(Non-)Existence of full shift factors for $\mathbb{Z}^d$ shifts of finite type

(joint work with Mike Boyle and Mike Boyle & Ronnie Pavlov)

Michael H. Schraudner

Centro de Modelamiento Matemático
Universidad de Chile

mschraudner@dim.uchile.cl
www.cmm.uchile.cl/~mschraudner

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Preliminaries

A some finite alphabet

d-dimensional full shift: \( A^\mathbb{Z}^d, \ \forall \vec{j} \in \mathbb{Z}^d : \sigma_{\vec{j}} : A^\mathbb{Z}^d \to A^\mathbb{Z}^d \ (x_{\vec{i}+\vec{j}})_{\vec{i} \in \mathbb{Z}^d} \mapsto (x_{\vec{i}})_{\vec{i} \in \mathbb{Z}^d} \)

d-dimensional subshift: \( X \subseteq A^\mathbb{Z}^d \) shift invariant, closed subset

d-dimensional shift of finite type (SFT):

\( X \) is a \( \mathbb{Z}^d \) SFT \( \iff \exists S \subset \mathbb{Z}^d \) finite set of coordinates, , \( P \subseteq A^S \) set of allowed patterns on \( S : X = \{ x \in A^{\mathbb{Z}^d} : \forall \vec{j} \in \mathbb{Z}^d : x|_{\vec{j}+S} \in P \} \)

\( \mathbb{Z}^d \) SFTs are much more complicated and diverse than \( \mathbb{Z} \) SFTs (undecidability results, large and very inhomogeneous class of shifts).

d-dimensional topological entropy of a \( \mathbb{Z}^d \) subshift \( X \):

\[
    h(X) = \limsup_{n \to \infty} \frac{1}{\text{card } F(n)} \log \text{card}\{ x|_{F(n)} : x \in X \}
\]

where \( F(n) = \{ \vec{j} \in \mathbb{Z}^d : ||\vec{j}||_\infty < n \} \)

A \( \mathbb{Z}^d \) shift \( (Y, \sigma_Y) \) is a (topological) factor of \( (X, \sigma_X) \)

\( \iff \exists \phi : X \to Y \) surjective continuous map such that \( \phi \circ \sigma_X = \sigma_Y \circ \phi \)
Some facts about $\mathbb{Z}$ shifts – some questions about $\mathbb{Z}^d$ shifts

$\mathbb{Z}^d$ SFTs are a much more complicated and diverse class of systems than $\mathbb{Z}$ SFTs. There are several undecidability results (existence and denseness of periodic points, emptiness problem, extension problem) for $\mathbb{Z}^d$ SFTs not present for $\mathbb{Z}$ SFTs. Properties of $\mathbb{Z}$ SFTs generalize, if at all, only to certain subclasses of $\mathbb{Z}^d$ SFTs when $d > 1$.

1. $\mathbb{Z}$ SFTs $X$ contain a tremendous diversity of pairwise disjoint $\mathbb{Z}$ SFTs with entropies dense in $[0, h(X)]$ (Jewett-Krieger-Theorem, Krieger-Embedding-Theorem); moreover those subSFTs can always be realized disjoint from any given proper subsystem of $X$.

For $d \geq 2$ a $\mathbb{Z}^d$ SFT contains SFTs with entropies dense in $[0, h(X)]$ (Desai).

**Question:** Can those $\mathbb{Z}^d$ subSFTs be chosen pairwise disjoint?

2. $\mathbb{Z}$ sofic shifts contain pairwise disjoint SFTs with entropies dense in $[0, h(S)]$ (Mixing $\mathbb{Z}$ sofics contain mixing $\mathbb{Z}$ SFTs of large entropy).

For $d \geq 2$ a $\mathbb{Z}^d$ sofic shift contains sofic shifts with entropies dense in $[0, h(S)]$ (Desai).

**Question:** Is there always a family of $\mathbb{Z}^d$ SFTs with dense entropies?

3. The abundance of subsystems in $\mathbb{Z}$ SFTs allows a complete characterization of lower entropy factors (see below).
The factoring-onto-full-shifts problem

We consider the following problem posed by Johnson and Madden:

**Question:** If $N$ is a positive integer and $X$ is a $\mathbb{Z}^d$ SFT with entropy $h(X) \geq \log N$, must there exist a continuous factor map from $X$ onto the full $\mathbb{Z}^d$ shift on $N$ symbols?

It is interesting to identify the factors (=building blocks) of a given system, topological analogue (for $\mathbb{Z}^d$ SFTs) of Sinai’s measurable Factor Theorem.

The case $d = 1$: The answer to the Question is YES and even a much stronger statement holds:

If $S$ is a $\mathbb{Z}$ sofic shift and $Y$ is a mixing $\mathbb{Z}$ SFT such that $h(S) > h(Y)$, then $Y$ is a factor of $S$ iff

$$\forall s \in S, n \in \mathbb{N} \text{ with } \sigma^n_S(s) = s \implies \exists y \in Y \text{ such that } \sigma^n_Y(y) = y.$$

A $\mathbb{Z}$ SFT with $h(X) \geq \log N$ factors onto the full $\mathbb{Z}$ shift on $N$ symbols (Boyle, Marcus).

The case $d > 1$: For $d > 1$ and $h(X) > \log N$, partial answer.

**Theorem [Johnson-Madden; Desai]:** Let $X$ be a corner gluing $\mathbb{Z}^d$ SFT with $h(X) > \log N$, then $X$ factors onto the full $N$ shift.
Results: The equal entropy case

For the equal entropy case we have produced counterexamples:

**Theorem [Boyle-Schraudner]:** Given integers $N, d > 1$, there are $\mathbb{Z}^d$ SFTs with entropy $\log N$ which DO NOT factor onto the full $N$ shift.

**Sketch of proof:** Produce a measure-of-clopen-sets-obstruction.

For our obstruction pick a prime $p$ that divides $N > 1$ and choose $K > N$ not divisible by $p$.

Start with a result of Hochman-Meyerovitch:

$r \in \mathbb{R}$ is **right recursively enumerable** if there exists a sequence $r(n) \geq r$ converging to $r$ and a Turing machine which given $n \in \mathbb{N}$ produces output $r(n)$.

The **upper frequency** of $\mathcal{A}' \subset \mathcal{A}$ in a point $x \in X$ is defined to be

$$\limsup_{n \to \infty} \frac{1}{\text{card } F(n)} \text{card}\{\vec{j} \in F(n) : x_{\vec{j}} \in \mathcal{A}'\}$$

If the lim sup is a limit, then it gives the **frequency** of $\mathcal{A}'$ in $x$. 
Theorem [Hochman-Meyerovitch]: Suppose $r \in [0, 1]$ is right recursively enumerable. Then there is a zero entropy $\mathbb{Z}^d$ SFT $Z$ and a subset $\mathcal{A}' \subset \mathcal{A}$ of the alphabet of $Z$ such that:

- for any point $z \in Z$, the upper frequency of $\mathcal{A}'$ is at most $r$
- there exists a point of $Z$ in which $\mathcal{A}'$ has frequency $r$.

Take $r = (\log N)/(\log K)$ for our example (right recursively enumerable by log power series).

Now replace the symbols of $\mathcal{A}'$ in every point $z \in Z$ independently with one of $K$ copies. Define

$$\tilde{\mathcal{A}} = (\mathcal{A} \setminus \mathcal{A}') \cup \{(a, i) : a \in \mathcal{A}' \land i \in \{1, \ldots, K\}\}$$

Define the subshift $W$ consisting of all configurations $w \in \tilde{\mathcal{A}}^{\mathbb{Z}^d}$ such that the one-block code $\pi : \tilde{\mathcal{A}} \to \mathcal{A}$ given by $a \mapsto a$ if $a \notin \mathcal{A}'$ and $(a, i) \mapsto a$ if $a \in \mathcal{A}'$ sends $W$ onto $Z$. 
Given $\mu$ on $W$ we denote by $\{\mu_z\}$ the $\nu$-a.e. unique family of Borel probabilities on the fibers $\pi^{-1}z$ such that $\mu(E) = \int \mu_z(E \cap \pi^{-1}z) \, d\nu(z)$, for all Borel sets $E$.

Given $\nu = \pi\mu$ on $Z$, let $\tilde{\nu}$ be the unique lift of $\nu$ such that $\tilde{\nu}_z = \beta_z$ for $\nu$-a.e. $z$.

**Lemma:** Suppose $Z$ is a $\mathbb{Z}^d$ subshift; $W$, $\pi$ and $\tilde{\nu}$ as above; $\mu \in \mathcal{M}(W)$; and $\pi\mu = \nu$. Then

$$h_\mu(W) \leq h_\nu(Z) + \nu(\bigcup_{a \in A'[a]0}) \log K$$

with equality holding if and only if $\mu = \tilde{\nu}$.

By the Lemma and the variational principle, we have $h(W) = \log N$ and $\mu = \tilde{\nu}$. 

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mschraudner@dim.uchile.cl
Using the disjointness of zero entropy and Bernoulli we get for $\nu$-almost all points $z \in Z$ that the factor map $\phi|_{\pi^{-1}z}$ maps $\tilde{\nu}_z$ to $m_B$.

Pick such a $z \in Z$.

\[ p \mid N \quad \implies \quad \exists C \subset B \text{ clopen: } m_B(C) = \frac{1}{p} \]
\[ \phi \text{ continuous} \quad \implies \quad D = (\phi|_{\pi^{-1}z})^{-1}(C) \subset \pi^{-1}z \text{ clopen} \]

But then $\tilde{\nu}_z(D) = m_B(C) = \frac{1}{p}$.

As $p \not| K$ there is no clopen set $D \subset \pi^{-1}z$ of $\tilde{\nu}_z$-measure $\frac{1}{p}$.

There is no factor map $\phi$ from $W$ onto $B$.

In general $\mathbb{Z}^d$ SFTs do not always provide equal entropy full shift factors.

**Lemma [Disjointness-Lemma]:** For $i = 1, 2, 3$ let $(X_i, \mathcal{B}_i, \mu_i, \alpha_i)$ denote a $\mu_i$-preserving $\mathbb{Z}^d$ action $\alpha_i$ on a Lebesgue space $(X_i, \mathcal{B}_i)$. Suppose $\alpha_1$ is zero entropy, $\alpha_2$ is Bernoulli ($\mu_2$ is the product measure of a measure on $A$) and there are measure preserving factor maps $p_i$ from $(X_3, \mathcal{B}_3, \mu_3, \alpha_3)$ to $(X_i, \mathcal{B}_i, \mu_i, \alpha_i)$, $i = 1, 2$. Then $p_1 \times p_2$ defines a measure preserving factor map from $(X_3, \mathcal{B}_3, \mu_3, \alpha_3)$ to the product system $(X_1 \times X_2, \mathcal{B}_1 \times \mathcal{B}_2, \mu_1 \times \mu_2, \alpha_1 \times \alpha_2)$. 
Results: The lower entropy case

**Definition:** A \( \mathbb{Z}^d \) subshift \( X \) is **block gluing at a distance** \( D \) if for every two (disjoint) solid blocks \( B_1 = [\vec{v}^{(1)}, \vec{w}^{(1)}], \ B_2 = [\vec{v}^{(2)}, \vec{w}^{(2)}] \subset \mathbb{Z}^d \) with distance

\[
d(B_1, B_2) := \min_{\vec{b}^{(1)} \in B_1, \vec{b}^{(2)} \in B_2} \| \vec{b}^{(1)} - \vec{b}^{(2)} \|_\infty > D
\]

any pair of patterns on \( B_1 \) and \( B_2 \) which occur in \( X \) can be put together to form a valid point of \( X \), i.e.

\[
\forall x, y \in X \ \exists z \in X : \ z|_{B_1} = x|_{B_1} \wedge z|_{B_2} = y|_{B_2}.
\]

A \( \mathbb{Z}^d \) subshift \( X \) is called **block gluing** if it is block gluing at distance \( D \) for some \( D \in \mathbb{N} \).

**Definition:** A general \( \mathbb{Z}^d \) shift \( X \) has a **safe symbol** if its alphabet contains an element that can be placed at any set of coordinates in any point of \( X \) such that the modified point is still valid in \( X \).

**Theorem** [BPS]: Suppose \( d \geq 1 \) and let \( X \) be a block gluing \( \mathbb{Z}^d \) shift. Then the following hold.

1. If \( N \in \mathbb{N} \) and \( h(X) > \log N \), then \( X \) factors topologically onto the \( \mathbb{Z}^d \) full shift on \( N \) symbols.
2. \( X \) factors topologically onto a family of strongly irreducible \( \mathbb{Z}^d \) SFTs with entropies dense in \( [0, h(X)] \).
3. \( X \) factors topologically onto any lower entropy \( \mathbb{Z}^d \) SFT having a safe symbol.
**Theorem [BPS]:** Given $M > 0$ and $d \geq 2$, there exists a $\mathbb{Z}^d$ sofic shift $S$ with the following properties.

1. $h(S) > M$.

2. $S$ (and thus every topological factor of $S$) contains a fixed point which is its unique minimal subsystem.

3. Every topological factor of $S$ (including $S$) contains a unique $\mathbb{Z}^d$ SFT, which is a fixed point. In particular, $S$ has no non-trivial SFT factor.

4. If $Y$ is any non-trivial subshift factor of $S$, then $Y$ cannot be block gluing. Hence $Y$ also cannot be strongly irreducible, cannot have the uniform filling property (UFP), and cannot be corner gluing. In particular, $Y$ cannot be a full shift.

5. In the case $d \geq 3$, $S$ can be chosen such that for every non-trivial topological factor $Y$ of $S$, there is no invariant Borel probability $\mu \in \mathcal{M}(Y)$ on $Y$ such that $(Y, \mu)$ as a measurable system has completely positive entropy.

6. $S$ has an equal entropy (sofic) subshift factor of topologically completely positive entropy.

7. There is a $\mathbb{Z}^d$ SFT $X$ and a factor map $\pi : X \to S$ such that $h(X) = h(S)$ and the preimage in $X$ of the unique fixed point in $S$ is a $\mathbb{Z}^d$ SFT $K \subsetneq X$ such that $h(K) = 0$.

8. In the case $d = 2$, $S$ can in addition be chosen to be mixing and of topologically completely positive entropy.

9. For $d \geq 3$, $S$ can be chosen to be mixing and satisfying all the properties (1)-(7) except (6).
Theorem [BPS]: Given $M > 0$ and $d \geq 2$, there exists a $\mathbb{Z}^d$ SFT $X$ with the following properties.

1. $h(X) > M$.

2. $X$ contains a zero entropy SFT $K$ which contains every minimal subsystem of $X$. In particular, every non-empty subsystem of $X$ has to intersect $K$.

3. If $Y$ is any block gluing subshift factor of $X$, then $Y$ is trivial. In particular, the only full shift factor of $X$ is the trivial shift.

4. If $Y$ is a subshift factor of $X$, and $Y_{\text{MIN}}$ is the orbit closure of its minimal subsystems, then $h(Y_{\text{MIN}}) = 0$.

5. In the case $d \geq 3$, $X$ can be chosen such that for every non-trivial topological factor $Y$ of $X$, there is no invariant Borel probability $\mu \in \mathcal{M}(Y)$ on $Y$ such that $(Y, \mu)$ as a measurable system has completely positive entropy.

6. $X$ has an equal entropy sofic factor with topologically completely positive entropy. In the case $d = 2$, $X$ itself can be chosen mixing with topologically completely positive entropy.
Implications of block gluing

**Proposition [BPS]:** If $X$ is a non-trivial block gluing $\mathbb{Z}^d$ subshift ($d > 1$) then

1. $h(X) > 0$.
2. Every topological factor of $X$ is block gluing.
3. $X$ has topologically completely positive entropy, i.e. every non-trivial topological factor has strictly positive entropy.
4. $X$ is topologically mixing.

Block gluing is **strictly stronger** than topological mixing ($\exists$ non-trivial top. mixing SFTs with entropy zero), but **strictly weaker** than corner gluing ($X \subset \{0, 1\}^{Z^2}$ with forbidden pattern $P = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$).