Arithmetic Convexity
\textit{Sails and Klein Polyhedra}

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References:
Contemporary Mathematics \textbf{210}, 1998
Book in progress
Klein polyhedra

Continued fractions

Felix Klein’s geometrical interpretation of continued fractions

- $D_i =$ half-line in $\mathbb{R}^2$ generated by $(1, \alpha_i)$ ($i = 1, 2$)
- $C =$ acute angular sector between $D_1$ and $D_2$.

Klein polygon $P$ and Klein line $L$:

$$P = \text{convex hull of } C \cap \mathbb{Z}^2, \quad L = \text{boundary of } P$$

![Figure 1: Drawing from Klein's lithographed lectures](image)

Extremal points of $P$ are the convergents vectors $x_n$ of odd (resp. even) order to $\alpha_1$ (resp. $\alpha_2$)

$$d(x_{2n+1}, D_1) \to 0, \quad d(x_{2n}, D_2) \to 0$$

$P$ contains the information on the approximation of $\alpha_1$ and $\alpha_2$

Generalization to higher dimensions?
Klein hull and sail of a convex set

$A \subset E = \mathbb{R}^d$ closed convex set, $\mathcal{M} = A \cap \mathbb{Z}^d$

**Klein set** of $A$ : $\text{Kl}_0 A = \text{Conv } \mathcal{M}$

**Klein hull** of $A$ : $\text{Kl } A = \overline{\text{Conv } \mathcal{M}}$

**Sail** of $A$ : $V(A) = \partial \text{Kl } A$

- $\text{Kl}_0 A$ is a polytope if $A$ is compact
- Mainly interested here in unbounded Klein hulls

![Figure 2: Artist’s view of a Klein hull (V.I. Arnold)](image-url)
Klein hull of a cone

Simplicial cone $C$ : convex conical hull of a basis of $E$

Blunted cone [fr. cônes épointés] : $C^\dagger = C \setminus \{0\}$

History : The Klein hull $KlC^\dagger$ of a blunted simplicial cone has been successively introduced by Félix Klein (1895, sunk into oblivion), T. Shintani (1976), P. Skubenko (1988), V.I. Arnold (1991)
Figure 4: Six faces of a sail (front view)

Figure 5: Twelve faces of a sail (front view)
Convex geometry

*Polytope*: convex hull of finitely many points

*Polyhedron*: convex hull of finitely many points and half-lines

*Generalized polyhedron* $P$: the intersection of $P$ with any polytope is a polytope. Alternately:

(i) The set $\text{Ext } P$ of extremal points of $P$ is locally finite

(ii) Given $x \in \text{Ext } P$, there are finitely many edges and extremal rays of $K$ with $x$ as a vertex.

A useful criterion:

**Theorem.** $A \subset E$ closed convex set, $x \in A$. Equivalent:

- $x$ is an isolated point of $\text{Ext } A$.
- The cone with vertex $x$ generated by $A$ is pointed and closed.

[pointed cone = fr. cône saillant]
**Topology of** $\text{Kl}_0 A$

For a convex set $A$, consider the following conditions:

(i) $\text{Kl}_0 A$ is a generalized polyhedron.

(In this case say that $A$ is *arithmetically regular*.)

(ii) $\text{Kl}_0 A$ is closed.

(iii) The closure $\text{Kl} A$ is a generalized polyhedron.

Needless to say : (i) $\iff$ [(ii) and (iii)]

- Counterexample :
  
  \[ C = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq z^2 \text{ and } z \geq 0\} \]

  Then neither $\text{Kl} C$ nor $\text{Kl} C^\dagger$ are generalized polyhedra.

- Counterexample : Take for $C$ the convex hull of the half lines with generators

  \[ (n^2 - 1, 2n, n^2 + 1), \quad n \in \mathbb{N}, \quad n \geq 1. \]

  Then $\text{Kl}_0 C = C$ is closed and is not a generalized polyhedron.

- We shall see in the next slide that if $C$ is a polyhedral cone, then (ii) $\iff$ (iii) for $\text{Kl} C^\dagger$

Hence, the three conditions above are equivalent for such a cone.
Closedness criteria

Theorem. \( A \subset E \) polyhedron of dimension \( d \), without lines.

\[(\partial A) \cap \mathbb{Z}^d \text{ empty } \Rightarrow Kl A \text{ is a generalized polyhedron}\]

\( F \subset E \) vector subspace

\( F \text{ rational} : \quad F \cap \mathbb{Z}^d \text{ contain a basis of } F \)

\( F \text{ irrational} : \quad F \cap \mathbb{Z}^d = \{0\} \).

Theorem (Moussafir). \( C \) polyhedral cone. Equivalent :

(i) \( Kl C^\dagger \) is closed (i.e. \( C \) is arithmetically regular).

(ii) \( Kl C^\dagger \) is a generalized polyhedron.

(iii) Any supporting hyperplane of \( C \) is either rational or irrational.
Geometry of Klein polyhedra

$C$ simplicial cone generated by $a_1, \ldots, a_d$; dual basis $w_1, \ldots, w_d$

$x \in C \iff x = \sum_{i=1}^{d} \lambda_i a_i \iff (w_i \mid x) \geq 0 \ (1 \leq i \leq d)$

The polar cone $C^*$ of $C$ is obtained by reversing the generators and the dual basis.

Define

$$H(a) = \{x \in E \mid (a \mid x) = 0\}$$

Assume $C$ and $C^*$ arithmetically regular, and let $P = Kl_0 C^\dagger$.

The following conditions are equivalent:

(i) $P \subset \text{Interior } C$.

(ii) The cone generated by $P$ is equal to $\text{Interior } C$.

(iii) $H(w_i) \cap C \cap \mathbb{Z}^d = \emptyset$ if $1 \leq i \leq d$.

(iv) $H(a_i) \cap C^* \cap \mathbb{Z}^d = \emptyset$ if $1 \leq i \leq d$.

These conditions are satisfied if (iii) or (iv) is satisfied for two indices only.

**Theorem.** Under the preceding conditions, the faces of the generalized polyhedron $P$ are polytopes and the light boundary of $P$ is equal to the sail $V = \partial P$ (hence, homomorphic to $\mathbb{R}^{d-1}$)

We say that $x \in V$ belongs to the **light boundary** of $P$ if a light ray starting from the origin does not meet any point of $P$ between 0 and $x$. 
Approximation

Modes of approximation of a line

Take $a = (a_1, \ldots, a_d) \in E$ ($a \neq 0$)

$D(a)$ : line generated by $a$

$(x_n)_{n \geq 0}$ : unbounded sequence of $\mathbb{Z}^d$

**Strong approximation of $D(a)$** : $\text{dist}(x_n, D(a)) = o(1)$ ($n \to \infty$)

**Bounded app.** : $\cdots = O(1)$

**Weak app.** : $\cdots = o(\text{dist}(x_n, H(a)))$

In terms of simultaneous approximations :

- **Strong approximation** : $||a \wedge x_n|| = o(1)$
- **Bounded app.** : $\cdots = O(1)$
- **Weak app.** : $\cdots = o((a \mid x_n))$

The Jacobi-Perron algorithm gives a weak approximation of $a$
Consequences of Kronecker’s theorem

$V$ vector subspace of $E = E$

$$V_Q = V \cap \mathbb{Q}^d, \quad V_Z = V \cap \mathbb{Z}^d$$

Define

$$\text{Rat } V = \bar{V}_Q = V_Z \otimes \mathbb{R}$$

$C = \text{simplicial cone generated by } a_1, \ldots, a_d; \text{ dual basis } w_1, \ldots, w_d$

By Kronecker’s theorem:

**Theorem.** Assume $C^* \cap \text{Rat } H(a_i) = \{0\}$. If $\varepsilon > 0$, there is $x \in \text{Ext } P$ such that

$$0 \leq \sup_{j \neq i} (w_j \mid x) < \varepsilon.$$ 

More precisely, by transfer theorems:

**Theorem.** for any $\delta > 0$, there is $x \in \text{Ext } P$ such that

$$0 \leq \sup_{j \neq i} (w_j \mid x) < C N^{-\frac{1}{\varepsilon} + \delta}, \quad (w_i \mid x) \leq N,$$

for a suitable $C > 0$ and $N$ sufficiently large, depending on $\delta$.

Hence, $\text{dist}(P, D(a_i)) = 0$, and:

**Corollary.** For every generator $a$ of $C$, there is a strong approximation of $D(a)$ in $\text{Ext } P$. 

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**Voronoï algorithm**

Approximation of the generator $a_i$ of $C$: Take $x_0 \in \text{Ext } P$.

- the set

$$Q_i(x_0) = \{ x \in \text{Ext } P \mid (w_j \mid x) < (w_j \mid x_0) \text{ if } j \neq i \}.$$  

is non-empty (Kronecker).

- The linear form $x \mapsto (w_i \mid x)$ reaches its minimum in $Q_i(x_0)$ in exactly one point $v_i(x_0)$.

Say that $x$ and $y$ in $\text{Ext } P$ are *neighbours* if there is no $z \in C^\dagger \cap \mathbb{Z}^d$ such that $(w_i \mid z) \leq \max((w_i \mid x), (w_i \mid y))$ for $1 \leq i \leq d$.

- Then $x_0$ and $v_i(x_0)$ are neighbours: we say that $v_i(x_0)$ is the *Voronoï neighbour* of $x_0$ towards $D(a_i)$.

This defines a map $v_i : \text{Ext } P \rightarrow \text{Ext } P$, and by induction the *Voronoï sequence*

$$x_{n+1} = v_i(x_n),$$

from $x_0$ towards $a_i$.

**Theorem.** For every generator $a$ of an irrational simplicial cone $C$, the Voronoï sequence towards $D(a)$ is a bounded approximation of $D(a)$. 


Split cones

Lagrange theorem : sequences

$K$ totally real number field of degree $d$ ; $\alpha_1, \ldots, \alpha_d$ basis of $K$
Choose a basis of $K/\mathbb{Q}$

$$a = (\alpha_1, \ldots, \alpha_d) \in E$$

Define a simplicial cone $C$ as the conical hull of $a$ and its conjugate vectors

$$a^{(i)} = (\alpha_1^{(i)}, \ldots, \alpha_d^{(i)}) \in E \quad (1 \leq i \leq d)$$

Such a simplicial cone $C$ is called a split cone.
Let $a \in E$ with $H(a)$ irrational. Let $T \in \text{GL}_d(\mathbb{Q})$ and $x_0 \in \mathbb{Q}^d$.
If the sequence $x_n = T^nx_0$ is a (strong, etc.) approximation of $a$, we call it a linearly periodic approximation, and $T$ is the period matrix.

Theorem (Lagrange I). Let $C$ be a simplicial cone generated by $a_1, \ldots, a_d$. Equivalent conditions :

(i) $C$ is split

(ii) For $1 \leq i \leq d$, there is a weak linearly periodic approximation of $D(a_i)$ in $\text{Ext } P$, and the period matrices are commuting.

(iii) For $1 \leq i \leq d$, the Voronoï sequence towards $D(a_i)$ is linearly periodic, and the period matrices are commuting.
Approximation in a split cone

The multiplication in the field $K$, read in the basis $\alpha_1, \ldots, \alpha_d$, gives the regular representation of the group $\mathcal{U}_K$ of totally positive units of $K$:

$$\begin{align*}
 T : \mathcal{U}_K & \longrightarrow \text{SL}_d(\mathbb{Z})
\end{align*}$$

and if $\lambda \in \mathcal{U}_K$, then

$$T(\lambda)a_i = \lambda^{(i)}a_i.$$ 

Define

$$m(\lambda) = \max(\lambda^{(2)}, \ldots, \lambda^{(d)}).$$

Then $T(\lambda)^n x_0$ is a linearly periodic approximation of $D(a)$ if $\lambda > 1$ and

- **Strong approximation:** $m(\lambda) < 1$ (PV-number)
- **Bounded app.:** $m(\lambda) \leq 1$ (Salem number)
- **Weak app.:** $m(\lambda) < \lambda$

Recall that there are $d - 1$ independent units in $\mathcal{U}_K$ which are PV-numbers.
Galleries in the sail

The sail as a building

A simplicial cone $C$, generated by $a_1, \ldots, a_d$ with dual basis $w_1, \ldots, w_d$ is \textit{doubly irrational} if $H(a_i)$ and $H(w_i)$ are irrationals for $1 \leq i \leq d$. For such a cone:

- $P = \text{Kl}_0 C^\dagger$ is closed (= arithmetic regularity)
- The faces of $P$ are polytopes and puts on $V$ the structure of an euclidean cellular complex, hence of a \textit{Tits building} with the facets as chambers.
- This building is a \textit{pseudovariety}: any face of dim. $d - 2$ is the divider (fr. \textit{cloison}) of exactly two chambers.
- $P$ defines as well an (infinite) \textit{fan}: associate to any face $\sigma$ the cone $\Gamma(\sigma)$ generated by $\sigma$. Then
  - the extremal rays are rational
  - if $\sigma_1$ and $\sigma_2$ are two faces in $V$, then

\[ \Gamma(\sigma_1) \cap \Gamma(\sigma_1) = \Gamma(\sigma_1 \cap \sigma_2) \]
Let $\Gamma_f(V)$ be the following graph:

- Vertices = chambers
- Edges = couples of chambers with a common divider

**Gallery**: a path in this graph

One can replace the complex of polytopes by a simplicial complex as well.

If $\sigma$ is a simplex in the building, let $M(\sigma)$ be the generating matrix of $\sigma$. For two simplices $\sigma$ and $\sigma'$ the *transition matrix* $R(\sigma, \sigma')$ is defined by

\[ M(\sigma') = M(\sigma)R(\sigma, \sigma') \]
Paths in $GL_d(\mathbb{Q})$

$GL_d(\mathbb{Q})$ generated by $U$ and $W$, where

- $W$ = Weyl group of $GL_d(\mathbb{Q})$ (permutation group)
- $U$ subgroup of matrices

$$U = \begin{bmatrix} 1_{d-1} & c \\ 0 & c_d \end{bmatrix} \quad (c \in \mathbb{Q}^{d-1}, c_d \in \mathbb{Q}^\times)$$

**Graph** of $GL_d(\mathbb{Q})$:

- Vertices = elements of $GL_d(\mathbb{Q})$
- Edges = couples $(M, MUW)$ $(U \in U, W \in W, UW \neq 1_d)$

**Path** in this graph: sequence $(M_0, \ldots, M_n, \ldots)$ s.t.

$$M_n = M_0 U_1 W_1 \ldots U_n W_n, \quad (U_n \in U, W_n \in W)$$

**Periodic path**: the sequence $(U_n W_n)$ is (ultimately) periodic

Such a path has a *period matrix* $T$:

$$M_{n+qm} = T^q M_n$$

A gallery $(\sigma_0, \ldots, \sigma_n, \ldots)$ is *normalized* if the chambers are ordered in such a way that

$$R(\sigma_{n-1}, \sigma_n) = U_n W_n \quad (U_n \in U, W_n \in W)$$

Any gallery can be normalized, and:

A normalized gallery maps to a path in the graph of $GL_d(\mathbb{Q})$
Lagrange theorem: galleries

Theorem (Lagrange II). A simplicial cone $C$ with dual basis $w_1, \ldots, w_d$ is split if and only if:

(i) For $1 \leq i \leq d$, there is a periodic normalized gallery in $V$ which is a weak approximation of $H(w_i)$.

(ii) The period matrices of these galleries are commuting.

Definition. Approximation of a hyperplane by a gallery:
Let $(\sigma_0, \ldots, \sigma_n, \ldots)$ be a gallery in $V$. Denote by

$$ (y_n | x) = 1, \quad y_n = \sigma_n^t M(\sigma_n)^{-1} e_0 $$

the support hyperplane of $\sigma_n$, with $e_0 = (1, \ldots, 1)$.

The gallery is a weak approximation of $H(w)$ if

- $(w | y_n) > 0$ and $y_n \to \infty$
- The projection of $y_n$ on $H(w)$ is $o((w | y_n))$

(idem for strong, bounded approximations)
Totally real number fields

Quotient torus and caliber

$K$: totally real number field of degree $d$

$M$: module of $K$, with basis $\alpha_1, \ldots, \alpha_d$; dual basis $\omega_1, \ldots, \omega_d$

$\mathcal{O}$: stabilizer of $M$

$\mathcal{U}$: group of totally positive units of $\mathcal{O}$

Recall the representation

$$T : \mathcal{U} \rightarrow\rightarrow G \subset \text{SL}_d(\mathbb{Z})$$

such that $T(\lambda)a = \lambda a$ if $\lambda \in \mathcal{U}$

Then $G$ is an abelian subgroup of rank $d - 1$

$C$: cone in $E$ generated by the basis $\alpha_1, \ldots, \alpha_d$

• The group $G$ operates on the sail $V$ of $KlC^\dagger$

**Theorem.** The quotient $X = V/G$ is a finite pseudovariety, homeomorphic to a torus of dimension $d - 1$.

**Caliber** of $C$ (or $M$):

$$m = |\text{Vert } X| = |(\text{Ext } P)/G|$$

where $\text{Vert } X$ = set of vertices of $X$

• Caliber of $M$ = Number of reduced modules equivalent to $M$

(in a suitable sense)
Example: Simplest cubic fields

$K$ field of roots of

$$f_m(x) = x^3 - (m - 2)x^2 - (m + 1)x - 1$$

known as “simplest cubic fields” ($m \in \mathbb{Z}$)

- Define $\omega = \text{root } > 1$ of $f_m$
- $1, \omega, \omega^2$ basis of $\mathcal{O}_K$ for most values of $m$ (incl. 1 and 2)
- $M(\omega)$ module with this basis

Theorem. The caliber of $M(\omega)$ is one.

For instance, the modules generated by

$$2 \cos \frac{2\pi}{7} \quad (m = 1), \quad 2 \cos \frac{\pi}{9} \quad (m = 2)$$

have caliber one.

They share this property with the golden number $2 \cos \frac{\pi}{5}$.
This picture shows the building of the sail for \( \omega = 2 \cos \frac{2\pi}{7} \).
The logarithmic embedding maps onto the hexagonal lattice.