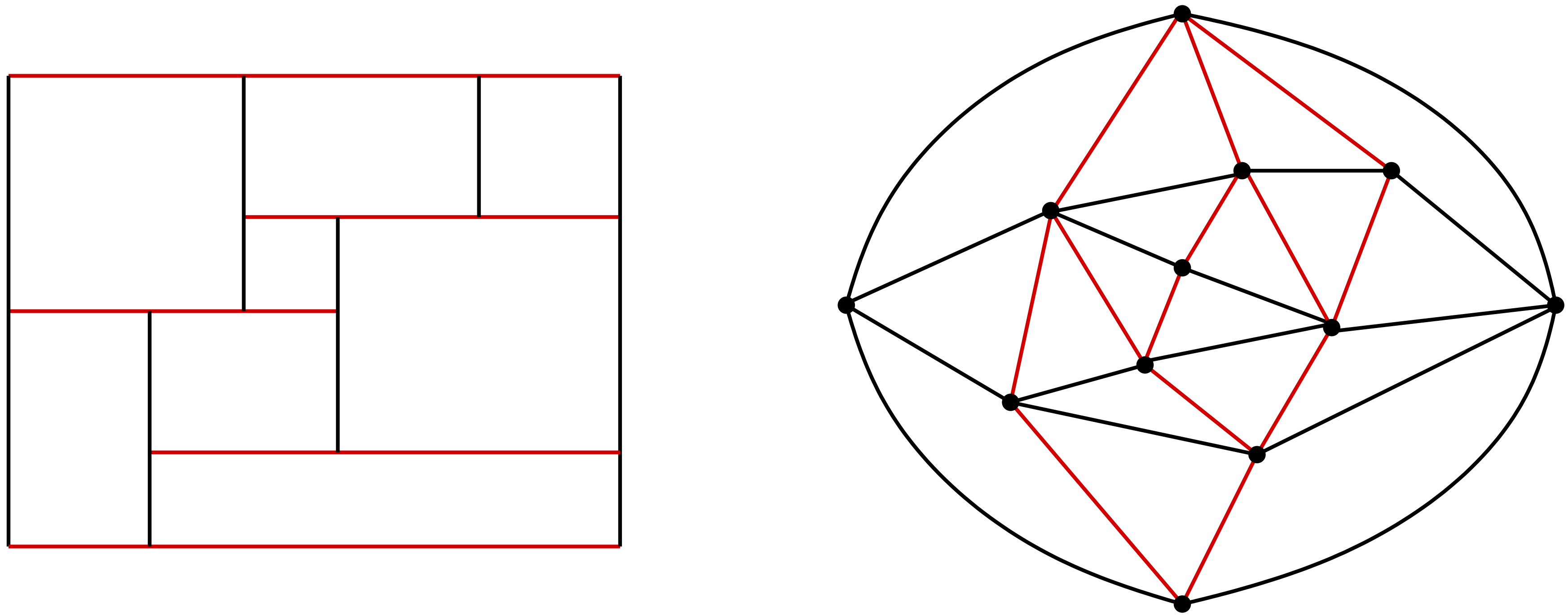
Rectangular Dissections and Graphs





A separating decomposition of the quadrangulation.

Rectangular Dissections and Graphs



The inner triangulation of a quadrangle.
R is the rectangular dual (a.k.a. floorplan).

Representation Problems

- G_B a bipolar graph
  find a rectangulation R representing G_B.

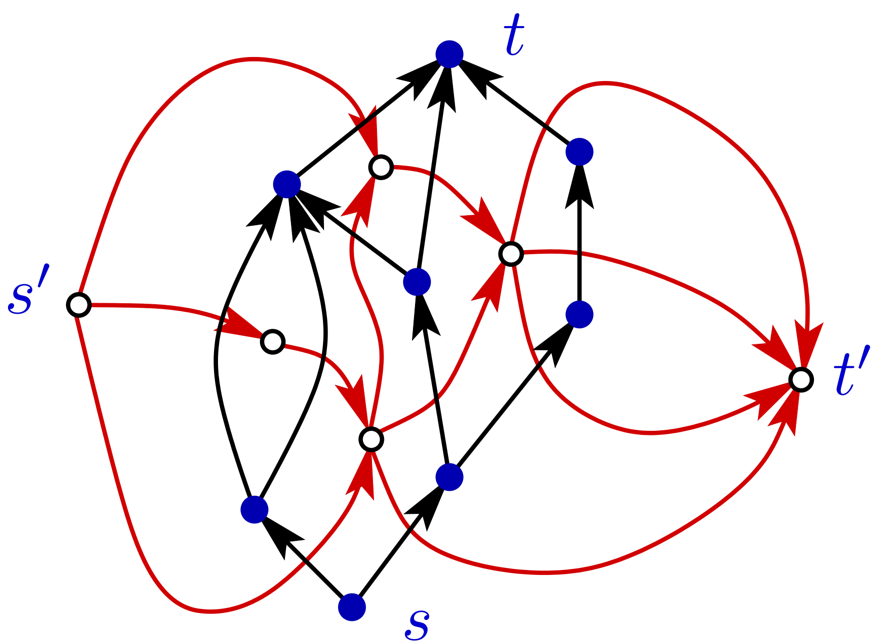
- Q a plane quadrangulation
  find some R representing Q as segment contact graph.

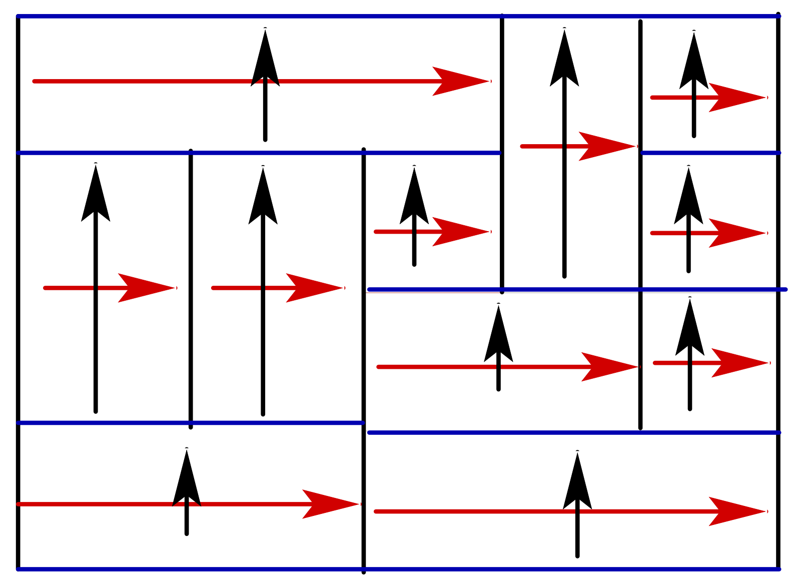
- G a triangulation of a quadrangle
  find some R representing G as rectangular dual.

S.F., *Rectangle and Square Representations of Planar Graphs*,
in *Thirty Essays in Geometric Graph Theory*,
Pach, János (Ed.), Springer 2013.

Sketch: Bipolar Orientation

From the bipolar orientation compute its dual orientation.
Together they yield a rectangular dissection.

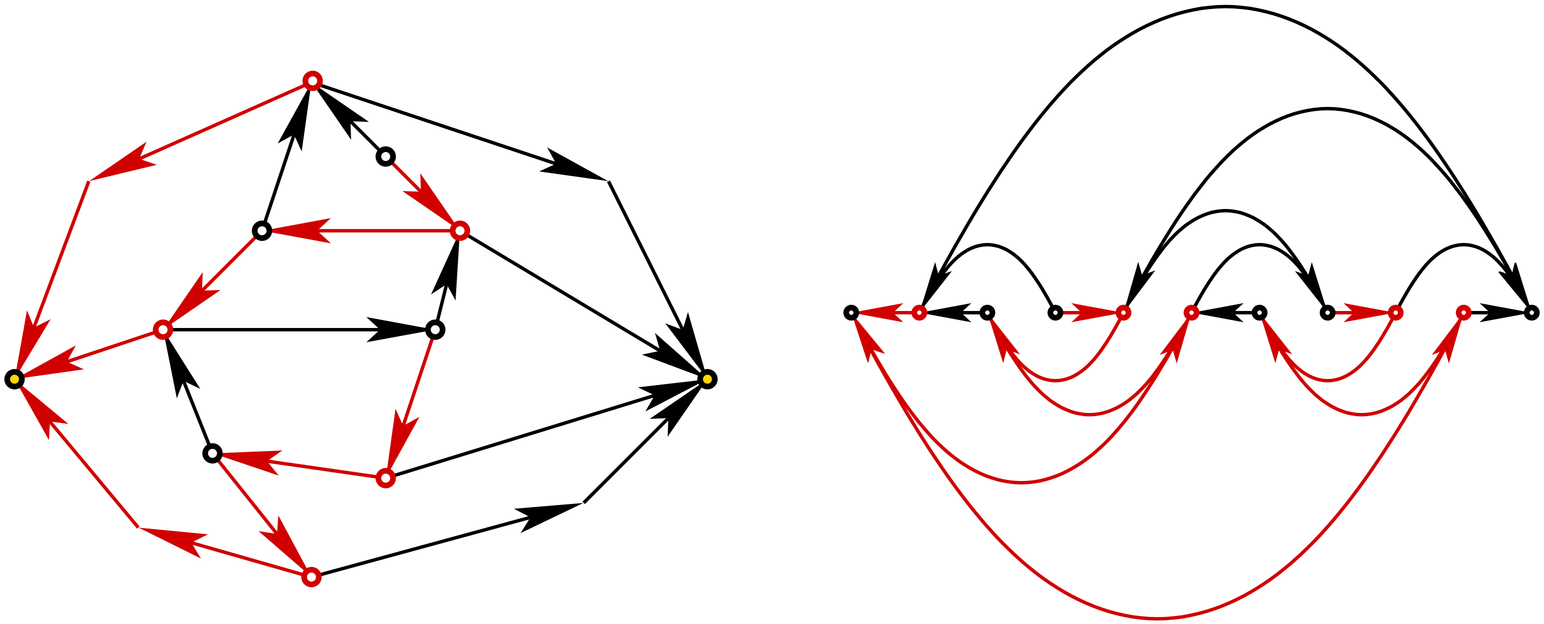




coordinates from longest paths

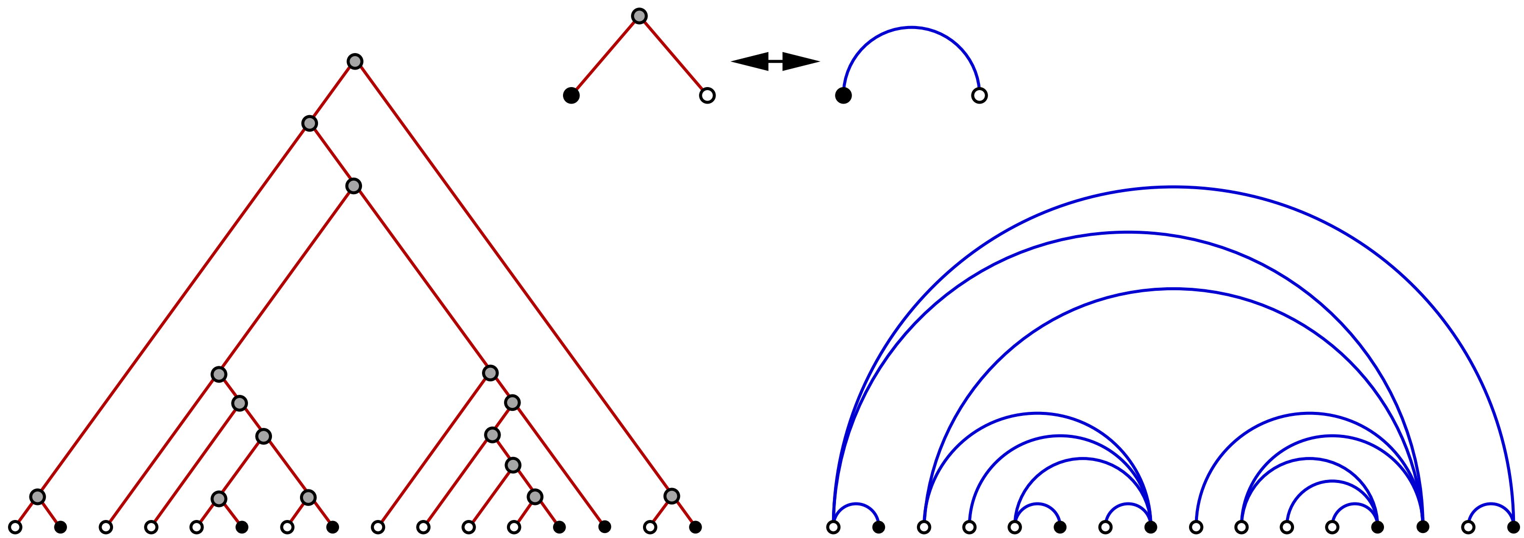
Sketch: Quadrangulation

- Compute a separating decomposition.
- Separate the two alternating trees.



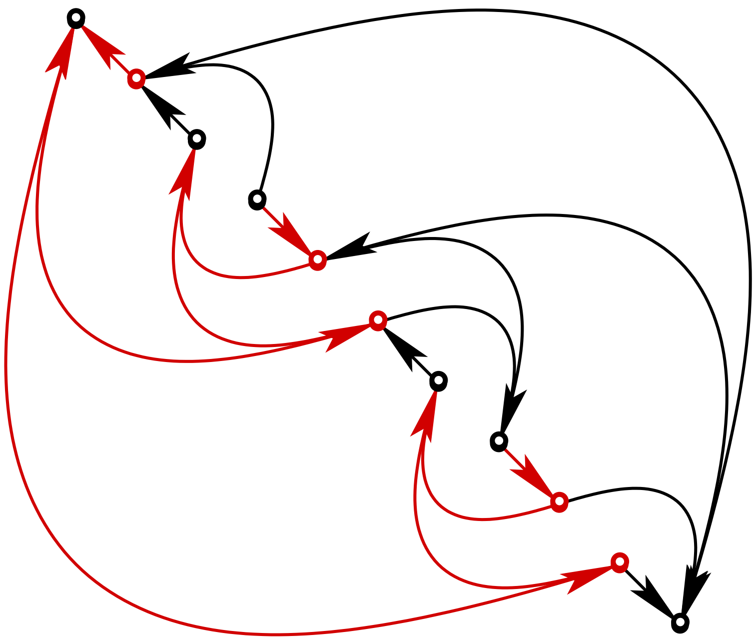
Alternating and Full Binary Trees

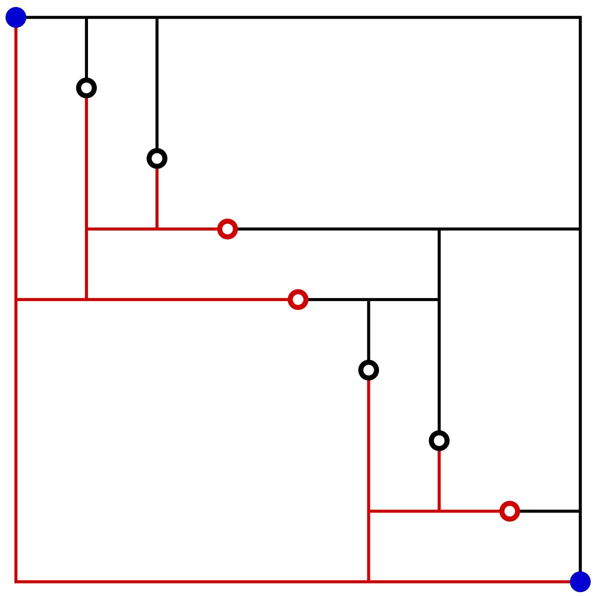
Proposition. There is bijection between alternating and binary trees that preserves types (left/right) of nodes.



Sketch: Quadrangulation

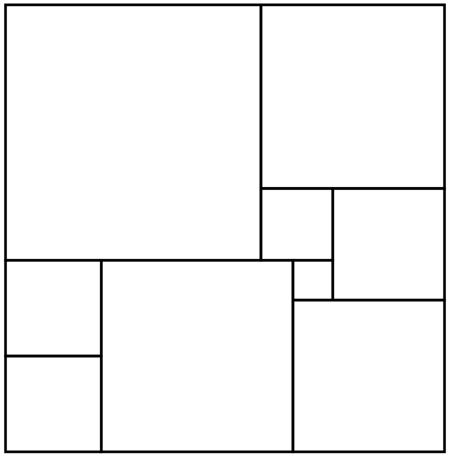
- The two binary trees obtained from the separating decomposition fit together.





Squarings

A squaring of a rectangle.



Representation Problems

- G_B a bipolar graph – find a corresponding squaring.

  The Dissection of Rectangles into Squares

  Brooks, Smith, Stone and Tutte 1940.

- Q a planar quadrangulation – find a squaring representing Q as segment contact graph.

- G a triangualation of a quadrangle – find a squaring representing G as rectangular dual.

  Square Tilings with Prescribed Combinatorics

  Oded Schramm 1993.

Rectangulations and Squarings

Segment Contact Representations

Segment Contacts on the Torus

Rectangular and Square Duals

Square Duals on the Torus

Squarings and Electricity

View the bipolar graph as electrical network with edge resistance $1\ \Omega$. Consider electrical $s \to t$ flow in this network. The distribution of flow/current in edges corresponds to sidelengths of a squaring.

- Kirchhoff's current law: flow conservation.
- Kirchhoff's potential law: rotation free flow, i.e., potentials exist.
- Ohm's law: $r_e f_e = \Delta p_e$, i.e., squares.

Squarings and Electricity

View the bipolar graph as electrical network with edge resistance $1\ \Omega$. Consider electrical $s \to t$ flow in this network. The distribution of flow/current in edges corresponds to sidelengths of a squaring.

The explicit solution:

$$\mathrm{flow}(i,j) = \\ \quad \#\text{ spanning trees } T \text{ with } (i,j) \text{ on the } s \to t \text{ path in } T \\ - \#\text{ spanning trees } T \text{ with } (j,i) \text{ on the } s \to t \text{ path in } T.$$

Squarings and Electricity

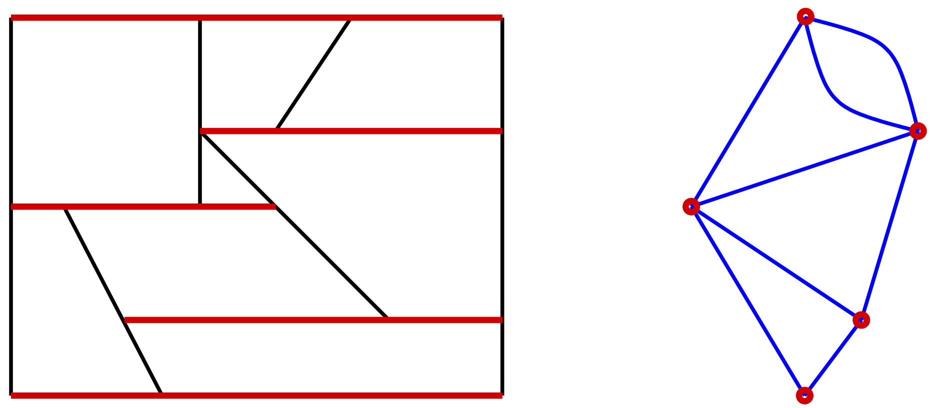
Instance of more general theory:

- Discrete harmonic functions.
- Rotation free flows.
- Random walks and Markov chains, e.g.

  Tilings and Discrete Dirichlet Problems

  Richard Kenyon 1998.

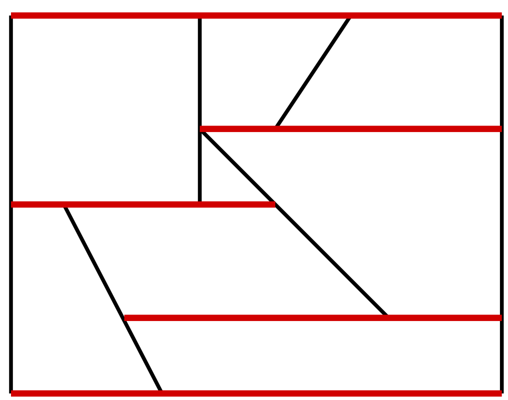
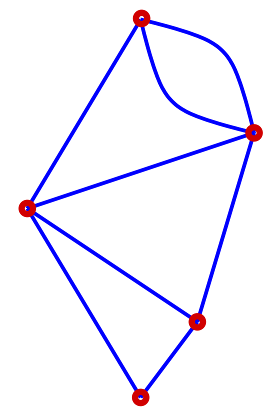
Trapezoidal Dissections and Markov Chains



Transition probabilities for G induced by a trapezoidal dissection:
For vertices i and j (horizontal segments) let

$$p(i,j) \propto m(i,j) = \frac{\mathrm{width}_i(T_{ij})}{\mathrm{height}(T_{ij})}.$$

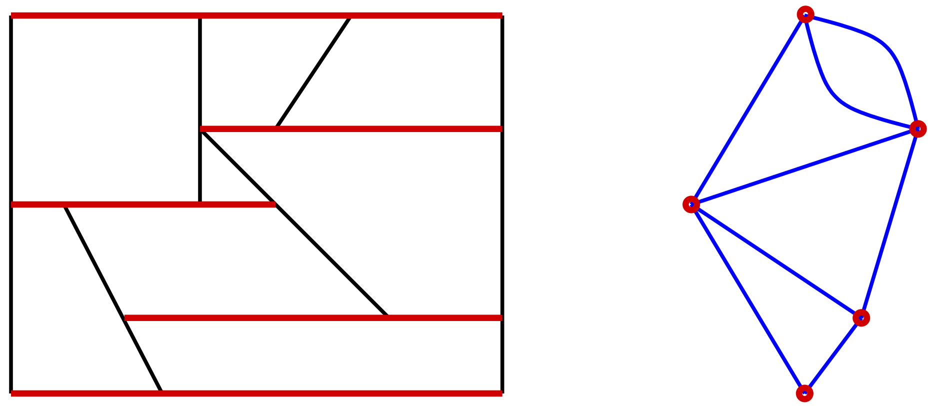
Trapezoidal Dissections and Markov Chains

Transition probabilities $p(i, j)$ are induced by a trapezoidal dissection.
The hights can be recovered:

Proposition. $f(i) = y_i$ is harmonic with respect to p for all $i \notin \{s, t\}$, i.e., $f(i) = \sum_j f(j)p(i, j)$.

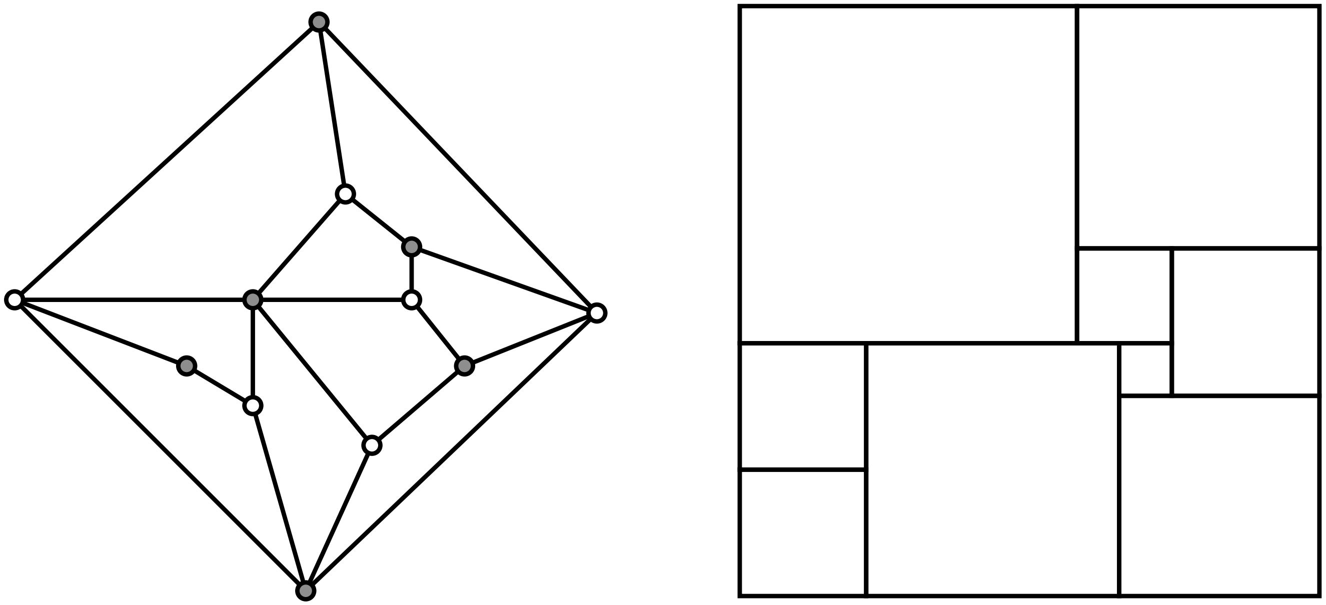
Trapezoidal Dissections and Markov Chains



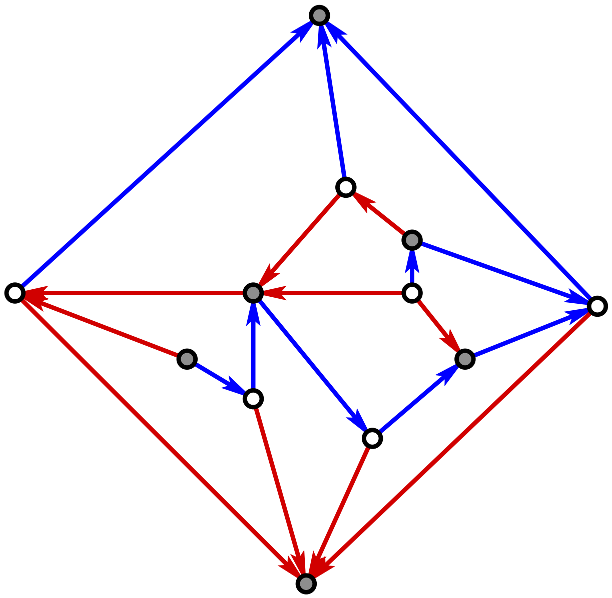
Theorem. G planar, p transition probabilities, s, t on the outer face $\implies$ the stationary distribution m on the edges together with the unique p-harmonic function f on $V \setminus \{s, t\}$ and the winding numbers (slopes) yield a trapezoidal dissection of a rectangle.

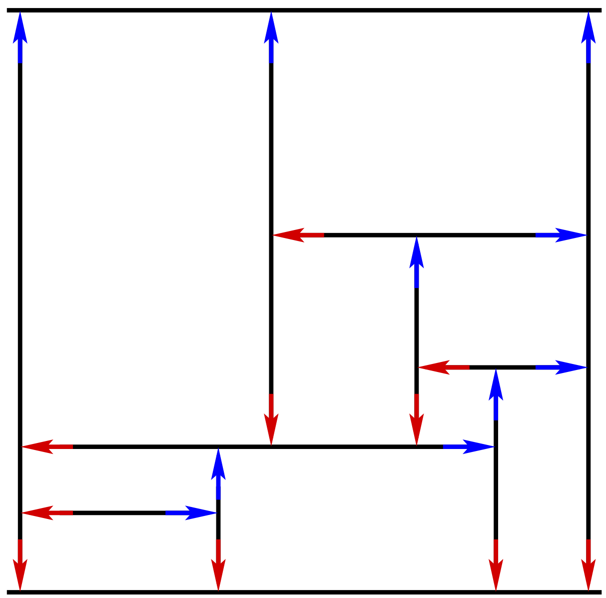
If $p(i, j) = \frac{1}{\deg(i)}$ the dissection is a squaring.

Squarings with Segment Contacts



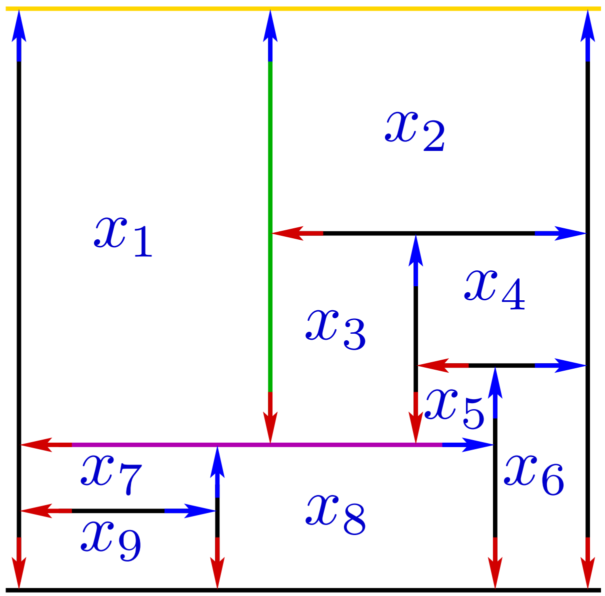
Squarings with Segment Contacts





Step I: Compute a separating decomposition on Q.
This corresponds to a rectangular dissection.

Squarings with Segment Contacts

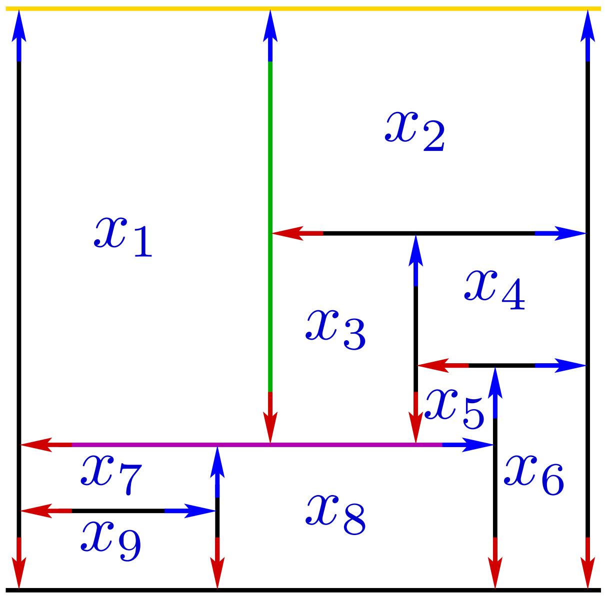


- $x_1 = x_2 + x_3$
- $x_1 + x_3 + x_5 = x_7 + x_8$
- $x_1 + x_2 = 1$

Step II: Set up a (quadratic) linear system of equations:

$$A_S \cdot x = e_1$$

Squarings with Segment Contacts



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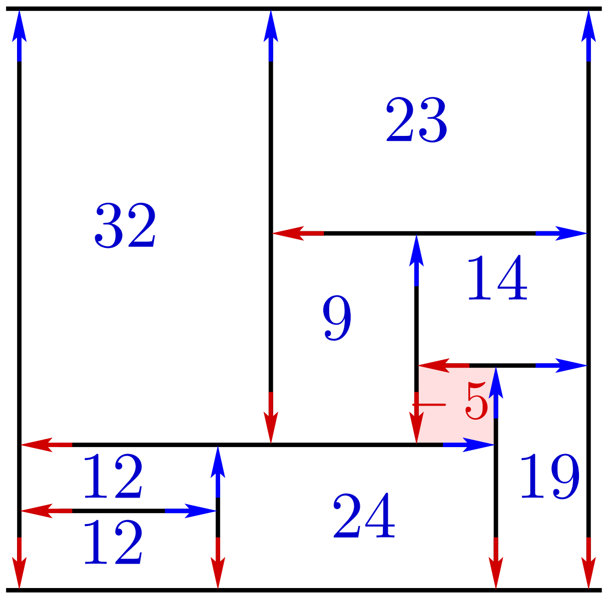
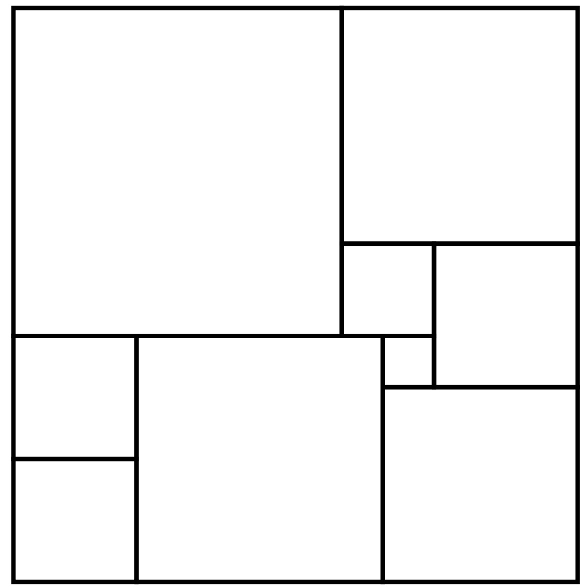
Step II: Set up a (quadratic) linear system of equations:

$$A_S \cdot x = e_1$$

$\det(A_S) = \pm$ # matchings of an auxiliary graph $\neq 0$.

Squarings with Segment Contacts





Step III: Flip negative faces to get the good separating decomposition and the squaring.

Rectangulations and Squarings

Segment Contact Representations

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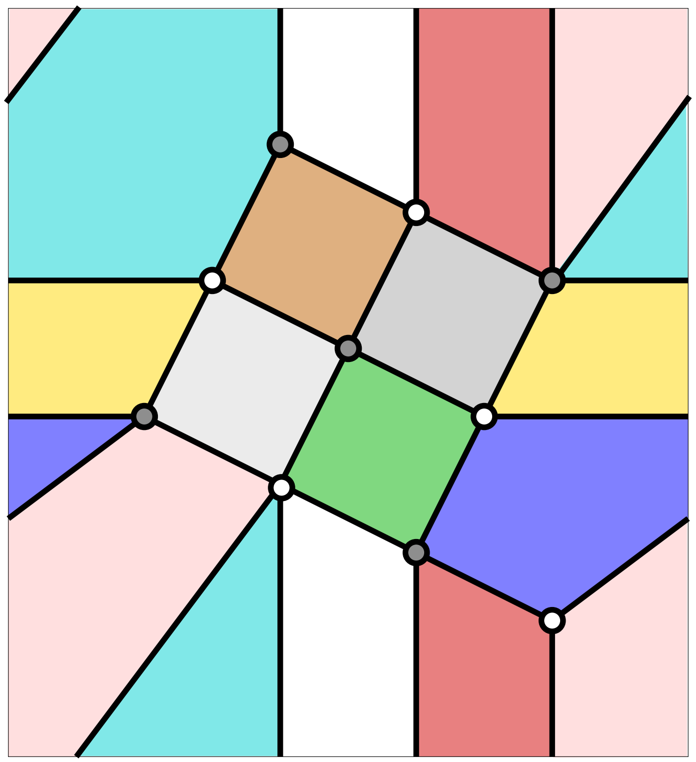
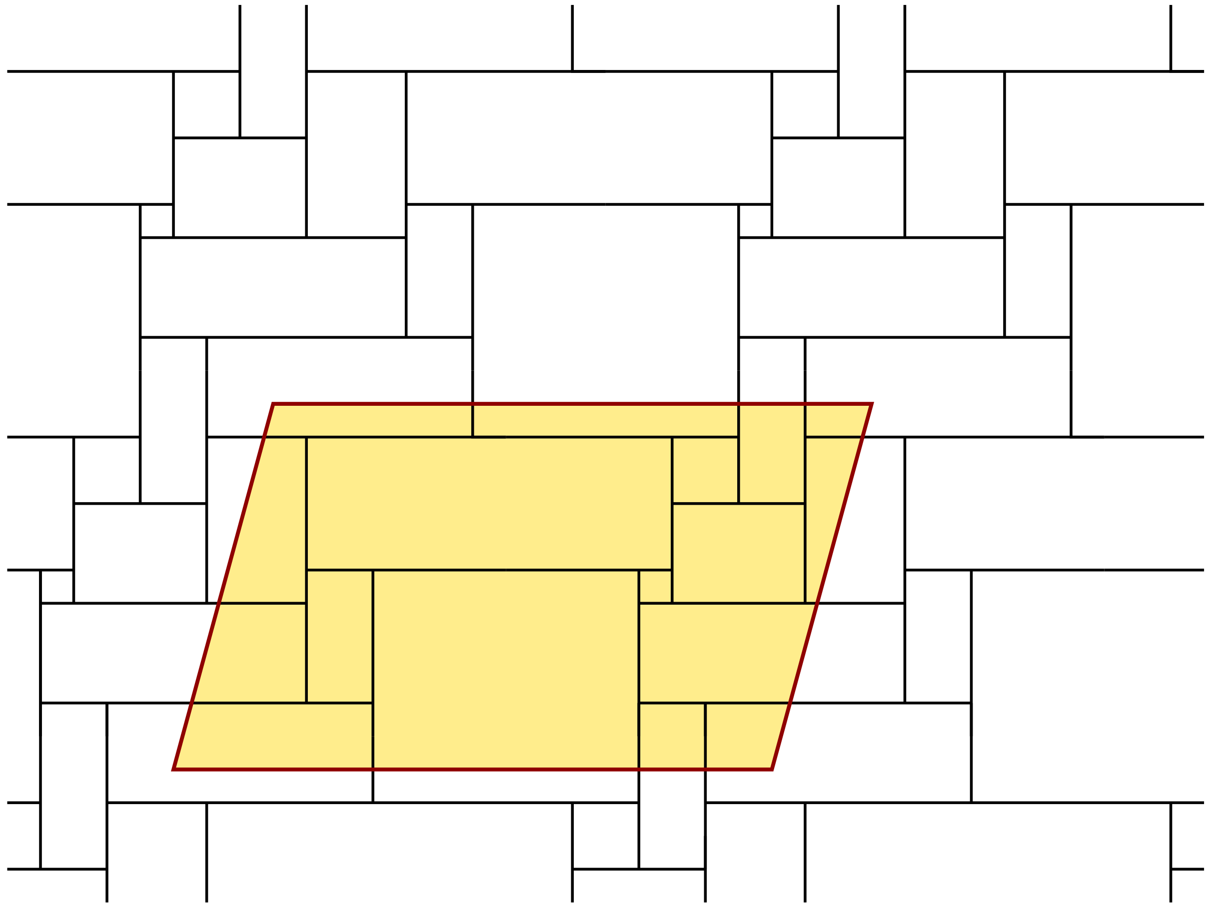
Torus Squarings with Segment Contacts

A torus quadrangulation.



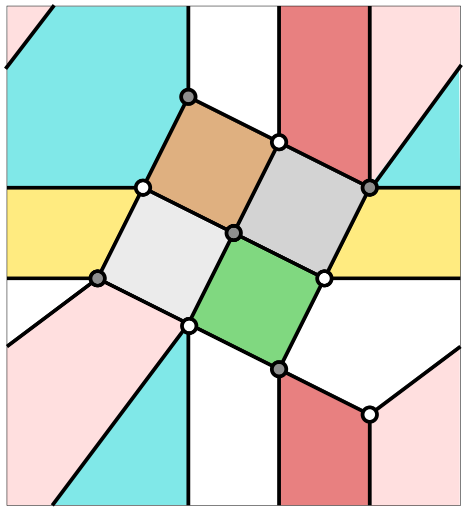
Segment Contacts on the Torus

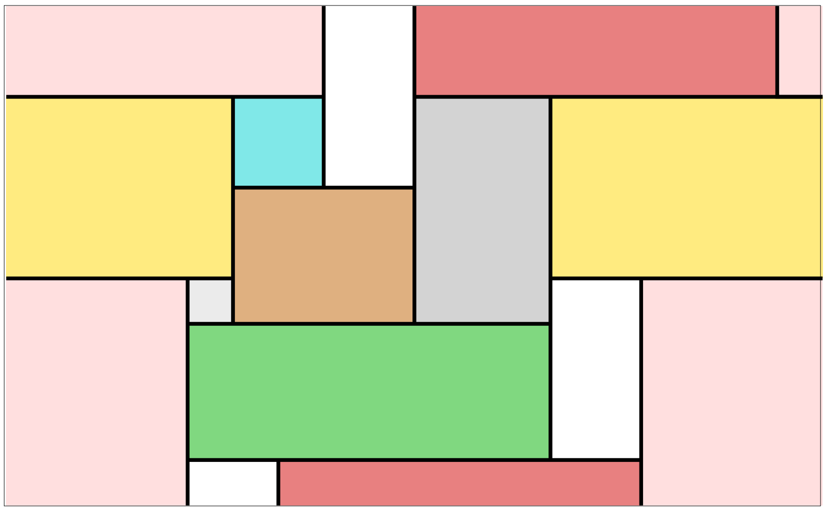


A torus rectangulation.

- Torus rectangulations are periodic tilings of the plane with a prallelogram as primitive cell.

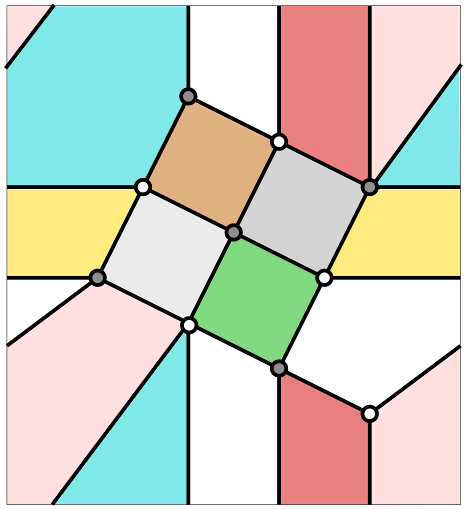
Segment Contacts on the Torus

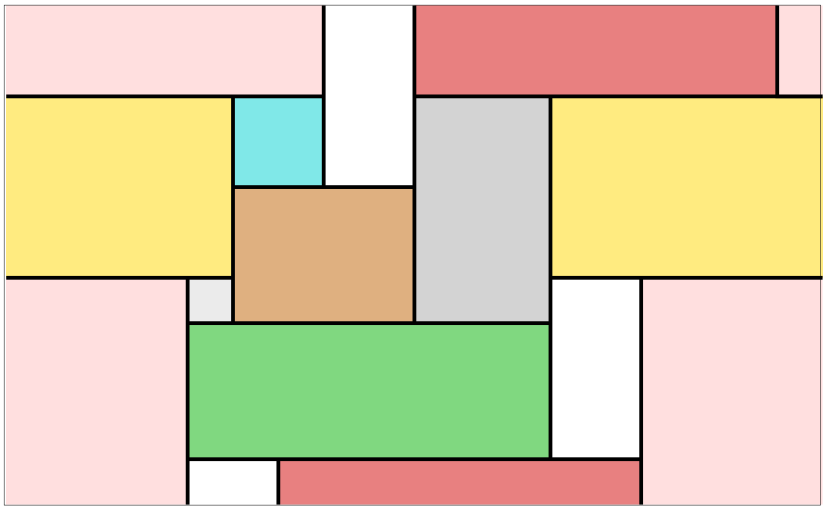




Torus quadrangulations can be represented by torus rectangulations. (Mohar, Rosenstiehl ’98)

Segment Contacts on the Torus





With Timo Strunk:

- A separating decomposition of the torus is a good separating decomposition if every every alternating cycle is crossing every monochromatic cycle.
- Good separating decompositions $\longleftrightarrow$ torus rectangulations.

Torus Squarings with Segment Contacts

Based on a torus rectangulation and two additional equations we can again set up a quadratic system of linear equations:

$$A \cdot x = e_1 + c\, e_2$$

Torus Squarings with Segment Contacts

Based on a torus rectangulation and two additional equations we can again set up a quadratic system of linear equations:

$$A \cdot x = e_1 + c\, e_2$$

A solution may have negative variables.

Lemma. The boundary of negative faces is a family of contractible cycles.

Flipping these cycles yields a torus rectangulation with a non-negative solution.

$\Longrightarrow$ A torus squaring.

Torus Squarings with Segment Contacts

Remains to show that there is a solution.

Want that A is non-degenerate.

- The proof for the plane case doesn't carry over (odd non-contractible cycles).

Torus Squarings with Segment Contacts

Proposition. A is non-degenerate.

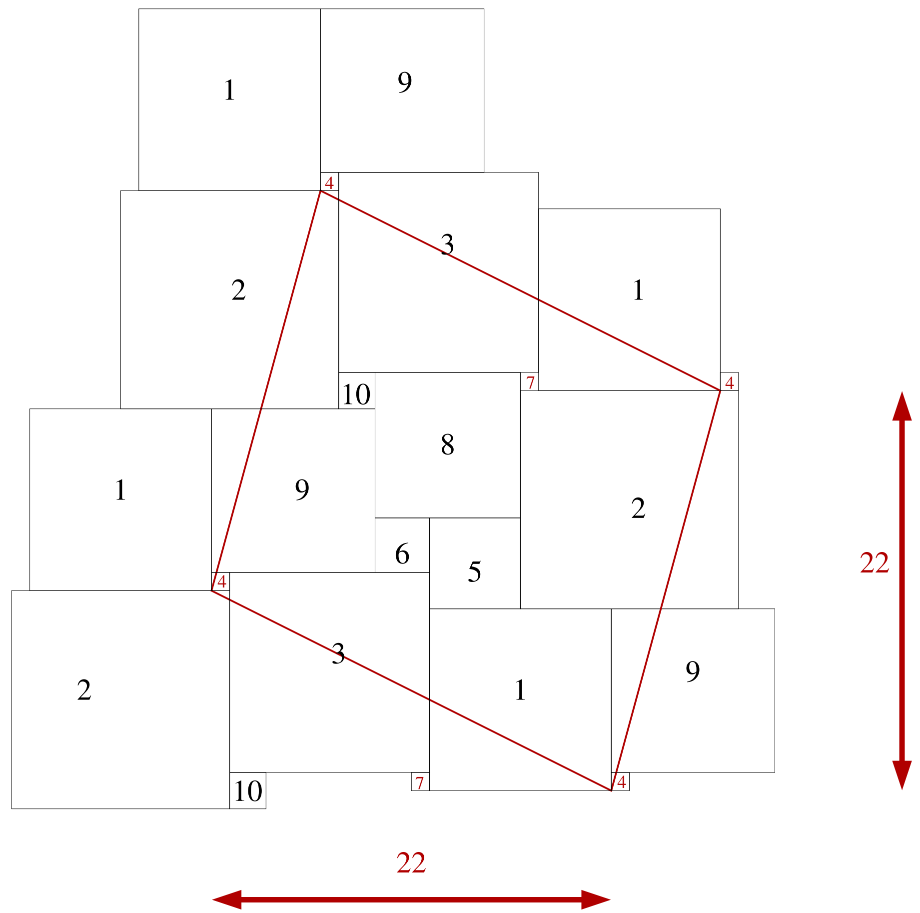
Proof. A nontrivial solution of $A \cdot x = 0$ yields a square tesselation with sidelength $|x_i|$.

Taking the lenght of two independent non-contractible dual cycles C_1, C_2 for the additional equations yields a contradiction:

> If $\ell(C_1) = \ell(C_1) = 0$ then the area Z of a fundamental cell is 0. However $Z = \sum_i x_i^2 > 0$.

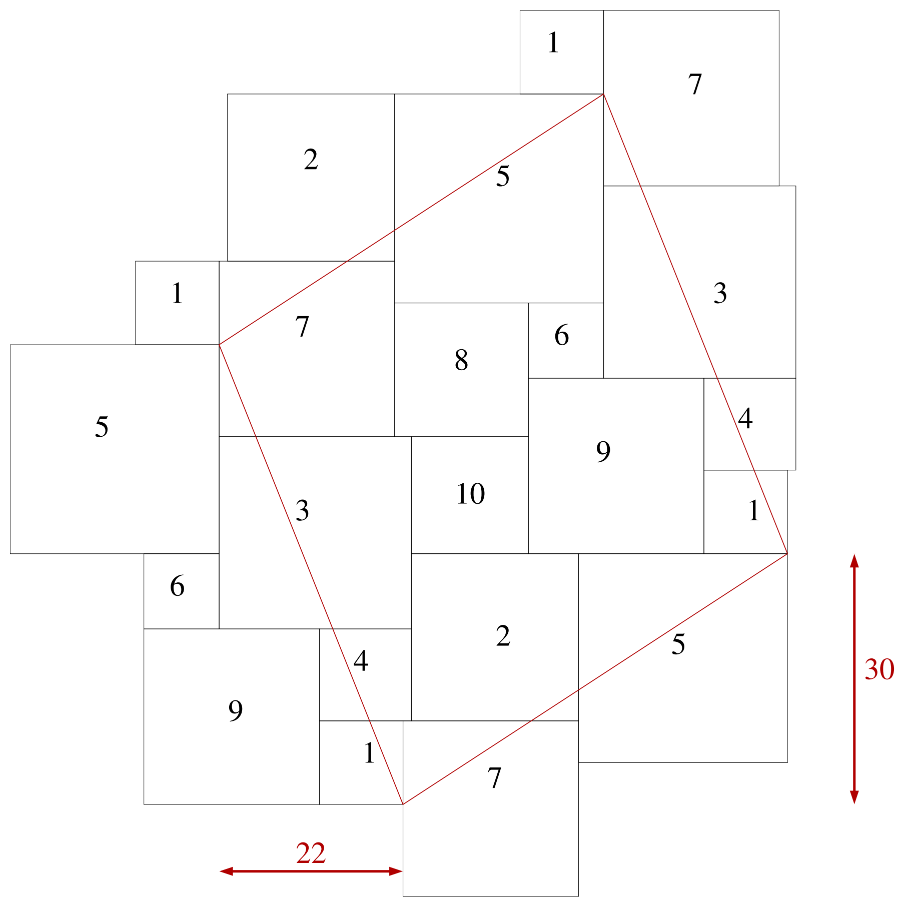
Torus Squarings with Segment Contacts

$C_1 = \{1, 2\}$ and $C_2 = \{1, 3, 4\}$ with length 22 and 22.



Torus Squarings with Segment Contacts

$C_1 = \{1, 2\}$ and $C_2 = \{1, 3, 4\}$ with length 30 and 22.



Torus Squarings with Segment Contacts

- Degeneracies. How to avoid squares of size 0?
  Sufficient conditions from connectivity known.
  Can cycle length be appropriately prescribed?

- Which cycles should be taken for the extra equations?
  Is it possible to prescribe properties of the fundamental cell?

Rectangulations and Squarings

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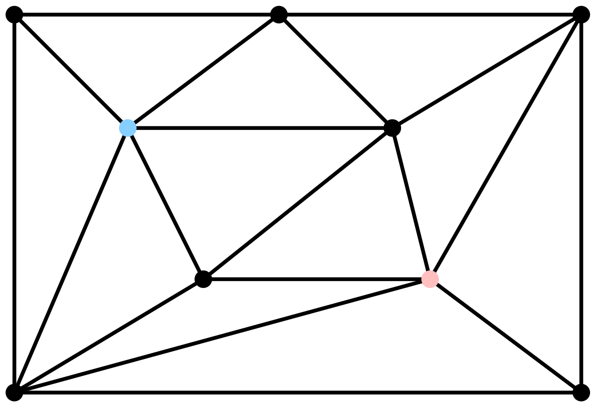
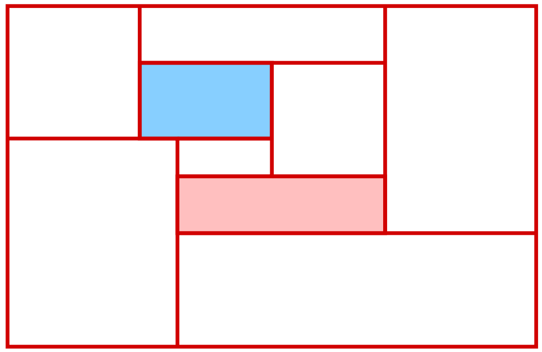
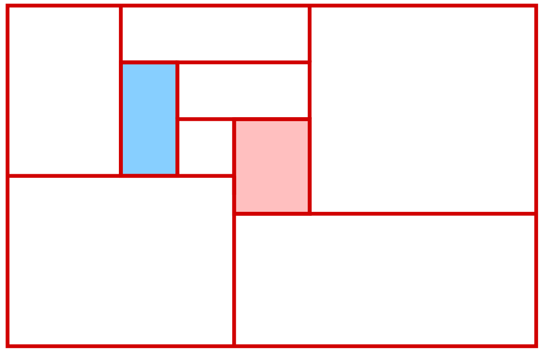
Rectangular Duals





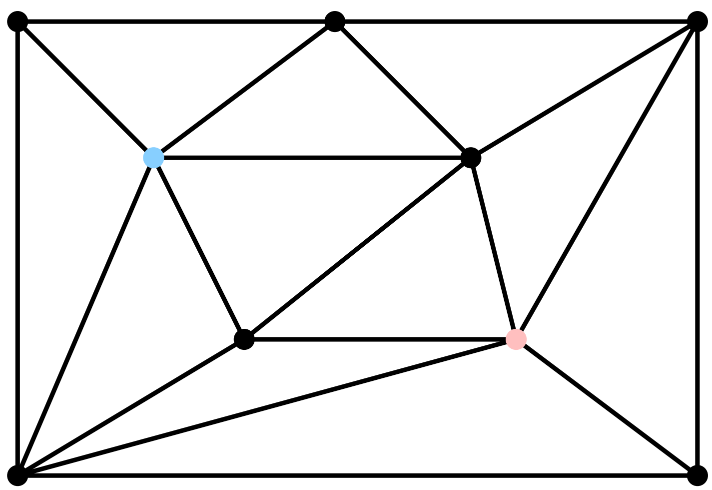


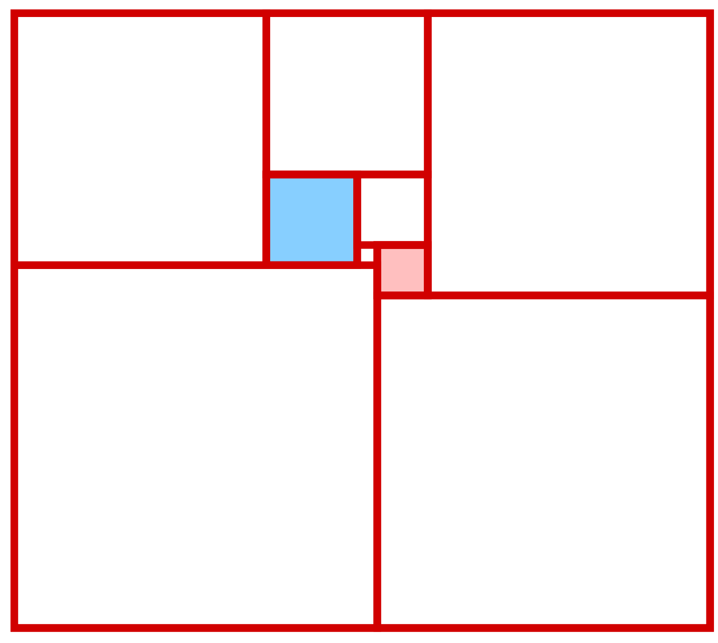
Prescribe corner rectangles.

Still there can be several rectangular duals.

Squarings for Inner Triangulations

The squaring is unique.





Extremal Length

O. Schramm, *Square Tilings with prescribed Combinatorics*, 1993.

- $m : V \to \mathbb{R}^+$ discrete metric on G.
- Length of a path: $\ell_m(\gamma) = \sum_{v \in \gamma} m(v)$.
- Distance between sets: $\ell_m(A, B) = \min_{\gamma \in \Gamma(A,B)} \ell_m(\gamma)$
- $\text{area}(m) = \sum_v m(v)^2 = ||m||^2$

Extremal Length

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- Distance between sets: $\ell_m(A, B) = \min_{\gamma \in \Gamma(A,B)} \ell_m(\gamma)$
- $\text{area}(m) = \sum_v m(v)^2 = ||m||^2$
- Normalized distance $\ell_m^*(A, B) = \frac{\ell_m(A,B)^2}{||m||^2}$
- Extremal length $L(A, B) = \sup_m \ell_m^*(A, B)$

Extremal Length and Squarings

Theorem. For G with A, B there is a unique extremal metric (up to scaling).

Proof. Normalized distance is invariant under scaling. Hence, we only have to look at metrics with

$$\ell_m(A,B) = \min_{\gamma \in \Gamma(A,B)} \ell_m(\gamma) = 1.$$

These m form a polyhedral set P (ineq. $\ell_m(\gamma) \geq 1$).

Extremal metric is the unique m with minimal norm in P.

Extremal Length and Squarings

Theorem. A squaring of G, with A and B at top and bottom induces an extremal metric.

Proof. Let $h = \text{height}(R)$ and $w = \text{width}(R)$ we may assume $h \cdot w = 1$.

For the side length $s(v)$:
$||s||^2 = \sum s(v)^2 = h \cdot w = 1$, hence $||s|| = 1$.

For $t \in [0, w]$ the squaring induces a path γ_t.
For all m we have:

$$\ell_m(A, B) \leq \sum_{v \in \gamma_t} m(v)$$

Extremal Length and Squarings

$$\begin{aligned}
w \cdot \ell_m(A,B) &\leq \int_0^w \sum_{v\in\gamma_t} m(v)dt \\
&= \int_0^w \sum_{v\in V} m(v)\delta_{[v\in\gamma_t]}dt \\
&= \sum_{v\in V} m(v) \int_0^w \delta_{[v\in\gamma_t]}dt \\
&= \sum_{v\in V} m(v)s(v) \\
&\leq \langle m, s\rangle \leq ||m|| \cdot ||s|| = ||m||
\end{aligned}$$

Hence:

$$\ell_m^*(A,B) = \frac{\ell_m(A,B)^2}{||m||^2} \leq \frac{1}{w^2} = h^2 = \frac{h^2}{||s||^2} = \ell_s^*(A,B)$$

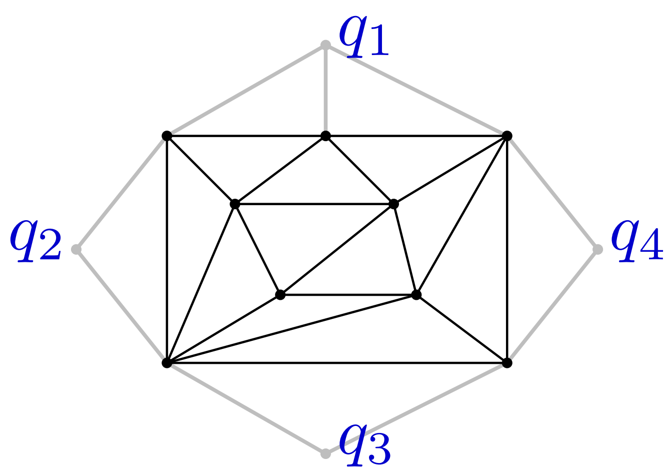
Extremal Length and Squarings

Theorem. An extremal metric of a triangulation yields a set of squares that fit together to a squaring representing G.

If there are no separating cycles of length ≤ 4 all squares have size ≥ 0.

The Polyhedral view on Squarings

L. Lovász, *Geometric Representations of Graphs*, 2009, Sec. 6.3.2.



$$P = \{x \in \mathbb{R}^V_{\geq 0} : \quad \sum_{i \in \gamma} x_i \geq 1 \quad \text{for all } q_1 \to q_3 \text{ paths } \gamma\}$$

Blocking Polyhedra

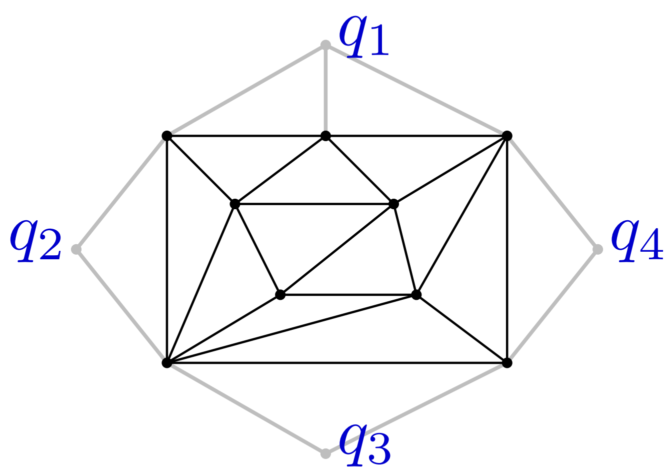
$P = \{x \in \mathbb{R}^n_{\geq 0} : \quad a_i^T x \geq 1 \quad \text{for } a_i \in \mathbb{R}^n_{\geq 0}, i = 1..k\}$

The blocker of P is:

$P^{\text{bl}} = \{y \in \mathbb{R}^V_{\geq 0} : \quad x^T y \geq 1 \quad \text{for all } x \in P\}$

- $(P^{\text{bl}})^{\text{bl}} = P$.
- $p \in \mathbb{R}^n_{\geq 0}$ is a vertex of $P \iff p$ is a facet of P^{bl}.

The Polyhedral view on Squarings



$P = \{x \in \mathbb{R}^V_{\geq 0} : \quad \sum_{i \in \gamma} x_i \geq 1 \quad \text{for all } q_1 \to q_3 \text{ paths } \gamma\}$

$Q = \{x \in \mathbb{R}^V_{\geq 0} : \quad \sum_{i \in \rho} x_i \geq 1 \quad \text{for all } q_2 \to q_4 \text{ paths } \rho\}$

Theorem. (P, Q) is a blocking pair of polyhedra.

The Polyhedral view on Squarings

$P = \{x \in \mathbb{R}^V_{\geq 0} : \quad \sum_{i \in \gamma} x_i \geq 1 \quad \text{for all } q_1 \to q_3 \text{ paths } \gamma\}$

$Q = \{x \in \mathbb{R}^V_{\geq 0} : \quad \sum_{i \in \rho} x_i \geq 1 \quad \text{for all } q_2 \to q_4 \text{ paths } \rho\}$

Theorem. (P, Q) is a blocking pair of polyhedra.

A criterion for blocking pairs: For all $w \in \mathbb{R}^V_{\geq 0}$

Minimum w-weight of a $q_1 \to q_3$ path $=$

Maximum w constrained packing of $q_2 \to q_4$ paths

Proof. Max-Flow Min-Cut together with the HEX-Lemma to show that Min-Cut corresponds to a $q_1 \to q_3$ path.

The Polyhedral view on Squarings

(P, Q) a blocking pair

$a \in P$ is minimizing $\sum_i x_i^2 \implies$

$\frac{1}{\sum_i a_i^2} a$ minimizes $\sum_i y_i^2$ over Q.

The Polyhedral view on Squarings

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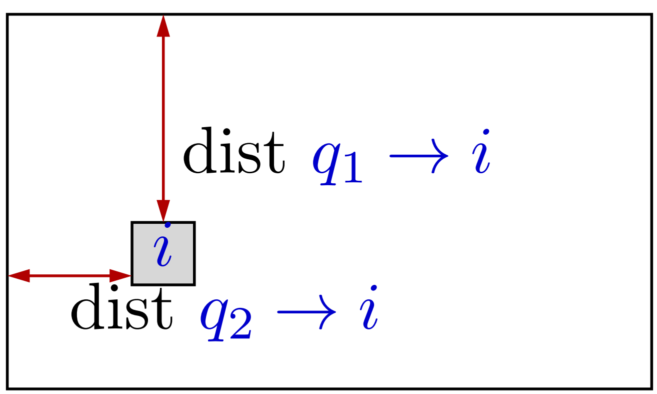
Theorem. There is a squaring of G inside a rectangle of height 1 and width $\frac{1}{\sum_i a_i^2}$ where the square of vertex i has sidelength a_i.



Rectangulations and Squarings

Segment Contact Representations

Segment Contacts on the Torus

Rectangular and Square Duals

Square Duals on the Torus

Blocking Polyhedra for the Torus

G a torus triangulation.
γ non-contractible circuit in G.
Γ the class of γ.

$$P = \{x \in \mathbb{R}^V_{\geq 0} : \quad \sum_{i \in \gamma} x_i \geq 1 \quad \text{for all } \gamma \in \Gamma\}$$

What is P^{bl}?

Blocking Polyhedra for the Torus

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$$P = \{x \in \mathbb{R}^V_{\geq 0} : \quad \sum_{i \in \gamma} x_i \geq 1 \quad \text{for all } \gamma \in \Gamma\}$$

What is P^{bl}?

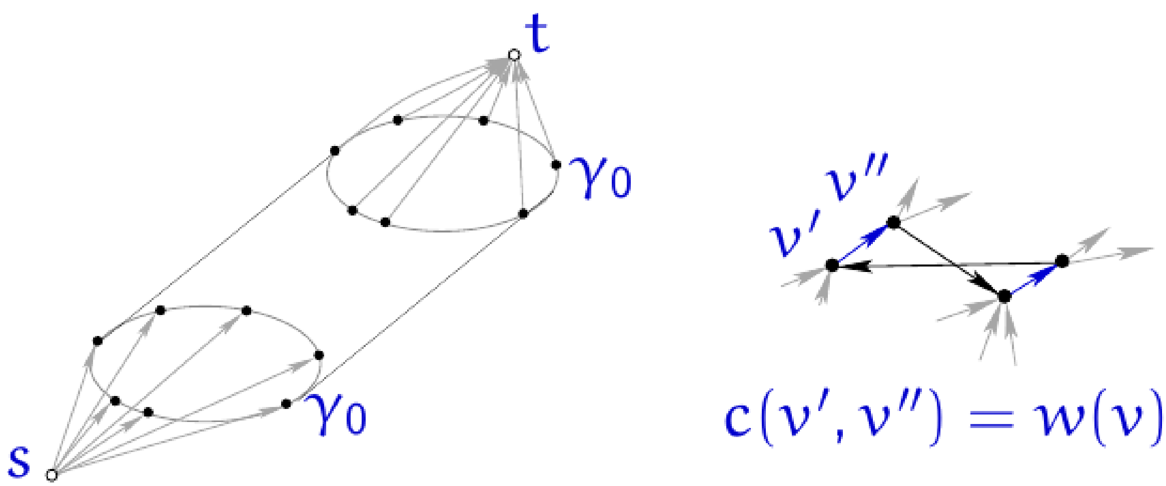
For a non-contractible circuit ρ let
$\mathsf{cr}_\Gamma(\rho) = \min \#($ of crossings between ρ and some $\gamma' \in \Gamma)$.

$$Q = \{y \in \mathbb{R}^V_{\geq 0} : \quad \sum_{i \in \rho} y_i \geq \mathsf{cr}_\Gamma(\rho) \quad \text{for all } \rho \in \overline{\Gamma}\}$$

Blocking Polyhedra for the Torus

Theorem. (P, Q) is a blocking pair of polyhedra.

Proof. Let γ_0 be a minimum weight circuit in Γ.



- A Max-Flow saturates all vertices on γ_0. (HEX Lemma on the sphere).
- Path decomposition of the flow induces weighted family of circuits such that $\sum_\rho \lambda_\rho \mathsf{cr}_\Gamma(\rho) = w(\gamma_0)$.

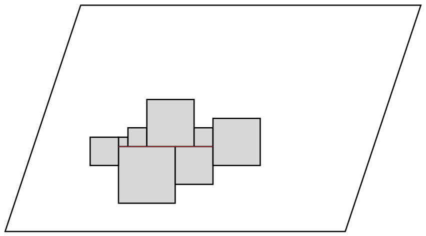
Square Duals on the Torus

(P, Q) a blocking pair

$a \in P$ is minimizing $\sum_i x_i^2 \implies$

$\frac{1}{\sum_i a_i^2} a$ minimizes $\sum_i y_i^2$ over Q.

Theorem. There is a torus squaring of G where the square of vertex i has sidelength a_i. The fundamental cell has a basis of width 1 parallel to the x-axis and height $\frac{1}{\sum_i a_i^2}$



Unique if there are no breaklines.

Square Duals on the Torus

- The proof yields a pairing $\Gamma \leftrightarrow \hat{\Gamma}$ of classes of non-contractible cycles. Independent description?

- Efficient computation of the squaring?

The End

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Thank you.