Bounded expansion: Introduction

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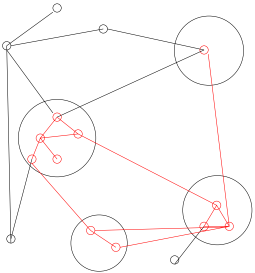
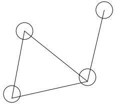
JCALM 2016

Outline

Minors



G

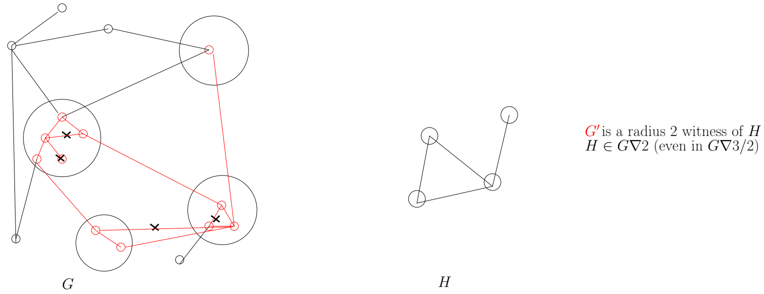


H

G' is a radius 2 witness of H
$H \in G\nabla 2$ (even in $G\nabla 3/2$)

- H minor of G iff exists subgraph $G' \subseteq G$ which is witness of H
- G' witness of H iff exists partition of $V_{G'}$ into connected $V_1, \ldots, V_{n_H}$ such that contracting G' gives H
- H is a r shallow minor of G ($H \in G\nabla r$) iff exists subgraph $G' \subseteq G$ such that G' is a radius (dist in $G[V_i]$) r witness of H
- G' radius r witness of H iff in addition we have $rad(V_i) \leq r$

Minors

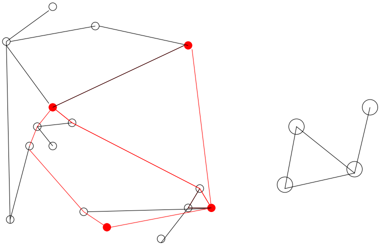


In the witness, we can suppose that

- V_i are rooted trees
- at most one external edge between any pair $\{V_i, V_j\}$
- all leaves are incident to an external edge
- $H \in G\nabla r \Leftrightarrow$ trees of height $\leq r$

$H \in G\nabla(r - \frac{1}{2})$ iff $H \in G\nabla r$ and no external edge between to leaves both at distance r of their root

Topological minors

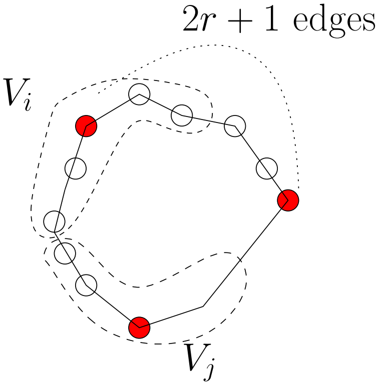


G $\qquad$ $H \in G\tilde{\nabla}2$

- H topological minor of G iff exists subgraph $G' \subseteq G$ such that G' is a subdivision of H ($\Leftrightarrow \exists v_1, \dots, v_{n_H}$ in V_G such that $\{v_i, v_j\} \in E_H \Rightarrow \exists$ path $P_{i,j}$ between v_i and v_j, where $P_{i,j}$ are verte disjoint paths)
- H r top. shallow minor of G ($H \in G\tilde{\nabla}r$) iff exists subgraph $G' \subseteq G$ G' is a $\leq 2r$ subdivision of H (path of length $\leq 2r + 1$)

Minor Vs Topological minor

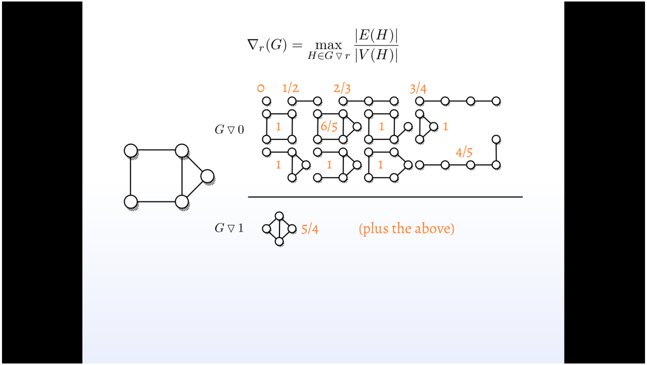
- $G\tilde{\nabla}0 = G\nabla 0 =$ subgraphs of G
- $G\tilde{\nabla}r \subseteq G\nabla r$
- beeing a topological minor is not a well quasi ordering relation



Grad and Top grad

- Greatest reduced average degree: $\nabla_r(G) = \max_{H \in G \nabla r} \frac{m_H}{n_H}$
- Top. Greatest reduced average deg: $\tilde{\nabla}_r(G) = \max_{H \in G \tilde{\nabla} r} \frac{m_H}{n_H}$

Thank you Felix Reidl!



- $\nabla_r(G)$ is the maximum external edges in a radius r witness G'
- $\nabla_0(G) = \tilde{\nabla}_0(G) = \frac{mad(G)}{2}$

Equivalence beetween grad and top grad

Corollary 4.1 of [dM$^+$12]

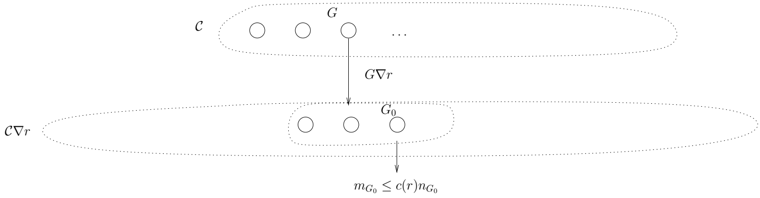
For any G and r, $\tilde{\nabla}_r(G) \leq \nabla_r(G) \leq 4(4\tilde{\nabla}_r(G))^{(r+1)^2}$

Bounded Expansion (BE)

Definitions

- $\mathcal{C}\nabla r = \bigcup_{G\in\mathcal{C}} G\nabla r$
- $\nabla_r(\mathcal{C}) = \sup_{G\in\mathcal{C}}(\nabla_r(G))$
- A class $\mathcal{C}$ is BE iff there exists a function $c < \infty$ such that $\forall r$, $\nabla_r(\mathcal{C}) \leq c(r)$ (or $\tilde{\nabla}_r(\mathcal{C}) \leq c'(r)$).

$\mathcal{C}$ is BE iff $\exists c$ such that $\forall r$, $\forall G \in \mathcal{C}$, $\forall G_0 \in G\nabla r$, $m_{G_0} \leq c(r)n_{G_0}$



Bounded Expansion (BE): examples

Remark

BE $\Rightarrow \nabla_0(\mathcal{C}) \leq c(0) \Rightarrow$ for any G: constant $mad(G) \Leftrightarrow$ constant degeneracy $\Rightarrow \chi(G)$ constant

Examples of BE class

- constant Δ ($\nabla_r(G) \leq \Delta^{r+1}$)
- H minor free $\Rightarrow$: implies K_{n_H} minor free, and thus for any minor G, $m_G \leq f(n_H)n_G$ (and thus $c(r)$ is even a constant)

$\Rightarrow$ (and thus planar graphs, bounded treewidth graphs are BE)

- bounded stack number, bounded queue number (see [dM+12])
- bounded crossing number

Examples

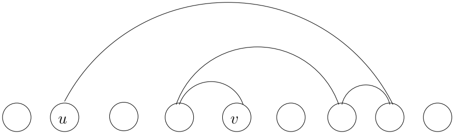
- A graph G has crossing number $cr(G) = k$ iff it can be drawn in the plane such that there is at most k crossing on each edge.
- Let $\mathcal{C} = \{G | cr(G) \leq k\}$. $\mathcal{C}$ has BE
- Let $H \in G \tilde{\nabla} r$. H has at most $cr' = k(2r+1)$ crossing per edge.
- thus $m' \leq f(r)n'$, and $\tilde{\nabla}_r(G) \leq f(r)$, and $\nabla_r(G) \leq g(r)$

Outline

The most beautiful slide

There are MANY characterizations of BE (Thm 13.2 in [dM$^+$12])

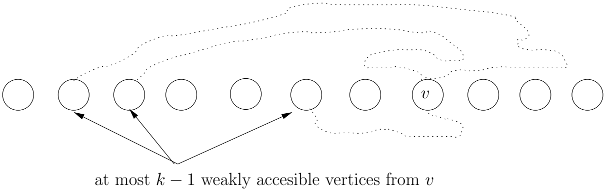
Consider a permutation π of the vertices of a graph G



u is weakly 4 accessible from v

- We say that u is weakly r-accessible from v iff $u < v$ and there exists a $u - v$ path P of length at most r with $u < min(P)$
- We denote $N_r^\pi(v) = \{u$ weakly r-accessible from $v\}$ the number of "backward" neighbors
- We denote $col_r^\pi(G) = \max_v N_r^\pi(v) + 1$.
- The weak r-coloring number of G is $wcol_r(G) = \min_\pi col_r^\pi(G)$.

Example of G with $wcol_r(G) = k$.



Observe that $\chi(G) \leq wcol_1(G)$

Characterization of BE with weak coloring

A class $\mathcal{C}$ have bounded generalized colouring number iff for any r, there exists $c(r)$ such that $wcol_r(G) \leq c(r)$ for any $G \in \mathcal{C}$.

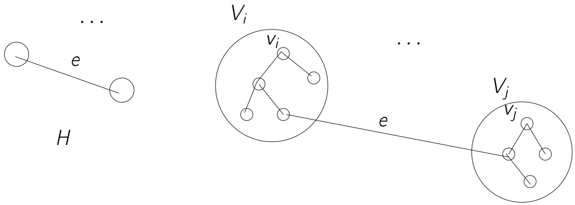
Theorem (in [Zhu09])

$BE \Leftrightarrow$ bounded generalized colouring number

Remarks:

- Goal $\forall r\ \nabla_r(G) \leq c(r) \Leftrightarrow \forall r'\ wcol_{r'}(G) \leq c'(r')$
- For example for $(r, r') = (0, 1)$:
- $\nabla_0(G) = \frac{mad(G)}{2}$ cst, and thus $\Leftrightarrow$ G has cst-degeneracy
- it remains to check that $wcol_1(G)$ cst $\Leftrightarrow$ G has cst-degeneracy

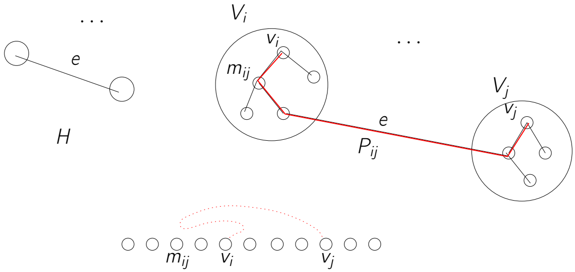
Proof of $\Leftarrow$



- goal: $\nabla_r(G) \leq c(r)$
- let $H \in G\nabla r$ such that $\frac{m_H}{n_H} = \nabla_r(G)$
- let G' be a witness of H: $G' = \{V_1, \ldots, V_H\}$ where V_i are trees of height $\leq r$
- suppose there is an external edge $e = \{V_i, V_j\}$

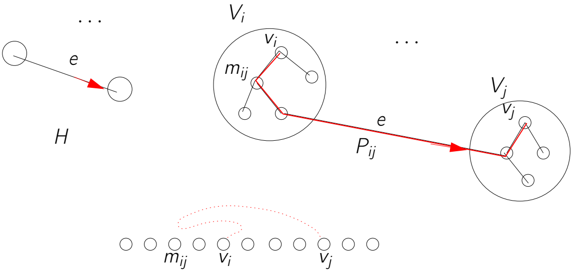
Characterization of BE with weak coloring

Proof of $\Leftarrow$



- this implies that there is in G a path P_{ij} of length at most $2r+1$ between v_i and v_j
- let m_{ij} be the minimum (in the best π) vertices of P_{ij}
- m_{ij} is weakly $2r+1$-accessible from v_i and from v_j
- orient e toward the V_l not containing m_{ij}
- now, given a V_j: each in arc means one disctinct $2r+1$-accessible vertex
- each V_j has indegree at most $wcol_{2r+1}(G)$

Proof of $\Leftarrow$



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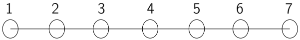
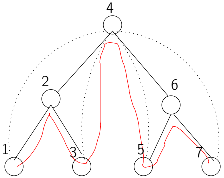
- now, given a V_j: each in arc means one disctinct $2r+1$-accessible vertex
- each V_j has indegree at most $wcol_{2r+1}(G)$

Characterization of BE with low tree-depth coloring

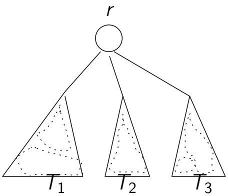
The tree-depth $td(G)$ of a connected graph G is the minimum height of a rooted tree T such that $G \subseteq clos(T)$ ($clos(T) = T+$ add an edge between any vertice and its ancestors)

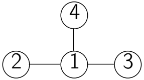


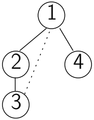


- $td(P_7) \leq 3$
- edges in T are not necessarily edges in G
- $tw(G) \leq pw(G) \leq td(G)$: pw decomposition from T: 421, 423, 465, 467

Characterization of BE with low tree-depth coloring







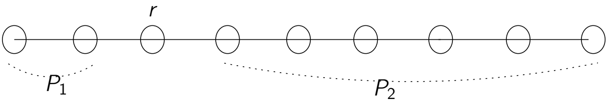
- no edge in G between T_i and T_j:
  - $td(K_n) = n$
  - the root of T separates T_i: the CC of $G \setminus \{r\}$ lie inside the T_i
  - we could have several CC in a T_i, but not interesting when minimizing the height of T

$\Rightarrow$ the T_i correspond exactly to the CC of $G \setminus \{r\}$

Characterization of BE with low tree-depth coloring

Tree-depth of path

$td(P_n) = \lceil log_2(n+1) \rceil$



- let T with root r such that $P_n \subseteq clos(T)$
- $td(P_n) \geq 1 + \max(td(P_1), td(P_2))$

$\Rightarrow$ choose r at the center of the path

Characterization of BE with low tree-depth coloring

Tree-depth coloring for a graph

- Motivation: coloring G such that every p color classes induce a "simple" graph
- $\chi_p(G)$ minimum number of colors such that each $i \leq p$ parts induce a graph with tree-depth at most i
- $\chi_1(G) = \chi(G)$
- $\chi_2(G) = \chi_s(G)$: star coloring: proper coloring and every two parts induces a star forest

Low tree-depth coloring for a class

A class $\mathcal{C}$ has low tree-depth coloring iff $\exists$ function c such that $\forall p$, $\forall G \in \mathcal{C}$, $\chi_p(G) \leq c(p)$

Characterization of BE with low tree-depth coloring

Succession of results described in [NdM08]

Minor closed class has low tree-width coloring

Minor closed class has low tree-depth coloring

Theorem [NdM08]

BE class has low tree-width coloring (in fact iff!)

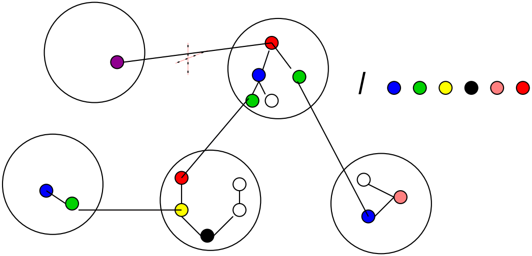
Let us prove the easy part of the last result:

Theorem 4

$\nabla_r(G) \leq (2r+1)\binom{2r+2}{\chi_{2r+2}(G)}$

Characterization of BE with low tree-depth coloring

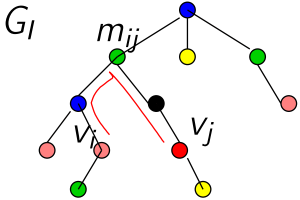
- Let $H \in G\nabla r$ such that $\frac{m_H}{n_H} = \nabla_r(G)$
- Let G' be a witness of H: $G' = \{V_1, \ldots, V_H\}$ where V_i are trees of height $\leq r$
- Let $N = \chi_{2r+2}(G)$, I be a subset of $2r+2$ colors among N
- Let $\{E_I\}$ be the external edges whose corresponding path P_{ij} (of length of at most $2r+2$ vertices) uses only colors of I
- We will prove that $|E_I| \leq 2r+1$



Characterization of BE with low tree-depth coloring

- let G_l be the graph induced by vertices of color l
- $td(G_l) \leq 2r + 2$
- let $e \in E_l$ between V_i and V_j
- let P_{ij} be the corresponding path between v_i and v_j, and m_{ij} be the highest vertex in this path
- orient e towards V_l not containing m_{ij}

$\Rightarrow$ each V_j has in-degree at most $2r + 1$ as each in arc corresponds to a distinct ancestor or v_j



Characterization of BE with χ

We define $\chi(G\tilde{\nabla}r)$ and $\chi(\mathcal{C}\tilde{\nabla}r) = \sup_{G\in\mathcal{C}}(\chi(G\tilde{\nabla}r))$.

Proposition 5.5 in [dM+12]

$\mathcal{C}$ BE $\Leftrightarrow \exists c$ such that $\forall r$, $\chi(\mathcal{C}\tilde{\nabla}r) \leq c(r)$
($\Leftrightarrow \exists c$ such that $\forall r$, $\chi(\mathcal{C}\nabla r) \leq c(r)$)

In fact, we will prove the following property.

Proposition 4.4 in [dM+12]

$\chi(G\tilde{\nabla}r) \leq 2(\tilde{\nabla}_r(G)) + 1$ and $\tilde{\nabla}_r(G) = \mathcal{O}((\chi(G\tilde{\nabla}(2r + \frac{1}{2}))^4)$

Characterization of BE with χ

Proposition 4.4 in [dM+12]

$\chi(G\tilde{\nabla}r) \leq 2(\tilde{\nabla}_r(G)) + 1$ and $\tilde{\nabla}_r(G) = \mathcal{O}((\chi(G\tilde{\nabla}(2r+\frac{1}{2}))^4)$

Proof of the first inequality.

- for $r = 0$ this can be rephrased as "any α degenerate graph can be $\alpha + 1$ colored".
- let $H \in G\tilde{\nabla}r$
- $\chi(H) \leq mad(H) + 1 = 2\tilde{\nabla}_0(H) + 1$
- as $\tilde{\nabla}_0(H) \leq \tilde{\nabla}_r(G)$, done!

Characterization of BE with χ

Proposition 4.4 in [dM+12]

$\chi(G\tilde{\nabla}r) \leq 2(\tilde{\nabla}_r(G)) + 1$ and $\tilde{\nabla}_r(G) = \mathcal{O}((\chi(G\tilde{\nabla}(2r + \frac{1}{2}))^4)$

Proof of the second one.

- No hope to bound $\tilde{\nabla}_r(G) \leq f(\chi(G\tilde{\nabla}r))$ (think of complete bipartite, even for $r = 0$)
- For $r = 0$: what contains $G\tilde{\nabla}\frac{1}{2}$?: graphs H whose 1-subdivision are subgraphs of G
- For $r = 0$ the inequality says (we consider the contrapositive) "if you have a lot of edges then you have one subgraph that is a 1-subdivision of a graph H with large χ"

Characterization of BE with χ

Proposition 4.4 in [dM+12]

$\chi(G\tilde{\nabla}r) \leq 2(\tilde{\nabla}_r(G)) + 1$ and $\tilde{\nabla}_r(G) = \mathcal{O}((\chi(G\tilde{\nabla}(2r+\frac{1}{2}))^4)$

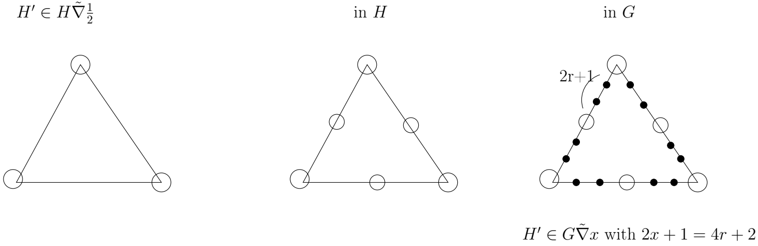
Thus, we will prove the following Lemma.

Lemma 4.5 in [dM+12]

Let $c \geq 4$, G with av degree $d > 56(c-1)^2 \frac{log(c-1)}{log(c)-log(c-1)}$. Then G contains a subgraph G' that is the 1-subdivision of a graph with chromatic number c.

This implies the result we want:

- Let $H \in G\tilde{\nabla}r$ such that $m_H/n_H = \tilde{\nabla}_r(G)$
- Lemma 4.5 says $d_{av}(H) \geq (c-1)^4 \Rightarrow \chi(H\tilde{\nabla}\frac{1}{2}) \geq c$, so $d_{av}(H) \leq \chi(H\tilde{\nabla}\frac{1}{2})^4$
- however $H\tilde{\nabla}\frac{1}{2} \subseteq G\tilde{\nabla}(2r+\frac{1}{2})$, so $\chi(H\tilde{\nabla}\frac{1}{2}) \leq \chi(G\tilde{\nabla}(2r+\frac{1}{2}))$.

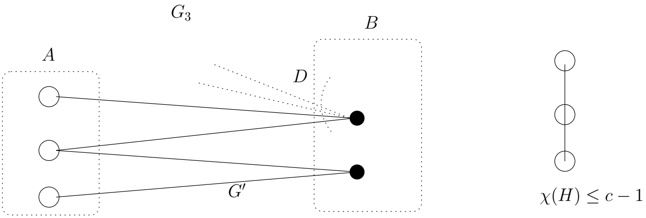


This implies the result we want as:

- Let $H \in G\tilde{\nabla}r$ such that $m_H/n_H = \tilde{\nabla}_r(G)$
- Proposition 4.4 says $d_{av}(H) \geq (c-1)^4 \Rightarrow \chi(H\tilde{\nabla}\frac{1}{2}) \geq c$, so $d_{av}(H) \leq \chi(H\tilde{\nabla}\frac{1}{2})^4$
- however $H\tilde{\nabla}\frac{1}{2} \subseteq G\tilde{\nabla}(2r+\frac{1}{2})$, so $\chi(H\tilde{\nabla}\frac{1}{2}) \leq \chi(G\tilde{\nabla}(2r+\frac{1}{2}))$.

Proof of large av deg $\Rightarrow$ contains G': a 1-sub of a graph with $\chi \geq c$

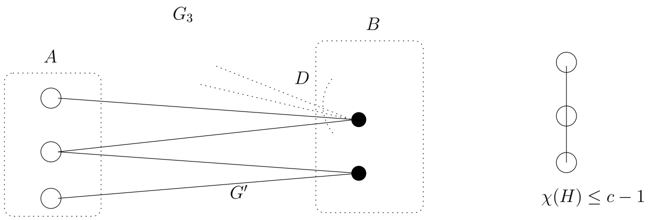
- There exists a bipartite subgraph $G_1 = (A, B) \subseteq G$ with ad degree $\frac{d}{2}$, and $G_2 \subseteq G_1$ with min degree $D \geq \frac{d}{2}$, and $G_3 \subseteq G_2$ with vertices of B having degree exactly D



Characterization of BE with χ

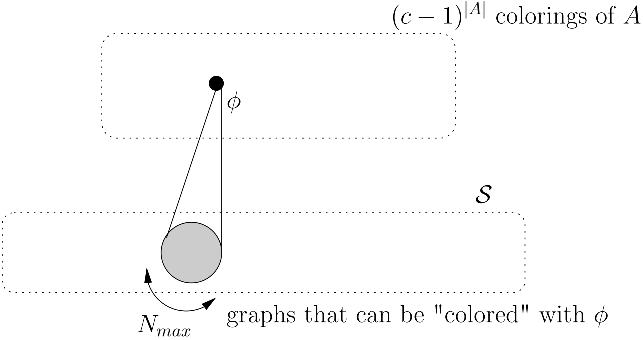
Proof of large av deg $\Rightarrow$ contains G': a 1-sub of a graph with $\chi \geq c$

- By contradiction: suppose that $\forall G' \subseteq G_3$ s.t. $sub(H) = G'$, $\chi(H) \leq c-1$.
  We forget H and say that G' has a "coloring" with $c-1$ colors, where "coloring" means coloring only vertices in A s.t..
- Let $\mathcal{S}$ be the subraphs of G_3 where vertices of B have degree 2
- In particular, $\forall G' \in \mathcal{S}$ have a "coloring" with $c-1$ colors
- Idea: if $c-1$ is to small (1 for example!) and D is big: contradiction



Proof of large av deg $\Rightarrow$ contains G': a 1-sub of a graph with $\chi \geq c$

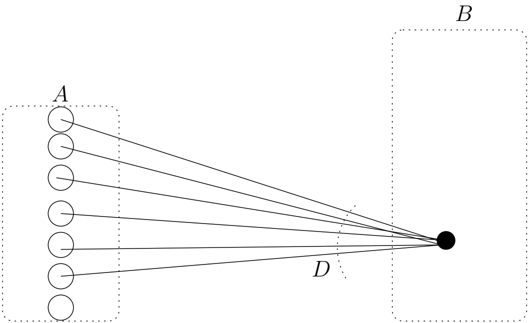
- Let $N_S = |\mathcal{S}|$
- Let $N_c = (c-1)^{|A|}$ be the number of coloring of A
- Let N_{max} be the maximum number of graphs of $\mathcal{S}$ that can be colored with a fixed coloring ϕ of A
- as all graphs of $\mathcal{S}$ can be colored, $N_S \leq N_C N_{max}$



Characterization of BE with χ

Proof of large av deg $\Rightarrow$ contains G': a 1-sub of a graph with $\chi \geq c$

- Let $N_S = |\mathcal{S}| = \binom{2}{D}^{|B|}$
- Let $N_{max} \leq \left(\binom{2}{c-1}\left(\frac{D}{c-1}\right)^2\right)^{|B|}$
- Now, writing $N_S \leq N_C N_{max}$ leads to a contradiction .. if $\frac{|B|}{|A|}$ is large enough



Characterization of BE with χ

Proof of large av deg $\Rightarrow$ contains G': a 1-sub of a graph with $\chi \geq c$

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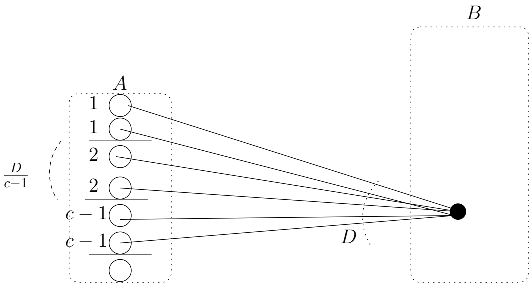


Outline

Equivalence beetween grad and top grad

Corollary 4.1 of [dM+12]

For any G and r, $\tilde{\nabla}_r(G) \leq \nabla_r(G) \leq 4(4\tilde{\nabla}_r(G))^{(r+1)^2}$

In fact, we will prove the following theorem.

Thm 3.9 in [Dvo07]

Let $r, d \geq 1$, $p = 4(4d)^{(r+1)^2}$. If $\nabla_r(G) \geq p$, then G contains a subgraph F' that is a $\leq 2r$ subdivision of a graph F with minimum degree d.

Theorem 2 says: if $\nabla_r(G) \geq p$, then $\tilde{\nabla}_r(G) \geq d$, and thus implies Theorem 1.

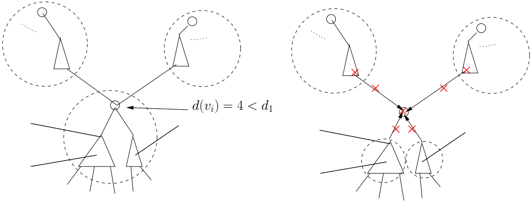
Equivalence beetween grad and top grad

Lemma in [Dvo07]

- Let G' be a radius r witness with min degree (of the corresponding contracted graph) is d.
- Let $d_1 = (\frac{d}{2})^{\frac{1}{r+1}}$.
- There exists a radius r witness $G' \subseteq G$ with min degree (of the corresponding contracted graph) is d_1, such that the degree in G' of each center $v_i \in V_i$ is also at least d_1. Moreover there is no useless leaf in G'.

Lemma says by loosing a factor $\sqrt[r+1]{.}$ on the density of the minor, we can assume that the centers of the witness have large degree.

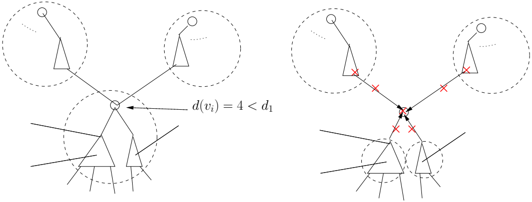
Equivalence beetween grad and top grad



Proof

- while there exists a center $v_i \in G$ with $d(v_i) < d_1$
  - remove v_i and adjacent edges and recursively remove useless leaves (this can decrease degree of other v_j)
  - define new trees corresponding to $V_i \setminus \{v_i\}$

Equivalence beetween grad and top grad

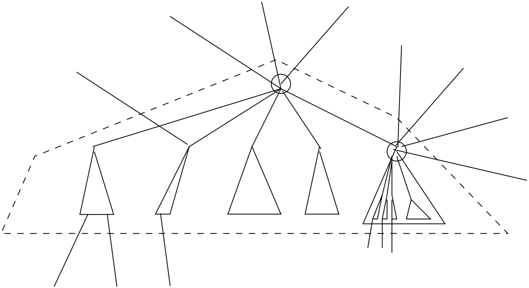


Proof

When we stop, the remaining graph G' is non empty:

- let k be the initial # trees in G, $e \geq \frac{d}{2}k$ be # external edges in G
- when removing v_i, its degree is at most $d_1 \Rightarrow$ at most $d_1 x$ external edges removed, where $x = \#$ suppressed vertices
- we bound x by looking what happen to a given tree

Equivalence beetween grad and top grad

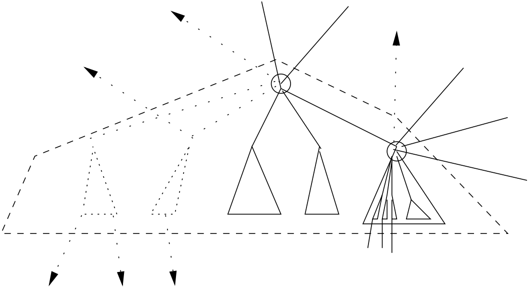


Proof

Upper bound on x:

- all the suppressed vertices belongs to the red subtree of degree at most d_1 and height at most $r \Rightarrow x < kd_1^r$
- we take d_1 such that $kd_1^{r+1} < \frac{d}{2}k$

Equivalence beetween grad and top grad



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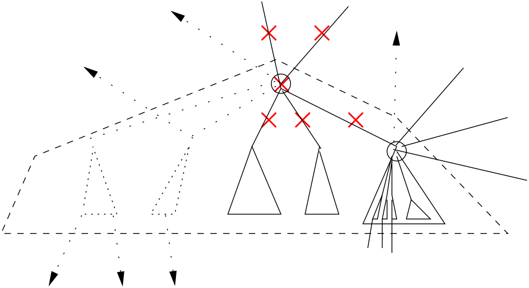

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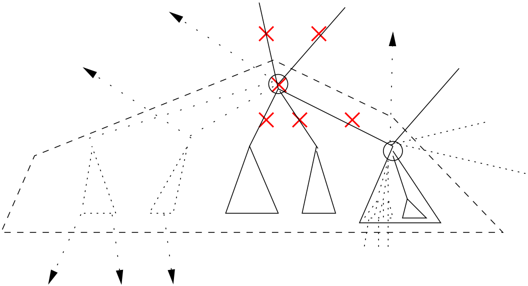

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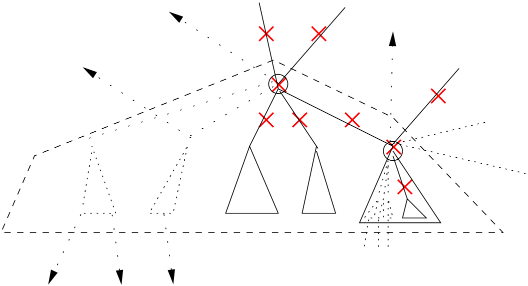
Equivalence beetween grad and top grad



Proof

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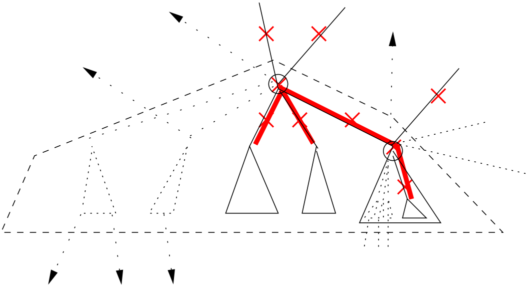
Equivalence beetween grad and top grad

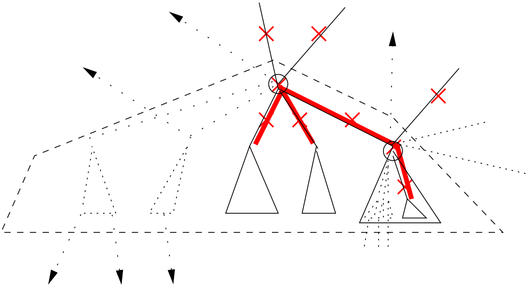


Proof

Upper bound on x:

- all the suppressed vertices belongs to the red subtree of degree at most d_1 and height at most $r \Rightarrow x < kd_1^r$
- we take d_1 such that $kd_1^{r+1} < \frac{d}{2}k$

Equivalence beetween grad and top grad



Proof

Upper bound on x:

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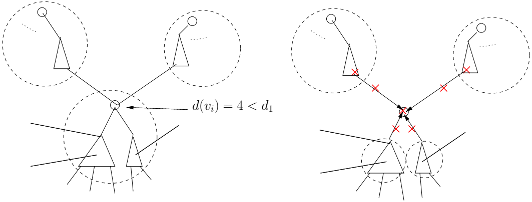
Equivalence beetween grad and top grad



Proof

When we stop, G' satisfies the two claimed properties:

- all centers v_i have $d(v_i) = d_{int} + d_{ext} \geq d_1$
- there is no useless leaf, implying that each of the d_{int} subtrees "produces" at least one external edge

Equivalence beetween grad and top grad

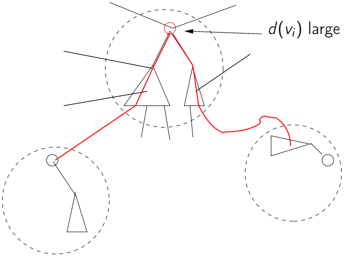
Back to Thm 3.9

Let $r, d \geq 1$, $p = 4(4d)^{(r+1)^2}$. If $\nabla_r(G) \geq p$, then G contains a subgraph F' that is a $\leq 2r$ subdivision of a graph F with minimum degree d.

Sketch of proof

- $\nabla_r(G) \geq p$ implies G contains a subgraph G_1 which is a radius r witness of min degree (in the contracted) p
- using previous lemma, let $G_2 \subseteq G_1$ be a radius r witness of min degree (in the contracted) d_1, such that the degree in G' of each center $v_i \in V_i$ is also at least d_1

Equivalence beetween grad and top grad



- get a subdivided graph $G' \subseteq G_2$ by keeping one external edge out of each subtree (and its corresponding path to the root)
- if you can indeed save these external edges:
    - large degree of center implies that we get many edges
    - the corresponding subgraph G' is a subdivided graph

Problems

- the other vertex of each edge may not be saved
- if the subtrees are very leafy, we have to bound the loss

Outline

Nowhere Dense (ND)

Definition

A class $\mathcal{C}$ is ND iff $\exists c$ such that $\forall r$, $\omega(\mathcal{C}\nabla r) \leq c(r)$

- BE $\subseteq$ ND (for BE we even require $\chi(\mathcal{C}\nabla r) \leq c(r)$)
- there exists several equivalent definitions of ND (Thm 13.2 in [dM+12]).
- in terms of number of edges: $\mathcal{C}$ is ND iff $\exists c$ such that $\forall r$, $\forall G \in \mathcal{C}$, $\forall H \in G\nabla r$, $m_H \leq n_H^{1+f_r(n_H)}$ (with $f_r = o_n(1)$)

Examples

Example of a class $\mathcal{C}$ ND but no BE (p105 [dM$^+$12])

- We want $\mathcal{C}$ such that for $r \geq r_0$ graphs of $\mathcal{C}\nabla r$ have big χ and small ω (Erdös classes).
- Let $\mathcal{C} = \{k$ cages (k-regular graphs with girth=k), $k \geq 0\}$)
- $\mathcal{C}$ is not BE are graphs do not have constant degeneracy
- $\mathcal{C}$ is ND:
  - Assume $K_n \in \mathcal{C}\nabla r$, let us wound $n \leq f(r)$
  - Let $G \in \mathcal{C}$ such that $K_n \in G\nabla r$
  - $K_3 \in G\nabla r \Rightarrow$ there exists a cycle of length at most $3(2r+1) \Rightarrow g(G) \leq 3(2r+1)$
  - $n - 1 \leq \Delta(G\nabla r) \leq \Delta(G)^{r+1}$

Bibliography

[dM$^+$12] Patrice Ossona de Mendez et al.
Sparsity: graphs, structures, and algorithms, volume 28.
Springer Science & Business Media, 2012.

[Dvo07] Zdenek Dvorák.
Asymptotical structure of combinatorial objects.
Charles University, Faculty of Mathematics and Physics, 2007.

[NdM08] Jaroslav Nešetřil and Patrice Ossona de Mendez.
Grad and classes with bounded expansion i. decompositions.
European Journal of Combinatorics, 29(3):760–776, 2008.

[Zhu09] Xuding Zhu.
Colouring graphs with bounded generalized colouring number.
Discrete Mathematics, 309(18):5562–5568, 2009.