Fiber-mixing codes between $\mathbb{Z}$-subshifts and relations of codes between $\mathbb{Z}^d$-subshifts

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This presentation is based on the following works.

- **U. Jung**
  Fiber-mixing codes between subshifts and their existence.

- **U. Jung**
  On the existence of open and bi-continuing codes.

- **U. Jung**
  Open maps between shift spaces.
Fiber-mixing codes

Relations between properties of codes
Let $\mathcal{A}$ be a finite set called an *alphabet*.

- The **full $\mathcal{A}$-shift** $\mathcal{A}^\mathbb{Z}$ is the set of all bi-infinite sequences over $\mathcal{A}$.

- The **shift map** $\sigma$ on $\mathcal{A}^\mathbb{Z}$ is defined by $\sigma(x)_i = x_{i+1}$. A **shift space**, or a **subshift** is a $\sigma$-invariant closed subset of a full shift.

- A **sliding block code** (simply, a **code**) is a $\sigma$-commuting continuous map between shift spaces.
Shift spaces and codes

Let $\mathcal{A}$ be a finite set called an alphabet.

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- A **sliding block code** (simply, a **code**) is a $\sigma$-commuting continuous map between shift spaces.

- A **topological Markov chain** determined by an $r \times r$, 0-1 matrix $A$ is the set of all $x = (x_i) \in \{1, \ldots, r\}^\mathbb{Z}$ with $A_{x_i x_{i+1}} = 1$ for $i \in \mathbb{Z}$.

- A subshift is an **SFT** if it is conjugate to a topological Markov chain.

- A **sofic shift** is a factor of an SFT.
Continuing codes

Let $\phi : X \to Y$ be a code.

- $\phi$ is **right continuing** if whenever $x \in X$, $y \in Y$ and $\phi(x)$ is left asymptotic to $y$, there exists $\bar{x} \in X$ such that $\bar{x}$ is left asymptotic to $x$ and $\phi(\bar{x}) = y$;

  $\xrightarrow{\phi} \quad x \quad \rightarrow \quad y$

- $\phi$ is **bi-continuing** if it is both left and right continuing.
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Let \( \phi : X \rightarrow Y \) be a code.

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A continuing code between irreducible SFTs lifts every Markov measure on $Y$ to a Markov measure on $X$. *(Boyle and Tuncel '84)*

On relations among of properties of codes (U.Jung) 

Fiber-mixing codes
Let $\phi : X \to Y$ be a code.

- $\phi$ is **strong fiber-mixing** if, for every $x, \bar{x} \in X$ and $y \in Y$ with $\phi(x)$ left asymptotic to $y$ and $\phi(\bar{x})$ right asymptotic to $y$, there is $z \in X$ such that $z$ is left asymptotic to $x$, right asymptotic to $\bar{x}$ and $\phi(z) = y$. 
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![Diagram]

- $x$ is left asymptotic to $x$. 

- $\bar{x}$ is right asymptotic to $\bar{x}$. 

- $y$ is an output of $\phi$. 

On relations among of properties of codes (U.Jung) Fiber-mixing codes 6
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- $\phi$ is **fiber-mixing** if, for every $x, \bar{x} \in X$ and $\phi(x) = \phi(\bar{x})$, there is $z \in X$ with $z$ left asymptotic to $x$, right asymptotic to $\bar{x}$ and $\phi(z) = \phi(x)$.

\[
\begin{align*}
\phi(x) &\to y = \phi(z) \\
\phi(\bar{x}) &\to y = \phi(z)
\end{align*}
\]

\[
\begin{align*}
x &\to z \\
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On relations among of properties of codes (U.Jung) Fiber-mixing codes
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Observations

- A strong fiber-mixing factor code is bi-continuing.
- Any injective code is fiber-mixing.

Proposition 1

1. A fiber-mixing bi-continuing code is strong fiber-mixing.
2. A factor code from an SFT is fiber-mixing if and only if strong fiber-mixing.
Fiber-mixing codes

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A continuing code between irreducible SFTs lifts every Markov measure on $Y$ to a Markov measure on $X$. *(Boyle and Tuncel ’84)*

A fiber-mixing factor code between mixing SFTs pulls down every Markov measure on $X$ to a Gibbs measure on $Y$. *(Shin and Yoo ’09, following Chazottes and Ugalde ’03)*
Let $\phi : X \rightarrow Y$ be a code.
Let $\phi : X \to Y$ be a \textit{factor} code.
Let $\phi : X \to Y$ be a factor code and $X$ be an SFT.
Let $\phi : X \to Y$ be a \textit{finite-to-one factor} code and $X$ be an \textit{irreducible SFT}.
The factor problem

Definitions

Let $X$ and $Y$ be shift spaces.

- The *period* of a periodic point $x \in X$ is denoted by $\text{per}(x)$.
- Define $\text{per}(X) = \gcd\{\text{per}(x) : x \in X \text{ is periodic}\}$.
- We say $P(X) \lesssim P(Y)$ if for any periodic point $x \in X$, there is a periodic point $y \in Y$ whose period divides the period of $x$.
- The *topological entropy* of $(X, \sigma)$ is denoted by $h(X)$.

On relations among properties of codes (U.Jung) Fiber-mixing codes
The factor problem

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If $\phi : X \to Y$ is a factor code, then $h(X) \geq h(Y)$ and $P(X) \leq P(Y)$.

Problem
Find a necessary and sufficient condition for $X$ to factor onto $Y$, through a code with certain properties.
Lower Entropy Factor Theorem (Boyle)

Let $X$, $Y$ be irreducible SFTs with $h(X) > h(Y)$ and $P(X) \downarrow P(Y)$. Then there is a factor code $\phi : X \rightarrow Y$. 

Lower entropy factor theorem (SFT case)
Lower Entropy Factor Theorem (Boyle)

Let $X, Y$ be irreducible SFTs with $h(X) > h(Y)$ and $P(X) \searrow P(Y)$. Then there is a factor code $\phi : X \rightarrow Y$.

Direction 1: Weaken the finite-type constraints.
Lower entropy factor theorem (SFT case)

Lower Entropy Factor Theorem (Boyle)

Let $X$, $Y$ be irreducible SFTs with $h(X) > h(Y)$ and $P(X) \downarrow P(Y)$. Then there is a factor code $\phi : X \to Y$.

Direction II: Impose additional properties to the factor code.

Theorem (Boyle and Tuncel)

Let $X$, $Y$ be irreducible SFTs with $h(X) > h(Y)$ and $P(X) \downarrow P(Y)$. Then there is a right continuing factor code $\phi : X \to Y$. 

On relations among properties of codes (U.Jung) Fiber-mixing codes
Lower entropy factor theorem (SFT case)

Lower Entropy Factor Theorem (Boyle)

Let $X, Y$ be irreducible SFTs with $h(X) > h(Y)$ and $P(X) \searrow P(Y)$. Then there is a **factor code** $\phi : X \to Y$.

**Direction II**: Impose additional properties to the factor code.

Theorem (J.)

Let $X, Y$ be irreducible SFTs with $h(X) > h(Y)$ and $P(X) \searrow P(Y)$. Then there is a **bi-continuing** factor code $\phi : X \to Y$. 
Lower Entropy Factor Theorem (Boyle)

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Let $X, Y$ be irreducible SFTs with $h(X) > h(Y)$ and $P(X) \downarrow P(Y)$. Then there is a bi-continuing factor code $\phi : X \rightarrow Y$.

**Theorem 2**

Let $X, Y$ be irreducible SFTs with $h(X) > h(Y)$ and $P(X) \downarrow P(Y)$ and $\text{per}(X) = \text{per}(Y)$. Then there is a strong fiber-mixing code $\phi : X \rightarrow Y$. 
Existence of fiber-mixing codes

**Proposition 3**

Let $\phi : X \rightarrow Y$ be a fiber-mixing factor code between irreducible SFTs.

1. $\text{per}(X) = \text{per}(Y)$.
2. If $\phi$ is finite-to-one, then $\phi$ is a conjugacy.

**Theorem 4**

Let $X$ and $Y$ be irreducible SFTs. Then there is a strong fiber-mixing code from $X$ to $Y$ if and only if

(i) $X$ is conjugate to $Y$, or

(ii) $h(X) > h(Y)$, $P(X) \searrow P(Y)$ and $\text{per}(X) = \text{per}(Y)$.
Existence of fiber-mixing codes

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Theorem 4
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(i) $X$ is conjugate to $Y$, or
(ii) $h(X) > h(Y)$, $\mathcal{P}(X) \searrow \mathcal{P}(Y)$ and $\text{per}(X) = \text{per}(Y)$.

The proof involves ideas and results from Denker-Grillenberger-Sigmund construction of mixing SFTs, Blowing-up Lemma, Extension Theorem, Krieger Embedding, and High-Low stretch construction of Boyle and Tuncel.
Almost specified shifts

Definition
A shift space $X$ is **almost specified** if there exists $N \in \mathbb{N}$ such that for all $u, v \in B(X)$, there exists $w \in B(X)$ with $uwv \in B(X)$ and $|w| \leq N$. 
Almost specified shifts

**Definition**
A shift space $X$ is *almost specified* if there exists $N \in \mathbb{N}$ such that for all $u, v \in \mathcal{B}(X)$, there exists $w \in \mathcal{B}(X)$ with $uwv \in \mathcal{B}(X)$ and $|w| \leq N$. 
Almost specified shifts

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A shift space $X$ is **almost specified** if there exists $N \in \mathbb{N}$ such that for all $u, v \in \mathcal{B}(X)$, there exists $w \in \mathcal{B}(X)$ with $uwv \in \mathcal{B}(X)$ and $|w| \leq N$.

Proposition (Thomsen)

Let $X$ be almost specified. Then there exist a unique $p \in \mathbb{N}$ and closed sets $D_i \subset X$, $i = 0, 1, \ldots, p - 1$, such that

i) $X = \bigcup_{i=0}^{p-1} D_i$,

ii) $\sigma(D_i) = D_{i+1}$,

iii) $\sigma^p|_{D_i}$ is mixing for all $i = 0, 1, \ldots, p - 1$, and

vi) $D_i \cap D_j$ has empty interior when $i \neq j$.

The collection of $D_i$’s is called the **cyclic cover** of $X$. $p$ is called the **essential period** of $X$ and denoted by $eper(x)$. 
Lower Entropy Factor Theorem (Boyle)

Let $X$, $Y$ be **irreducible SFTs** with $h(X) > h(Y)$ and $P(X) \searrow P(Y)$. Then there is a factor code $\phi : X \to Y$.

**Direction I**: Weaken the finite-type constraints.
Lower entropy factor theorem II

Lower Entropy Factor Theorem (Boyle)

Let $X$, $Y$ be irreducible SFTs with $h(X) > h(Y)$ and $P(X) \narrow P(Y)$. Then there is a factor code $\phi : X \to Y$.

Direction I: Weaken the finite-type constraints.

Theorem (Thomsen)

Let $X$ be almost specified and $Y$ an irreducible SFT with $h(X) > h(Y)$. Then $Y$ is a factor of $X$ if and only if $P(X) \narrow P(Y)$ and

* if $\{D_0, \cdots, D_{p-1}\}$ is the cyclic cover of $X$, then $\text{per}(Y) \mid p$ and

$$\left( \bigcup_{j=0}^{\frac{p-1}{q}} D_{i+jq} \right) \cap \left( \bigcup_{j=0}^{\frac{p-1}{q}} D_{k+jq} \right) = \emptyset \text{ for } i \neq k.$$
Lower Entropy Factor Theorem (Boyle)

Let $X, Y$ be irreducible SFTs with $h(X) > h(Y)$ and $P(X) \searrow P(Y)$. Then there is a factor code $\phi : X \to Y$.

**Direction I**: Weaken the finite-type constraints.

**Theorem (Thomsen)**

Let $X$ be almost specified and $Y$ an irreducible SFT with $h(X) > h(Y)$. Then $Y$ is a factor of $X$ if and only if $P(X) \searrow P(Y)$ and $\text{cover}(X, Y)$. 
Lower Entropy Factor Theorem (Boyle)

Let $X, Y$ be irreducible SFTs with $h(X) > h(Y)$ and $P(X) \downarrow P(Y)$. Then there is a factor code $\phi : X \to Y$.

**Direction I**: Weaken the finite-type constraints.

**Theorem (J.)**

Let $X$ be almost specified and $Y$ an irreducible SFT with $h(X) > h(Y)$. Then $Y$ is a bi-continuing factor of $X$ iff $P(X) \downarrow P(Y)$ and $\text{cover}(X,Y)$. 
Lower Entropy Factor Theorem (Boyle)

Let $X$, $Y$ be irreducible SFTs with $h(X) > h(Y)$ and $P(X) \downarrow P(Y)$. Then there is a factor code $\phi : X \to Y$.

Direction I: Weaken the finite-type constraints.

Theorem (J.)

Let $X$ be almost specified and $Y$ an irreducible SFT with $h(X) > h(Y)$. Then $Y$ is a bi-continuing factor of $X$ iff $P(X) \downarrow P(Y)$ and $\text{cover}(X,Y)$.

Theorem 5

Let $X$ be almost specified and $Y$ an irreducible SFT with $h(X) > h(Y)$. Then $Y$ is a factor of $X$ by a (strong) fiber-mixing code if and only if $P(X) \downarrow P(Y)$, $\text{cover}(X,Y)$ and $\text{eper}(X) = \text{eper}(Y)$. 
Fiber-mixing codes

- Relations between properties of codes
Let $\phi : X \to Y$ be a code between $\mathbb{Z}^d$-shift spaces, $d \in \mathbb{N}$.

**Definitions**

- $\phi$ is **finite-to-one** if $\phi^{-1}(y)$ is a finite set for all $y \in Y$.
- $\phi$ is **constant-to-one** if $|\phi^{-1}(y)| \in \mathbb{N}$ is independent of $y$.
- $\phi$ is **open** if images of open sets are open.

**Definitions**

Let $\phi$ be a code between $\mathbb{Z}$-shift spaces.

- $\phi$ is **right closing** if it does not collapse left asymptotic points.

$$x_{(-\infty,N]} = y_{(-\infty,N]}$$

- $\phi$ is **bi-closing** if it is both left and right closing.
Let $X$ and $Y$ be $\mathbb{Z}$-shift spaces.

**Theorem (Nasu)**

Let $X$ and $Y$ be irreducible SFTs and $\phi : X \to Y$ a finite-to-one factor code. Then the following are equivalent.

i) $\phi$ is open.

ii) $\phi$ is constant-to-one.

iii) $\phi$ is bi-closing.
Let $X$ and $Y$ be $\mathbb{Z}$-shift spaces.

**Theorem (Nasu)**

Let $X$ and $Y$ be irreducible SFTs and $\phi : X \to Y$ a finite-to-one factor code. Then the following are equivalent.

i) $\phi$ is open.

ii) $\phi$ is constant-to-one.

iii) $\phi$ is bi-closing.

**Theorem (J.)**

Let $\phi$ be a code from a shift space $X$ to an irreducible shift space $Y$. Then any two of the following conditions imply the third:

i) $\phi$ is open.

ii) $\phi$ is constant-to-one.

iii) $\phi$ is bi-closing.
Let $X$ and $Y$ be $\mathbb{Z}$-shift spaces.

**Theorem (Nasu)**

Let $X$ and $Y$ be irreducible SFTs and $\phi : X \to Y$ a finite-to-one factor code. Then the following are equivalent.

i) $\phi$ is open.

ii) $\phi$ is constant-to-one.

iii) $\phi$ is bi-closing.

**Theorem (J.)**

Let $\phi$ be a code from a shift space $X$ to an irreducible sofic shift $Y$. Then any two of the following conditions imply the third:

i) $\phi$ is open.

ii) $\phi$ is constant-to-one.

iii) $\phi$ is right closing (or left closing).

If these conditions hold, then $X$ is a nonwandering sofic shift.
Fiber-separating codes

Let $\phi : X \to Y$ be a code between $\mathbb{Z}^d$-shift spaces, $d \in \mathbb{N}$.

**Definition**

$\phi$ is *fiber-separating* if there is an $\epsilon > 0$ such that whenever $y \in Y$ and $x, \bar{x} \in \phi^{-1}(y)$ with $x \neq \bar{x}$, we have $d(x, \bar{x}) \geq \epsilon$.

**Observation**

If $d = 1$, then a code is *fiber-separating* if and only if it is *bi-closing*. 
Fiber-separating codes

Let $\phi : X \to Y$ be a code between $\mathbb{Z}^d$-shift spaces, $d \in \mathbb{N}$.

Definition

$\phi$ is fiber-separating if there is an $\epsilon > 0$ such that whenever $y \in Y$ and $x, \bar{x} \in \phi^{-1}(y)$ with $x \neq \bar{x}$, we have $d(x, \bar{x}) \geq \epsilon$.

Proposition 6

Let $Y$ be a transitive shift space, i.e., $Y$ contains a dense orbit.

1. If $\phi$ is finite-to-one open, then there is $d \in \mathbb{N}$ with $|\phi^{-1}(y)| = d$ for each transitive point $y$ of $Y$, and $|\phi^{-1}(y)| \leq d$ for all $y \in Y$.

2. If $\phi$ is fiber-separating onto, then there is a $d \in \mathbb{N}$ with $|\phi^{-1}(y)| = d$ for each transitive point $y$ of $Y$, and $|\phi^{-1}(y)| \geq d$ for all $y \in Y$. 
Cross sections

Definitions
Let $\phi : X \rightarrow Y$ be a code.

- A continuous map $f : Y \rightarrow X$ is called a cross section of $\phi$ if $\phi(f(y)) = y$ for all $y$ in $Y$.
- $\phi$ has disjoint covering cross sections if there exist finitely many cross sections $f_i : Y \rightarrow X$ such that $f_i(Y) \cap f_j(Y) = \emptyset$ for all $i \neq j$ and $\bigcup_i f_i(Y) = X$.

Theorems
- **(Hedlund)** An open code has a cross section.
- **(Nasu)** Let $X$ and $Y$ be irreducible $\mathbb{Z}$-SFTs and $\phi$ a finite-to-one factor code. Then $\phi$ is open if and only if it has a cross section.
Cross sections

Definitions
Let \( \phi : X \to Y \) be a code.

- A continuous map \( f : Y \to X \) is called a **cross section** of \( \phi \) if \( \phi(f(y)) = y \) for all \( y \) in \( Y \).

- \( \phi \) has **disjoint covering cross sections** if there exist finitely many cross sections \( f_i : Y \to X \) such that \( f_i(Y) \cap f_j(Y) = \emptyset \) for all \( i \neq j \) and \( \bigcup_i f_i(Y) = X \).

Proposition 7

Among the following properties of \( \phi \),

i) \( \phi \) has disjoint covering cross sections;

ii) For any \( x \in X \), there exists a cross section \( f \) with \( x \in f(X) \);

iii) \( \phi \) is open;

iv) \( \phi \) has a cross section,

The implications i) \( \Rightarrow \) ii) \( \Rightarrow \) iii) \( \Rightarrow \) iv) hold.
Let $X$ and $Y$ be $\mathbb{Z}^d$-shift spaces, $d \in \mathbb{N}$.

**Proposition 8**

*For any constant-to-one fiber-separating code between shift spaces, disjoint covering cross sections can be found.*
Let $X$ and $Y$ be $\mathbb{Z}^d$-shift spaces, $d \in \mathbb{N}$.

**Proposition 8**

*For any constant-to-one fiber-separating code between shift spaces, disjoint covering cross sections can be found.*

**Theorem 9**

*Let $\phi$ be a code from a shift space $X$ to a transitive shift space $Y$. Then any two of the following conditions imply the third:*

1. \( \phi \) is open.
2. \( \phi \) is constant-to-one.
3. \( \phi \) is fiber-separating.*
Thank You!