Cellular Automata and Tilings

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Outline of the talk

(1) Cellular automata (CA)
   - Introduction and examples
   - General definitions
   - Topology & Curtis-Hedlund-Lyndon theorem
   - Reversible CA
   - Surjective CA: balance, Garden-of-Eden theorem

(2) Wang tiles
   - Aperiodicity and undecidability
   - NW-determinism & one-dimensional CA
   - Robinson’s aperiodic tile set
Cellular Automata (CA): Introduction

Cellular automata are among the oldest models of natural computing. They are versatile objects of study, investigated

- in physics as discrete models of physical systems,
- in computer science as models of massively parallel computation under the realistic constraints of locality and uniformity,
- in mathematics as endomorphisms of the full shift in the context of symbolic dynamics.
Cellular automata possess several fundamental properties of the physical world: they are

- massively parallel,
- homogeneous in time and space,
- all interactions are local,
- time reversibility and conservation laws can be obtained by choosing the local update rule properly.
Example: the **Game-of-Life** by John Conway.

- Infinite checker-board whose squares (=cells) are colored black (=**alive**) or white (=**dead**).

- At each discrete time step each cell counts the number of living cells surrounding it, and based on this number determines its new state.

- All cells change their state simultaneously.
The local update rule asks each cell to check the present states of the eight surrounding cells.

- If the cell is **alive** then it stays alive (survives) iff it has two or three live neighbors. Otherwise it dies of loneliness or overcrowding.

- If the cell is **dead** then it becomes alive iff it has exactly three living neighbors.
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A typical snapshot of a time evolution in Game-of-Life:

Initial uniformly random configuration.
A typical snapshot of a time evolution in Game-of-Life:

The next generation after all cells applied the update rule.
A typical snapshot of a time evolution in Game-of-Life:

Generation 10
A typical snapshot of a time evolution in Game-of-Life:

Generation 100
A typical snapshot of a time evolution in Game-of-Life:

GOL is a computationally universal two-dimensional CA.
Another famous universal CA: **rule 110** by S. Wolfram.

A one-dimensional CA with binary state set \( \{0, 1\} \), i.e. a two-way infinite sequence of 0’s and 1’s.

Each cell is updated based on its old state and the states of its left and right neighbors as follows:

\[
\begin{align*}
111 & \rightarrow 0 \\
110 & \rightarrow 1 \\
101 & \rightarrow 1 \\
100 & \rightarrow 0 \\
011 & \rightarrow 1 \\
010 & \rightarrow 1 \\
001 & \rightarrow 1 \\
000 & \rightarrow 0
\end{align*}
\]
**Space-time diagram** is a pictorial representation of a time evolution in one-dimensional CA, where space and time are represented by the horizontal and vertical direction:
General definition of $d$-dimensional CA

- **Finite state set** $S$.
- **Configurations** are elements of $S^{\mathbb{Z}^d}$, i.e., functions $\mathbb{Z}^d \rightarrow S$ assigning states to cells,
- **A neighborhood vector**
  \[ N = (\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n) \]
  is a vector of $n$ distinct elements of $\mathbb{Z}^d$ that provide the relative offsets to neighbors.
- The **neighbors** of a cell at location $\vec{x} \in \mathbb{Z}^d$ are the $n$ cells at locations
  \[ \vec{x} + \vec{x}_i, \text{ for } i = 1, 2, \ldots, n. \]
Typical two-dimensional neighborhoods:

Von Neumann neighborhood
\{ (0, 0), (±1, 0), (0, ±1) \}

Moore neighborhood
\{ -1, 0, 1 \} \times \{ -1, 0, 1 \}
The **local rule** is a function

\[ f : S^n \rightarrow S \]

where \( n \) is the size of the neighborhood.

State \( f(a_1, a_2, \ldots, a_n) \) is the new state of a cell whose \( n \) neighbors were at states \( a_1, a_2, \ldots, a_n \) one time step before.
The local update rule determines the global dynamics of the CA: Configuration $c$ becomes in one time step the configuration $e$ where, for all $\vec{x} \in \mathbb{Z}^d$,

$$e(\vec{x}) = f(c(\vec{x} + \vec{x}_1), c(\vec{x} + \vec{x}_2), \ldots, c(\vec{x} + \vec{x}_n)).$$

The transformation

$$G : S^{\mathbb{Z}^d} \rightarrow S^{\mathbb{Z}^d}$$

that maps $c \mapsto e$ is the **global transition function** of the CA.
It is convenient to endow $S^\mathbb{Z}^d$ with the product topology. The topology is compact and induced by a metric.
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The topology is generated by the **cylinder sets**

$$Cyl(c, M) = \{ e \in S^{\mathbb{Z}^d} \mid e(\vec{x}) = c(\vec{x}) \text{ for all } \vec{x} \in M \}$$

for $c \in S^{\mathbb{Z}^d}$ and finite $M \subset \mathbb{Z}^d$.

All cylinder sets are clopen, i.e. closed and open. Cylinders for fixed finite $M \subset \mathbb{Z}^d$ form a finite partitioning of $S^{\mathbb{Z}^d}$. 
Under this topology, a sequence $c_1, c_2, \ldots$ of configurations converges to $c \in S^{\mathbb{Z}^d}$ if and only if for all cells $\vec{x} \in \mathbb{Z}^d$ and for all sufficiently large $i$ holds
\[ c_i(\vec{x}) = c(\vec{x}). \]

Compactness of the topology means that all infinite sequences $c_1, c_2, \ldots$ of configurations have converging subsequences.
All cellular automata are **continuous** transformations

\[ S^\mathbb{Z}^d \rightarrow S^\mathbb{Z}^d \]

under the topology. Indeed, locality of the update rule means that if

\[ c_1, c_2, \ldots \]

is a converging sequence of configurations then

\[ G(c_1), G(c_2), \ldots \]

converges as well, and

\[ \lim_{i \to \infty} G(c_i) = G(\lim_{i \to \infty} c_i). \]
The **translation** $\tau$ determined by vector $\vec{r} \in \mathbb{Z}^d$ is the transformation

$$S^{\mathbb{Z}^d} \longrightarrow S^{\mathbb{Z}^d}$$

that maps $c \mapsto e$ where

$$e(\vec{x}) = c(\vec{x} - \vec{r})$$

for all $\vec{x} \in \mathbb{Z}^d$.

(It is the CA whose local rule is the identity function and whose neighborhood consists of $-\vec{r}$ alone.)

Translations determined by unit coordinate vectors $(0, \ldots, 0, 1, 0 \ldots, 0)$ are called **shifts**
Since all cells of a CA use the same local rule, the CA commutes with all translations:

\[ G \circ \tau = \tau \circ G. \]
We have seen that all CA are continuous, translation commuting maps $S^\mathbb{Z}_d \rightarrow S^\mathbb{Z}_d$.

The **Curtis-Hedlund- Lyndon theorem** from 1969 states that also the converse is true:

**Theorem**: A function $G : S^\mathbb{Z}_d \rightarrow S^\mathbb{Z}_d$ is a CA function if and only if

(i) $G$ is continuous, and

(ii) $G$ commutes with translations.
• The set $S^\mathbb{Z}_d$, together with the shift maps, is the $d$-dimensional **full shift**.

• Topologically closed, shift invariant subsets of $S^\mathbb{Z}_d$ are called **subshifts**.

• Cellular automata are the endomorphisms of the full shift.
A CA is called

- **injective** if $G$ is one-to-one,
- **surjective** if $G$ is onto,
- **bijective** if $G$ is both one-to-one and onto.
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A CA $G$ is a **reversible** (RCA) if there is another CA function $F$ that is its inverse, i.e.

$$G \circ F = F \circ G = \text{identity function}.$$ 

RCA $G$ and $F$ are called the **inverse automata** of each other.
Game-of-Life and Rule 110 are irreversible: Configurations may have several pre-images.
Two-dimensional **Q2R** Ising model by G. Vichniac (1984) is an example of a reversible cellular automaton.

Each cell has a spin that is directed either up or down. The direction of a spin is swapped if and only if among the four immediate neighbors there are exactly two cells with spin up and two cells with spin down:
The twist that makes the Q2R rule reversible: Color the space as a checker-board. On even time steps only update the spins of the white cells and on odd time steps update the spins of the black cells.
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Q2R is **reversible**: The same rule (applied again on squares of the same color) reconstructs the previous generation.

Q2R rule also exhibits a local **conservation law**: The number of neighbors with opposite spins remains constant over time.
Evolution of Q2R from an uneven random distribution of spins:

Initial random configuration with 8% spins up.
Evolution of Q2R from an uneven random distribution of spins:

After approx. one million steps. Notice the clustering.
From the Curtis-Hedlund-Lyndon theorem we get

**Corollary:** A cellular automaton $G$ is reversible if and only if it is bijective.

**Proof:** If $G$ is a reversible CA function then $G$ is by definition bijective.

Conversely, suppose that $G$ is a bijective CA function. Then $G$ has an inverse function $G^{-1}$ that clearly commutes with the shifts. The inverse function $G^{-1}$ is also continuous because the space $S^{\mathbb{Z}^d}$ is compact. It now follows from the Curtis-Hedlund-Lyndon theorem that $G^{-1}$ is a cellular automaton. □
The point of the corollary is that in bijective CA each cell can determine its previous state by looking at the current states in some bounded neighborhood around them.
Garden-Of-Eden and orphans

Configurations that do not have a pre-image are called Garden-Of-Eden configurations. Only non-surjective CA have GOE configurations.

A finite pattern consists of a finite domain \( M \subseteq \mathbb{Z}^d \) and an assignment

\[
p : M \longrightarrow S
\]

of states.

Finite pattern is called an orphan for CA \( G \) if every configuration containing the pattern is a GOE.
From the compactness of $S^{\mathbb{Z}^d}$ we directly get:

**Proposition.** Every GOE configuration contains an orphan pattern.

Non-surjectivity is hence equivalent to the existence of orphans.
Balance in surjective CA

All surjective CA have balanced local rules: for every $a \in S$

$$|f^{-1}(a)| = |S|^{n-1}.$$
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$$|f^{-1}(a)| = |S|^{n-1}.$$ 

Indeed, consider a non-balanced local rule such as rule 110 where five contexts give new state 1 while only three contexts give state 0:

- $111 \rightarrow 0$
- $110 \rightarrow 1$
- $101 \rightarrow 1$
- $100 \rightarrow 0$
- $011 \rightarrow 1$
- $010 \rightarrow 1$
- $001 \rightarrow 1$
- $000 \rightarrow 0$
Consider finite patterns where state 0 appears in every third position. There are $2^{2(k-1)} = 4^{k-1}$ such patterns where $k$ is the number of 0’s.

\[
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & \ldots & 0 & 0
\end{array}
\]
Consider finite patterns where state 0 appears in every third position. There are $2^{2(k-1)} = 4^{k-1}$ such patterns where $k$ is the number of 0’s.

A pre-image of such a pattern must consist of $k$ segments of length three, each of which is mapped to 0 by the local rule. There are $3^k$ choices.

As for large values of $k$ we have $3^k < 4^{k-1}$, there are fewer choices for the red cells than for the blue ones. Hence some pattern has no pre-image and it must be an orphan.
One can also verify directly that pattern

01010

is an orphan of rule 110. It is the shortest orphan.
Balance of the local rule is not sufficient for surjectivity. For example, the **majority** CA (Wolfram number 232) is a counter example. The local rule

\[ f(a, b, c) = 1 \text{ if and only if } a + b + c \geq 2 \]

is clearly balanced, but 01001 is an orphan.
The balance property of surjective CA generalizes to finite patterns of arbitrary shape:

**Theorem:** Let $G$ be surjective. Let $M, D \subseteq \mathbb{Z}^d$ be finite domains such that $D$ contains the neighborhood of $M$. Then every finite pattern with domain $M$ has the same number

$$n|D| - |M|$$

of pre-images in domain $D$, where $n$ is the number of states. $\square$
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The balance property means that the uniform probability measure is **invariant** for surjective CA. (Uniform randomness is preserved by surjective CA.)
Let us call configurations $c_1$ and $c_2$ **asymptotic** if the set

$$\text{diff}(c_1, c_2) = \{ \vec{n} \in \mathbb{Z}^d \mid c_1(\vec{n}) \neq c_2(\vec{n}) \}$$

of positions where $c_1$ and $c_2$ differ is finite.

A CA is called **pre-injective** if any asymptotic $c_1 \neq c_2$ satisfy $G(c_1) \neq G(c_2)$. 
The **Garden-Of-Eden -theorem** by Moore (1962) and Myhill (1963) connects surjectivity with pre-injectivity.

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**Theorem:** CA $G$ is surjective if and only if it is pre-injective.

The proof idea can be easily explained using rule 110 as a running example.
1) $G$ not surjective $\implies G$ not pre-injective:

Since rule 110 is not surjective it has an orphan 01010 of width five. Consider a segment of length $5k - 2$, for some $k$, and configurations $c$ that are in state 0 outside this segment. There are $2^{5k-2} = 32^k/4$ such configurations.
1) $G$ not surjective $\implies G$ not pre-injective:

The non-0 part of $G(c)$ is within a segment of length $5k$. Partition this segment into $k$ parts of length 5. Pattern 01010 cannot appear in any part, so only $2^5 - 1 = 31$ different patterns show up in the subsegments. There are at most $31^k$ possible configurations $G(c)$.
1) \( G \) not surjective \( \implies \) \( G \) not pre-injective:

The non-0 part of \( G(c) \) is within a segment of length \( 5k \). Partition this segment into \( k \) parts of length 5. Pattern 01010 cannot appear in any part, so only \( 2^5 - 1 = 31 \) different patterns show up in the subsegments. There are at most \( 31^k \) possible configurations \( G(c) \).

As \( 32^k/4 > 31^k \) for large \( k \), there are more choices for red than blue segments. So there must exist two different red configurations with the same image.
2) \( G \) not pre-injective \( \implies G \) not surjective:

In rule 110

\[
\begin{array}{cccccc}
0 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 0
\end{array}
\] \quad \begin{array}{cccccc}
0 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0
\end{array}

so patterns \( p \) and \( q \) of length 8 can be exchanged to each other in any configuration without affecting its image. There exist just

\[ 2^8 - 1 = 255 \]

essentially different blocks of length 8.
2) $G$ not pre-injective $\implies G$ not surjective:

Consider a segment of $8k$ cells, consisting of $k$ parts of length 8. Patterns $p$ and $q$ are exchangeable, so the segment has at most $255^k$ different images.
2) $G$ not pre-injective $\implies$ $G$ not surjective:

Consider a segment of $8k$ cells, consisting of $k$ parts of length 8. Patterns $p$ and $q$ are exchangeable, so the segment has at most $255^k$ different images.

There are, however, $2^{8k-2} = 256^k/4$ different patterns of size $8k - 2$. Because $255^k < 256^k/4$ for large $k$, there are blue patterns without any pre-image.
Garden-Of-Eden -theorem: CA $G$ is surjective if and only if it is pre-injective.
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Corollary: Every injective CA is also surjective. Injectivity, bijectivity and reversibility are equivalent concepts.

Proof: If $G$ is injective then it is pre-injective. The claim follows from the Garden-Of-Eden -theorem.
Surjective = Pre-injective

Injective = Bijection = Reversible
Examples:

The majority rule is not surjective: finite configurations

\[ \ldots 0000000 \ldots \quad \text{and} \quad \ldots 0001000 \ldots \]

have the same image, so $G$ is not pre-injective. Pattern

\[ 01001 \]

is an orphan.
Examples:

In Game-Of-Life a lonely living cell dies immediately, so $G$ is not pre-injective. GOL is hence not surjective.
Interestingly, no small orphans are known for Game-Of-Life. Currently, the smallest known orphan consists of 113 cells:
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   - Topology & Curtis-Hedlund-Lyndon theorem
   - Reversible CA
   - Surjective CA: balance, Garden-of-Eden theorem

(2) Wang tiles
   - Aperiodicity and undecidability
   - NW-determinism & one-dimensional CA
   - Robinson’s aperiodic tile set
Wang tiles and decidability questions

Suppose we are given a CA (in terms of its local update rule) and want to know if it is reversible or surjective? Is there an algorithm to decide this? Or is there an algorithm to determine if the dynamics of a given CA is trivial in the sense that after a while all activity has died?

It turns out that many such algorithmic problems are undecidable. In some cases there is an algorithm for one-dimensional CA while the two-dimensional case is undecidable.

A useful tool to obtain undecidability results is the concept of Wang tiles and the undecidable tiling problem.
A **Wang tile** is a unit square tile with colored edges. A tile set $T$ is a finite collection of such tiles. A valid tiling is an assignment

$$\mathbb{Z}^2 \rightarrow T$$

of tiles on infinite square lattice so that the abutting edges of adjacent tiles have the same color.
A **Wang tile** is a unit square tile with colored edges. A tile set $T$ is a finite collection of such tiles. A valid tiling is an assignment

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of tiles on infinite square lattice so that the abutting edges of adjacent tiles have the same color.

For example, consider Wang tiles

![Wang tiles](image)
With copies of the given four tiles we can properly tile a $5 \times 5$ square...

... and since the colors on the borders match this square can be repeated to form a valid periodic tiling of the plane.
Configuration $c \in T^\mathbb{Z}^2$ is **valid inside** $M \subseteq \mathbb{Z}^2$ if the colors match between any two neighboring cells, both of which are inside region $M$.

If here $M = \mathbb{Z}^2$, the configuration is a **valid tiling** of the plane.
The set of valid tilings over $T$ is a translation invariant, compact subset of the configuration space $T^\mathbb{Z}^2$, i.e., it is a subshift.
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More precisely, valid tilings form a subshift of finite type because they are defined via a finite collection of patterns that are not allowed in any valid tiling.

Moreover, any two-dimensional subshift of finite type is conjugate to the set of valid tilings under a suitable Wang tile set.
A configuration $c \in T^{\mathbb{Z}^2}$ (doubly) periodic if there are two linearly independent translations $\tau_1$ and $\tau_2$ that keep $c$ invariant:

$$\tau_1(c) = \tau_2(c) = c.$$

Then $c$ is also invariant under some horizontal and vertical translations.
A configuration $c \in T^\mathbb{Z}^2$ \textbf{(doubly) periodic} if there are two linearly independent translations $\tau_1$ and $\tau_2$ that keep $c$ invariant:

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Then $c$ is also invariant under some horizontal and vertical translations.

\textbf{Proposition:} If a tile set admits a tiling that is invariant under some non-zero translation then it admits a valid doubly periodic tiling.
More generally, a $d$-dimensional configuration $c \in S^{\mathbb{Z}^d}$ is (d-ways) periodic if it is invariant under $d$ linearly independent translations.
The **tiling problem** of Wang tiles is the decision problem to determine if a given finite set of Wang tiles admits a valid tiling of the plane.

**Theorem (R. Berger 1966):** The tiling problem of Wang tiles is undecidable.
Observations:

(1) If $T$ admits valid tilings inside squares of arbitrary size then it admits a valid tiling of the whole plane.
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Follows from compactness: Let $C_n$ be the set of configurations that satisfy the tiling constraint inside the $(2n + 1) \times (2n + 1)$ square centered at the origin. Then

$$C_1 \supseteq C_2 \supseteq C_3 \supseteq \ldots$$

is a decreasing chain of non-empty compact sets. Hence their intersection is non-empty.
Observations:

(1) If $T$ admits valid tilings inside squares of arbitrary size then it admits a valid tiling of the whole plane.

(2) There is a semi-algorithm to recursively enumerate tile sets that do not admit valid tilings of the plane.
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(1) If \( T \) admits valid tilings inside squares of arbitrary size then it admits a valid tiling of the whole plane.

(2) There is a \textit{semi-algorithm} to recursively enumerate tile sets that do not admit valid tilings of the plane.

Follows from (1): Just try tiling larger and larger squares until (if ever) a square is found that can not be tiled.
Observations:

(1) If $T$ admits valid tilings inside squares of arbitrary size then it admits a valid tiling of the whole plane.

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(1) If $T$ admits valid tilings inside squares of arbitrary size then it admits a valid tiling of the whole plane.

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(3) There is a semi-algorithm to recursively enumerate tile sets that admit a valid periodic tiling.

Reason: Just try tiling rectangles until (if ever) a valid tiling is found where colors on the top and the bottom match, and left and the right sides match as well.
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(1) If $T$ admits valid tilings inside squares of arbitrary size then it admits a valid tiling of the whole plane.

(2) There is a semi-algorithm to recursively enumerate tile sets that do not admit valid tilings of the plane.

(3) There is a semi-algorithm to recursively enumerate tile sets that admit a valid periodic tiling.

(4) There exist aperiodic sets of Wang tiles. These

- admit valid tilings of the plane, but
- do not admit any periodic tiling
Observations:

(1) If $T$ admits valid tilings inside squares of arbitrary size then it admits a valid tiling of the whole plane.

(2) There is a **semi-algorithm** to recursively enumerate tile sets that do not admit valid tilings of the plane.

(3) There is a **semi-algorithm** to recursively enumerate tile sets that admit a valid periodic tiling.

(4) There exist **aperiodic** sets of Wang tiles. These

  - admit valid tilings of the plane, but
  - do not admit any periodic tiling

Follows from (2), (3) and undecidability of the tiling problem.
The tiling problem can be reduced to various decision problems concerning (two-dimensional) cellular automata, so that the undecidability of these problems then follows from Berger’s result.

This is not so surprising since Wang tilings are “static” versions of “dynamic” cellular automata.
**Example:** Let us prove that it is undecidable whether a given two-dimensional CA $G$ has any fixed point configurations, that is, configurations $c$ such that $G(c) = c$.

**Proof:** Reduction from the tiling problem. For any given Wang tile set $T$ (with at least two tiles) we effectively construct a two-dimensional CA with state set $T$, the von Neumann -neighborhood and a local update rule that keeps a tile unchanged if and only if its colors match with the neighboring tiles.

Trivially, $G(c) = c$ if and only if $c$ is a valid tiling. □
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Trivially, $G(c) = c$ if and only if $c$ is a valid tiling. □

**Note:** For one-dimensional CA it is decidable whether fixed points exist. Fixed points form a subshift of finite type that can be effectively constructed.
More interesting reduction: A CA is called **nilpotent** if all configurations eventually evolve into the quiescent configuration.

**Observation:** In a nilpotent CA all configurations must become quiescent within a bounded time, that is, there is number $n$ such that $G^n(c)$ is quiescent, for all $c \in \mathcal{S} \mathbb{Z}^d$. 
More interesting reduction: A CA is called **nilpotent** if all configurations eventually evolve into the quiescent configuration.

**Observation:** In a nilpotent CA all configurations must become quiescent within a bounded time, that is, there is number $n$ such that $G^n(c)$ is quiescent, for all $c \in S^{\mathbb{Z}^d}$.

**Proof:** Suppose contrary: for every $n$ there is a configuration $c_n$ such that $G^n(c_n)$ is not quiescent. Then $c_n$ contains a finite pattern $p_n$ that evolves in $n$ steps into some non-quiescent state. A configuration $c$ that contains a copy of every $p_n$ never becomes quiescent, contradicting nilpotency.
Theorem (Culik, Pachl, Yu, 1989): It is undecidable whether a given two-dimensional CA is nilpotent.
Theorem (Culik, Pachl, Yu, 1989): It is undecidable whether a given two-dimensional CA is nilpotent.

Proof: For any given set $T$ of Wang tiles the goal is to construct a two-dimensional CA that is nilpotent if and only if $T$ does not admit a tiling.
For tile set $T$ we make the following CA:

- State set is $S = T \cup \{q\}$ where $q$ is a new symbol $q \not\in T$, 
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- State set is $S = T \cup \{q\}$ where $q$ is a new symbol $q \not\in T$,
- Von Neumann neighborhood,
- The local rule keeps state unchanged if all states in the neighborhood are tiles and the tiling constraint is satisfied. In all other cases the new state is $q$. 
For tile set $T$ we make the following CA:

- State set is $S = T \cup \{q\}$ where $q$ is a new symbol $q \not\in T$,
- Von Neumann neighborhood,
- The local rule keeps state unchanged if all states in the neighborhood are tiles and the tiling constraint is satisfied. In all other cases the new state is $q$. 

![Diagram of a Von Neumann neighborhood with states B, C, A, and D, with A transitioning to A.]
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$\implies$ If $T$ admits a tiling $c$ then $c$ is a non-quiescent fixed point of the CA. So the CA is not nilpotent.

$\impliedby$ If $T$ does not admit a valid tiling then every $n \times n$ square contains a tiling error, for some $n$. State $q$ propagates, so in at most $2n$ steps all cells are in state $q$. The CA is nilpotent. \qed
If we do the previous construction for an aperiodic tile set $T$ we obtain a two-dimensional CA in which every periodic configuration becomes eventually quiescent, but there are some non-periodic fixed points.
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Another interesting observation is that while in nilpotent CA all configurations become quiescent within bounded time, that transient time can be very long: one cannot compute any upper bound on it as otherwise nilpotency could be effectively checked.
While tilings relate naturally to two-dimensional CA, one can strengthen Berger’s result so that the nilpotency can be proved undecidable for one-dimensional CA as well.

The basic idea is to view space-time diagrams of one-dimensional CA as tilings: they are two-dimensional subshifts of finite type.
Tile set $T$ is **NW-deterministic** if no two tiles have identical colors on their top edges and on their left edges. In a valid tiling the left and the top neighbor of a tile uniquely determine the tile.

For example, our sample tile set

![Tile set ABCD](image)

is NW-deterministic.
In any valid tiling by NW-deterministic tiles, NE-to-SW diagonals uniquely determine the next diagonal below them. The tiles of the next diagonal are determined locally from the previous diagonal:
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More precisely, for any given NW-deterministic tile set $T$ we construct a one-dimensional CA whose

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More precisely, for any given NW-deterministic tile set $T$ we construct a one-dimensional CA whose

- state set is $S = T \cup \{q\}$ where $q$ is a new symbol $q \not\in T$,
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More precisely, for any given NW-deterministic tile set $T$ we construct a one-dimensional CA whose

- state set is $S = T \cup \{q\}$ where $q$ is a new symbol $q \notin T$,
- neighborhood is $(0, 1)$,
- local rule $f : S^2 \rightarrow S$ is defined as follows:

  - $f(A, B) = C$ if the colors match in $\begin{array}{c} A \\ B \\ C \end{array}$
  - $f(A, B) = q$ if $A = q$ or $B = q$ or no matching tile $C$ exists.
Claim: The CA is nilpotent if and only if $T$ does not admit a tiling.
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Proof:

$\implies$ If $T$ admits a tiling $c$ then diagonals of $c$ are configurations that never evolve into the quiescent configuration. So the CA is not nilpotent.
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Proof:

$\implies$ If $T$ admits a tiling $c$ then diagonals of $c$ are configurations that never evolve into the quiescent configuration. So the CA is not nilpotent.

$\impliedby$ If $T$ does not admit a valid tiling then every $n \times n$ square contains a tiling error, for some $n$. Hence state $q$ is created inside every segment of length $n$. Since $q$ starts spreading once it has been created, the whole configuration becomes eventually quiescent.
Now we just need the following strengthening of Berger’s theorem:

**Theorem:** The tiling problem is undecidable among NW-deterministic tile sets.

and we have

**Theorem:** It is undecidable whether a given one-dimensional CA (with spreading state $q$) is nilpotent. □
NW-deterministic tile sets exist (and we design such a set later in the talk).

If we do the previous construction using an aperiodic set then we have an interesting one-dimensional CA:

- all periodic configurations eventually die, but
- there are non-periodic configurations that never create a quiescent state in any cell.
NW-deterministic tile sets exist (and we design such a set later in the talk).

If we do the previous construction using an aperiodic set then we have an interesting one-dimensional CA:

- all periodic configurations eventually die, but
- there are non-periodic configurations that never create a quiescent state in any cell.

As in the two-dimensional case, the transient time before a one-dimensional nilpotent CA dies can be very long: it cannot be bounded by any computable function.
The construction also provides the following result (due to Culik, Hurd, Kari):

**Theorem:** The topological entropy of a one-dimensional CA cannot be effectively computed, or even approximated.
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**Theorem:** The topological entropy of a one-dimensional CA cannot be effectively computed, or even approximated.

**Proof:** Add to the previous construction as a second layer a CA $A$ with positive entropy $h(A)$. States of the new layer are killed whenever the tiling layer enters state $q$.

This CA is still nilpotent (and has zero entropy) if the tiles do not admit a tiling, but otherwise contain all orbits of $A$ and hence have entropy at least as high as $h(A)$.
Analogously we can define NE-, SW- and SE-determinism of tile sets. A tile set is called 4-way deterministic if it is deterministic in all four corners.

Our sample tile set is 4-way deterministic

A B C D
Analogously we can define NE-, SW- and SE-determinism of tile sets. A tile set is called 4-way deterministic if it is deterministic in all four corners.

Our sample tile set is 4-way deterministic

\[
\begin{array}{cccc}
A & B & C & D \\
\end{array}
\]

Recently V. Lukkarila showed the following:

**Theorem:** The tiling problem is undecidable among 4-way deterministic tile sets.

This result provides some undecidability results for dynamics of reversible one-dimensional CA.