Avoiding Abelian Powers in Binary Words with Bounded Abelian Complexity

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Outline

1. Abelian Complexity: Definitions and Examples
2. Bounded Abelian Complexity
3. Avoiding Abelian Powers
4. Avoiding Abelian Powers at Some Position
5. Avoiding Abelian Powers at Infinitely Many Positions
6. Conclusion
Definitions

- If $x \in \{a_1, a_2, \ldots, a_n\}^*$, then its Parikh vector is
  \[ \Psi(x) = (|x|_1, |x|_2, \ldots, |x|_n). \]

- Words $x$ and $y$ are Abelian equivalent if $\Psi(x) = \Psi(y)$.

- We identify Abelian equivalent words; thus 00111 and 01011 are the same.

- Abelian complexity (shortened to ab-complexity) of an infinite word $x$ is the mapping $\rho_x^{ab} : \mathbb{N} \to \mathbb{N}$ such that $\rho_x^{ab}(n)$ gives the number of Abelian inequivalent factors of $x$ of length $n$. 
Examples

Example

If $s$ is a Sturmian word, then

$$\rho_{ab}^s(n) = 2 \quad \text{for all } n \in \mathbb{N}.$$ 

Example

If $TM$ is the Thue–Morse word, then

$$\rho_{ab}^{TM}(n) = \begin{cases} 
2 & \text{if } n \text{ is odd}, \\
3 & \text{if } n \text{ is even}.
\end{cases}$$

In fact, the Thue–Morse subshift is characterized by its factor and Abelian complexity together.
Example

Let $k \geq 1$ and $x_k = 123 \cdots k^\omega$. Then $\rho_{x_k}^{ab}(n) = k$ for all $n \in \mathbb{N}$.

This is why we usually focus on recurrent words.

Theorem (Richomme, S., Zamboni)

There exist uniformly recurrent words $w$ with $\rho_{w}^{ab}(n) = 3$.

Theorem (Currie and Rampersad)

For any $k \geq 4$, there are no recurrent words $w$ with $\rho_{w}^{ab}(n) = k$. 
Bounded Ab-Complexity and Balance

An infinite word \( x \) is \( C \)-balanced, where \( C > 0 \), if for all \( u, v \in \text{Fact}(x) \) and \( b \in \text{Alph}(x) \), we have

\[ |u|_b - |v|_b | \leq C. \]

Lemma

The ab-complexity of \( x \) is bounded iff \( x \) is \( C \)-bounded for some \( C \).

Proof.

- If ab-complexity is bounded by \( M \), then \( x \) is \((M - 1)\)-balanced.
- If \( x \) is \( C \)-balanced, then the ab-complexity is bounded by \((C + 1)\#\text{Alph}(x)\).
Bounded Abelian Complexity

- Let $k \geq 1$, and let $u_1, u_2, \ldots, u_k$ be Abelian equivalent words. The word $u_1 u_2 \cdots u_k$ is called an Abelian $k$-power.
- The upper density of a set $D \subset \mathbb{N}$ is the number

$$\limsup_{n \to \infty} \frac{\#(D \cap \{1, 2, \ldots, n\})}{n}.$$  

**Theorem (RSZ)**

Let $\omega$ be an infinite word with bounded Abelian complexity. Let $I \subset \mathbb{N}$ have positive upper density, and let $k \geq 1$. Then some position of $\omega$ in $I$ has an occurrence of an Abelian $k$-power.
Denote $r := \#\text{Alph}(\omega)$. Because of the bounded ab-complexity, $\omega$ is $M$-balanced for some $M$.

**Lemma**

There exist positive integers $\alpha_1, \alpha_2, \ldots, \alpha_r$ and $N$ such that if

$$\sum_{i=1}^{r} c_i \alpha_i \equiv 0 \pmod{N}$$

for integers $c_i$ with $|c_i| \leq M$ for $1 \leq i \leq r$, then $c_1 = \cdots = c_r = 0$.

We may assume that $\text{Alph}(\omega) = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$. 

Outline of the proof I
Outline of the proof II

- Denote
  \[ \omega[i,j] = \omega_i \ldots \omega_j \quad \text{and} \quad \sum \omega[i,j] = \omega_i + \cdots + \omega_j. \]

- Consider the function: \( \nu : \mathbb{N} \rightarrow \{0, 1, \ldots, N - 1\} \) defined by
  \[ \nu(t) = \sum \omega[1,t] \quad \text{(mod } N). \]

- For some \( i \in \{0, 1, \ldots, N - 1\} \), the set \( D := \{ j - 1 \in I \mid \nu(j) = i \} \) has positive upper density.
Theorem (Szmerédi 1976)

If $D \subset \mathbb{N}$ has positive upper density, then it contains arbitrarily long arithmetic progressions.

Thus for every $k \geq 1$ there exist $t_0 \geq 1$ and $s \geq 1$ such that $t_0, t_0 + s, t_0 + 2s, \ldots, t_0 + ks \in D$ and

$$\nu(t_0) = \nu(t_0 + s) = \nu(t_0 + 2s) = \cdots = \nu(t_0 + ks). \quad (1)$$

For each $1 \leq j \leq k$, set $\omega[j] = \omega[t_0 + (j-1)s + 1, t_0 + js]$. Then

$$\sum \omega[j] \equiv 0 \pmod{N}. \quad (2)$$
Set $\Psi(\omega[j]) = (a_1^{[j]}, a_2^{[j]}, \ldots, a_r^{[j]})$. Then since

$$\sum_{i=1}^{r} a_i^{[j]} \alpha_i = \sum \omega^{[j]} \equiv 0 \pmod{N},$$

we have

$$\sum_{i=1}^{r} (a_i^{[j]} - a_i^{[1]}) \alpha_i \equiv 0 \pmod{N}.$$

As $|\omega^{[j]}| = |\omega^{[1]}|$ for each $1 \leq j \leq k$, we have $|a_i^{[j]} - a_i^{[1]}| \leq M$.

Thus $a_i^{[j]} - a_i^{[1]} = 0$ and hence $\Psi(\omega^{[j]}) = \Psi(\omega^{[1]})$ for every $1 \leq j \leq k$.

Thus the factor $\omega^{[1]}\omega^{[2]} \ldots \omega^{[k]}$ is an Abelian $k$-power of $\omega$ starting at position $t_0 + 1 \in I$. 
Examples

Theorem (RSZ)

Every position of a Sturmian word has an occurrence of an Abelian $k$-power for all $k$.

Theorem (RSZ)

Every position of the Thue–Morse word has an occurrence of an Abelian $k$-power for all $k$. 
Question

- What about nonempty sets of positions with zero density?
- This is trivial: take 0111....
- But a good example must be at least recurrent.
- The proper question is therefore:
- Does there exist a minimal binary word with bounded ab-complexity with
  1. a position where Abelian squares do not occur, or
  2. infinitely many positions where Abelian squares do not occur.
     (Any such set of positions must have upper density 0.)
**Theorem (RSZ)**

The subshift generated by any infinite binary overlap-free word contains a word that does not have an Abelian cube as a prefix.

**Outline of the proof.**

- If an Abelian cube occurs in $TM$ at position $2^{n+2} - 1$, then its period is at least $2^{n+1}$.

- Thus the subshift of $TM$ contains a word with no Abelian cubes as a prefix.

- The general case follows because any infinite binary overlap-free word contains all factors of $TM$. (Due to Allouche, Currie, and Shallit 1998.)
Avoiding Ab-squares with a binary fixed point

Theorem (RSZ)

*The fixed point $w$ of the morphism $\tau: \{0, 1\}^* \rightarrow \{0, 1\}^*$ defined by $0 \mapsto 011110$ and $1 \mapsto 01110$ is minimal, has bounded ab-complexity, and avoids Abelian squares in the prefix.*

Proof.

- $w$ is minimal because the morphism is primitive.
- The morphism is of Pisot type. Thus $w$ is $C$-balanced for some $C$ (by Adamczewski 2003), and thus has bounded ab-complexity.
- It is easy to check that $w$ has no Abelian squares as a prefix.
Theorem (RSZ)

There exists a uniformly recurrent infinite word with bounded ab-complexity and infinitely many positions in which Abelian squares do not occur.

- Let $h$ be the uniform morphism defined by
  \[ h(0) = 01011111 \quad \text{and} \quad h(1) = 11101111 \]

- Let $f : \{0,1\}^* \rightarrow \{0,1\}^*$ be the morphism defined by
  \[ f(0) = 00011 \quad \text{and} \quad f(1) = 01100. \]

- We prove the claim for the word $f(h^\omega(0))$. 
The morphism $h: 0 \mapsto 01011111, 1 \mapsto 11101111$ is primitive.
Therefore $h^\omega(0)$ is uniformly recurrent.
Consequently, so is $f(h^\omega(0))$. 

$f(h^\omega(0))$ is uniformly recurrent
$f(h^\omega(0))$ has bounded ab-complexity

**Theorem (RSZ)**

A morphism $f : A \to B$ maps all words to words with bounded ab-complexity if and only if there exists $\vec{v} \in \mathbb{N}^B$ such that, for all letters $a \in A$, there exists an integer $K_a$ such that $\Psi(f(a)) = K_a \vec{v}$.

- The morphism $f : 0 \mapsto 00011, 1 \mapsto 01100$ satisfies

  $$\Psi(f(0)) = \Psi(f(1)) = (3, 2).$$

- Therefore $f(h^\omega(0))$ has bounded ab-complexity.
Two lemmas on $h$ and $f$

**Lemma**

The word $h^\omega(0)$ does not have a prefix of the form $h^n(01)0xy0$ with $|x| = |y|$.

**Lemma**

Let $w$ be an infinite binary word. Suppose that $f(w)$ begins in a word of the form $0001uv$, where $u$ and $v$ are nonempty Abelian equivalent words. Then $w$ has a prefix of the form $0xy0$ for some words $x$ and $y$ with $|x| = |y|$.
Let $w_n$ be given by $h^\omega(0) = h^n(01)w_n$

Then $w_n$ does not begin with $0x0y0$ with $|x| = |y|$.

Thus $f(w_n)$ does not begin with $0001uv$ with $u$ and $v$ Abelian equivalent.

Since $f(h^\omega(0)) = f(h^n(01))f(w_n)$ for all $n \geq 1$, the word has infinitely many positions without Abelian squares.
Conclusion

- The set of positions in $f(h^\omega(0))$ in which no Abelian squares occurs have upper density 0.
- Such set cannot have positive upper density.
- Hence our result is optimal.

Problem

Does there exist any infinite binary words avoiding Abelian squares at a set of positions with positive upper density.