Augmented Lattice Reduction for MIMO decoding

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Multiple antenna systems allow to improve data rates and reliability.

In order to increase data rates, both the number of antennas and the size of the signal set can be increased.

This entails a high **decoding complexity** with is a real challenge for practical implementation.
MIMO systems

System model: spatial multiplexing

$m$ transmit antennas, $n$ receive antennas

\[
y_{n \times 1} = H_{n \times m} x_{m \times 1} + w_{n \times 1}
\]

- $H, w$ random with i.i.d. complex Gaussian entries
- $x \in S$ signal constellation, $S \subset \mathbb{Z}[i]^m$
Decoding

\[ \hat{x} = \arg\min_S \|y - Hx\|^2 \]  

ML solution

- **ML Decoders** (Sphere Decoder, Schnorr-Euchner...)
  - optimal performance, but worst-case complexity is exponential in the number of antennas

- **Suboptimal decoders** (ZF, SIC...)
  - polynomial complexity, but poor performance

Channel preprocessing improves the performance of suboptimal decoders:
- Right preprocessing:
  - lattice-reduce the channel matrix
Decoding

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**Channel preprocessing** improves the performance of suboptimal decoders:
- **Right preprocessing**: lattice-reduce the channel matrix
Preprocessing using LLL reduction

$m$ transmit antennas, $n$ receive antennas

\[
\mathbf{y}_{n \times 1} = \mathbf{H}_{n \times m} \mathbf{x}_{m \times 1} + \mathbf{w}_{n \times 1}
\]

- received signal
- channel
- codeword
- noise

\[
\mathbf{H}_{\text{red}} = \mathbf{HU} \quad \text{LLL-reduced form}
\]
Preprocessing using LLL reduction

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received signal \hspace{1cm} channel \hspace{1cm} codeword \hspace{1cm} noise

\[
H_{\text{red}} = HU \quad \text{LLL-reduced form}
\]

**LLL-ZF decoder**

- compute the pseudo-inverse $H_{\text{red}}^\dagger$

\[
\hat{x}_{\text{LLL-ZF}} = U \left( \left\lfloor H_{\text{red}}^\dagger y \right\rfloor \right)
\]
Preprocessing using LLL reduction

\( m \) transmit antennas, \( n \) receive antennas

\[
\begin{align*}
\mathbf{y}_{n \times 1} &= \mathbf{H}_{n \times m} \mathbf{x}_{m \times 1} + \mathbf{w}_{n \times 1} \\
& \quad \text{received signal} \quad \text{channel} \quad \text{codeword} \quad \text{noise}
\end{align*}
\]

\( \mathbf{H}_{\text{red}} = \mathbf{H} \mathbf{U} \quad \text{LLL-reduced form} \)

**LLL-ZF decoder**

- compute the pseudo-inverse \( \mathbf{H}_{\text{red}}^\dagger \)
- \( \hat{\mathbf{x}}_{\text{LLL-ZF}} = \mathbf{U} \left( \left[ \mathbf{H}_{\text{red}}^\dagger \mathbf{y} \right] \right) \)

**LLL-SIC decoder**

- QR decomposition of \( \mathbf{H}_{\text{red}} \)
- \( \tilde{\mathbf{y}} = \mathbf{Q}^T \mathbf{y} \)
- recursively compute \( \tilde{x}_m = \left[ \frac{\tilde{y}_m}{r_{mm}} \right], \quad i = m - 1, \ldots, 1 \)
- \( \hat{\mathbf{x}}_{\text{LLL-SIC}} = \mathbf{U} \tilde{\mathbf{x}} \)
Augmented lattice reduction - principle

- new decoding technique which doesn’t require the inversion of the channel matrix

- MIMO decoding amounts to solving the closest vector problem (CVP) in the lattice generated by the channel matrix

- reduce the CVP to the SVP (shortest vector problem)

- use LLL reduction to solve the SVP in polynomial time
Embedding method to reduce the CVP to the SVP

- Follow Kannan’s approach (1987):
  - embed the \( m \)-dimensional lattice generated by \( H \) into a suitable \((m + 1)\)-dimensional lattice

\[
\tilde{H} = \begin{pmatrix} H & -y \\ 0 & t \end{pmatrix}
\]

- \( v = \begin{pmatrix} Hx - y \\ t \end{pmatrix} = \begin{pmatrix} w \\ t \end{pmatrix} = \tilde{H} \begin{pmatrix} x \\ 1 \end{pmatrix} \)

- if \( \|w\| \) and \( t \) are “small”, \( v \) is a shortest lattice vector
LLL reduction finds the shortest lattice vector

Property of an LLL-reduced basis in dimension $m$:

$$\|h_1\| \leq 2^{\frac{m-1}{2}} d_H$$

- LLL-reduce $\tilde{H}$: $\tilde{H}_{\text{red}} = \tilde{H}U$

$v$ shortest vector in $\mathcal{L}$ is called $\alpha$-unique if $\forall u \in \mathcal{L}$,

$$\|u\| \leq \alpha \|v\| \implies u, v \text{ linearly dependent}$$

$v$ is $2^{\frac{m}{2}}$-unique $\implies \pm v$ is the first column of $\tilde{H}_{\text{red}}$
LLL reduction finds the shortest lattice vector

\[ a(H) \triangleq \min_{1 \leq i \leq m} \| h_i^* \| \] minimum norm of the Gram-Schmidt vectors

**Lemma**

Let \( t = \frac{a(H_{\text{red}})}{2^{m+1}} \), and suppose that \( \| w \| \leq \frac{d_H}{2^{m+1}} \).

Then \( v = \begin{pmatrix} Hx - y \\ t \end{pmatrix} \) is a \( 2^{\frac{m}{2}} \)-unique shortest vector in \( L(\tilde{H}) \).

**Sketch of the proof:**

\[ \frac{d_H}{\alpha^{m-1}} \leq a(H_{\text{red}}) \leq d_H \]

Suppose \( \exists u = \begin{pmatrix} Hx' - qy \\ qt \end{pmatrix} \) such that \( \| u \| \leq 2^\frac{m}{2} \| v \| \), and \( u, v \) linearly independent

- \( \| u \| \geq |q| t \Rightarrow |q| \leq \frac{2^\frac{m}{2} \| v \|}{t} \)

- \( \| H(x' - qx) \| \leq \| Hx' - qy \| + |q| \| y - Hx \| \leq 2^\frac{m}{2} \| v \| + \frac{2^\frac{m}{2} \| v \|}{t} \| w \| \leq \]

\[ \leq 2^\frac{m}{2} \sqrt{\| w \|^2 + t^2 \left( 1 + \frac{\| w \|}{t} \right)} < d_H \Rightarrow \text{contradiction.} \]
Augmented lattice reduction - Decoding

- Choose $t = \frac{a(H_{\text{red}})}{2^m+1} \implies$ the augmented lattice has an exponential gap

- LLL-reduce $\tilde{H}$: $\tilde{H}_{\text{red}} = \tilde{H}\tilde{U}$

$\implies v = \tilde{H} \begin{pmatrix} x \\ 1 \end{pmatrix}$ is the first column of $\tilde{H}_{\text{red}}$

- find the transmitted message $x$ on the first column of $\tilde{U}$:

\[
\hat{x} = \frac{1}{u_{m+1,1}}(\tilde{u}_1, 1, \ldots, \tilde{u}_m, 1)^T
\]
Performance

Diversity gain

\[ d = - \lim_{\text{SNR} \to \infty} \frac{\log(P_e)}{\log(\text{SNR})} \]

Proposition

If \( t = \frac{a(H_{\text{red}})}{2^{m+1}} \), then augmented lattice reduction achieves the maximum receive diversity \( n \).

Sketch of the proof.

\[ P_e(H) \leq P \left\{ \|w\| > \frac{dH}{2^{m+1}} \right\} \leq \frac{C(\log(\text{SNR}))^{n+1}}{\text{SNR}^n} \]
$6 \times 6$ MIMO system, spatial multiplexing

![Graph showing SER vs. SNR for different detection algorithms in a MIMO 6x6 system with 16-QAM modulation. The graph compares ML, MMSE-GDFE+LLL+ZF, MMSE-GDFE+LLL-SIC, MMSE-GDFE+ALR, and MMSE-GDFE+ILR detection methods. The x-axis represents SNR (in dB), and the y-axis represents SER (in dB). The legend indicates the performance of each algorithm.](image_url)
MIMO 8x8, spatial multiplexing, 16-QAM

SNR

SER

\begin{align*}
\text{ML} & \quad \text{MMSE−GDFE+LLL−ZF} \\
\text{ML} & \quad \text{MMSE−GDFE+LLL−SIC} \\
\text{ML} & \quad \text{MMSE−GDFE+ALR}
\end{align*}
Comparison with Kim and Park’s Method

[N. Kim, H. Park, “Improved lattice reduction aided detections for MIMO systems”, *Vehicular Technology Conference* 2006]

- **Same form** of the augmented matrix $\tilde{H}$

- **No exponential gap technique:**
  - the parameter $t$ is “big” ($t > \max |r_{ii}|$, where $H_{\text{red}} = QR$)
  - the solution is found on the last column of $\tilde{U}$

- **no guarantee** that LLL reduction will find the right vector
  $\Rightarrow$ performance is the same as LLL-SIC
Complexity

- # iterations of the LLL algorithm for fixed $H$:
  \[ K(H) \leq m^2 \log \frac{2}{\sqrt{3}} \left( \frac{A(H)}{a(H)} \right) + m \]  
  (Daudé, Vallée 1994)

- $\mathbb{E}(K(H)) \sim O \left( m^2 \ln \left( \frac{m}{n-m+1} \right) \right)$  
  (Jalden et al. 2008)

Comparison with LLL-SIC:

<table>
<thead>
<tr>
<th>LLL-SIC</th>
<th>ALR</th>
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</thead>
<tbody>
<tr>
<td>- LLL on $H$: $\sim O \left( m^2 \ln \left( \frac{m}{n-m+1} \right) \right)$</td>
<td>- LLL on $\tilde{H}$: $\sim O(m^3)$</td>
</tr>
<tr>
<td>- QR of $H_{\text{red}}$: $\sim O(nm^2)$</td>
<td></td>
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</tbody>
</table>
Complexity: numerical simulations

Average number of flops, SNR=12

LLL−ZF
LLL−SIC
ALR
Sphere Decoder