Computational Topology and Dynamics

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Question: How can we quantify the difference between geometric objects/topological spaces?
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**Answer:** It depends on what differences are important!
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Which is the most different?
Question: How can we quantify the difference between geometric objects/topological spaces?

Answer: It depends on what differences are important!

Which is the most different?

Question: Are there computational cheap & robust methods for answering this?
We want to distinguish differences up to deformations

Homotopy
We want to distinguish differences up to deformations

**Homotopy**

\[ f, g : X \to Y \text{ continuous maps} \]

\( f \) is **homotopic** to \( g \) if there exists a continuous map \( H : [0, 1] \times X \to Y \) such that

\[
H(0, \cdot) = f(\cdot) \quad \text{and} \quad H(1, \cdot) = g(\cdot)
\]

\[ f \sim g \]
$X$ is **homotopic** to $Y$ if there exist continuous maps

$$f: X \to Y$$ and $$g: Y \to X$$

such that

$$g \circ f \sim \text{id}_X$$ and $$f \circ g \sim \text{id}_Y$$
$X$ is **homotopic** to $Y$ if there exist continuous maps

\[ f : X \to Y \quad \text{and} \quad g : Y \to X \]

such that

\[ g \circ f \sim \text{id}_X \quad \text{and} \quad f \circ g \sim \text{id}_Y \]

\[ \mathbb{R}^2 \setminus \{0\} \sim S^1 \]
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\[ \mathbb{R}^2 \setminus \{0\} \sim S^1 \]

\[ f: \mathbb{R}^2 \setminus \{0\} \to S^1 \quad \begin{array}{ccc} x & \mapsto & \frac{x}{\|x\|} \end{array} \]

\[ g: S^1 \to \mathbb{R}^2 \setminus \{0\} \quad \begin{array}{ccc} x & \mapsto & x \end{array} \]

**Computing homotopy type is too hard!**
$X$

topological space

\[ H_*(X) = \{ H_k(X) \mid k = 0, 1, 2, \ldots \} \]
abelian groups
A topological space $X$ has homology groups defined as:

$$H_*(X) = \{H_k(X) \mid k = 0, 1, 2, \ldots\}$$

These groups form an abelian group structure.
Homology groups

$H_*(X) = \{ H_k(X) \mid k = 0, 1, 2, \ldots \}$

abelian groups

$H_k(S^2) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0, 2 \\ 0 & \text{otherwise} \end{cases}$
$X$
topological
space

Homology groups

$H_\ast(X) = \{H_k(X) \mid k = 0, 1, 2, \ldots\}$
abelian groups

$H_k(S^2) \cong \begin{cases} 
\mathbb{Z} & \text{if } k = 0, 2 \\
0 & \text{otherwise}
\end{cases}$

$H_k(T^2) \cong \begin{cases} 
\mathbb{Z} & \text{if } k = 0, 2 \\
\mathbb{Z}^2 & \text{if } k = 1 \\
0 & \text{otherwise}
\end{cases}$
Homology groups

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If $X \sim Y$ then $H_*(X) \cong H_*(Y)$
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Converse is not true!
If $X \sim Y$ then $H_*(X) \cong H_*(Y)$

Converse is not true!

Homology is computable
Computing Homology

The Classical Approach
Geometry

Algebra

\[ C_0 := \{ x = \sum \alpha_i v_i \mid \alpha_i \in \mathbb{Z}, \, v_i \text{ vertex} \} \]
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\[ C_1 := \left\{ x = \sum \alpha_i e_i \mid \alpha_i \in \mathbb{Z}, \; e_i \text{ edge} \right\} \]
Geometry

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\[ \partial_k : C_k \rightarrow C_{k-1} \]

Linear map
\[ C_0 := \left\{ x = \sum \alpha_i v_i \mid \alpha_i \in \mathbb{Z}, \ v_i \text{ vertex} \right\} \]

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\[ \partial_k : C_k \rightarrow C_{k-1} \quad \text{Linear map} \]

\[ \partial_0 (v_i) := 0 \]
**Geometry**

**Algebra**

\[ C_0 := \{ x = \sum \alpha_i v_i \mid \alpha_i \in \mathbb{Z}, \ v_i \ \text{vertex} \} \]

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\[ \partial_k : C_k \rightarrow C_{k-1} \]

\[ \partial_0 (v_i) := 0 \]

\[ \partial_1 (\text{edge}) := \bullet - \bullet \]
Geometry

Algebra

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Linear map

\[ \partial_0 (v_i) := 0 \]

\[ \partial_1 (\ ) := - \]

\[ \partial_2 (\ ) := - + \]
Geometry

Algebra

\[ C_0 := \left\{ x = \sum \alpha_i v_i \mid \alpha_i \in \mathbb{Z}, \ v_i \ \text{vertex} \right\} \]
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\[ \partial_k : C_k \to C_{k-1} \] \text{Linear map}

\[ \partial_0 (v_i) := 0 \]
\[ \partial_1 (v_i \rightarrow v_j) := \bullet - \bullet \]
\[ \partial_2 (v_i \rightarrow v_j \rightarrow v_k) := \bullet - \bullet + \bullet \]

Integer Matrices
Formal Algebra

$C_k$, finitely generated free abelian group ($k$-chains)
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$\partial_k : C_k \rightarrow C_{k-1}$ group homomorphism

\[ \partial_k \circ \partial_{k+1} = 0 \]
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\[ Z_k := \ker \partial_k \] ($k$-cycles)

\{ boundary operator \}
Formal Algebra

$C_k$, finitely generated free abelian group ($k$-chains)

$\partial_k : C_k \rightarrow C_{k-1}$ group homomorphism

$\partial_k \circ \partial_{k+1} = 0$

$Z_k := \text{kernel } \partial_k$ ($k$-cycles)

$B_k := \text{image } \partial_{k+1}$ ($k$-boundaries)
Formal Algebra

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$k$-th Homology group:

$H_k := \frac{Z_k}{B_k}$
Intuition
\[ C_k = \begin{cases} \mathbb{Z} & k = 0 \\ 0 & \text{otherwise} \end{cases} \]
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\[ C_k = \begin{cases} \mathbb{Z}^3 & k = 0, 1 \\ 0 & \text{otherwise} \end{cases} \]

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\[ \partial_1( ) = 0 \]
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$C_k = \begin{cases} \mathbb{Z} & k = 0 \\ 0 & \text{otherwise} \end{cases}$

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\[ \partial_1(\mathcal{C}) = 0 \]
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\[ \partial_1 ( ) = 0 \]

\[ \partial_2 ( ) = \]

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\[
B_k = 0
\]

\[
\partial_1(\ ) = 0
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H_k = \begin{cases} \mathbb{Z} & k = 0 \\ 0 & \text{otherwise} \end{cases}
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\]

\[
\partial_2(\ ) = \]

\[
H_k = \begin{cases} \mathbb{Z} & k = 0 \\ 0 & \text{otherwise} \end{cases}
\]

\[
C_k = \begin{cases} \mathbb{Z}^3 & k = 0, 1 \\ \mathbb{Z} & k = 2 \\ 0 & \text{otherwise} \end{cases}
\]

\[
Z_k = \begin{cases} \mathbb{Z}^3 & k = 0 \\ \mathbb{Z} & k = 1 \\ 0 & \text{otherwise} \end{cases}
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B_k = \begin{cases} \mathbb{Z}^2 & k = 0 \\ \mathbb{Z} & k = 1 \\ 0 & \text{otherwise} \end{cases}
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k-th Homology group:

\[ H_k := \frac{\text{kernel } \partial_k}{\text{image } \partial_{k+1}} \]
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Gaussian Elimination
k-th Homology group:

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Gaussian Elimination \quad \rightarrow \quad Smith Normal Form

Integers not Real numbers!
k-th Homology group:

\[ H_k := \frac{\text{kernel } \partial_k}{\text{image } \partial_{k+1}} \]

Gaussian Elimination

Integers not Real numbers!

Smith Normal Form

Complexity: \( O(n^{3.376\ldots}) \)
k-th Homology group: $H_k := \frac{\text{kernel } \partial_k}{\text{image } \partial_{k+1}}$

Complexity: $O(n^{3.376...})$

Delfinado & Edelsbrunner 95
(linear, dim < 4)
Cahn-Hilliard Equation

\[ \frac{\partial u}{\partial t} = -\Delta \left( \epsilon^2 \Delta u + u - u^3 \right) \]

+ no flux boundary conditions
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\frac{\partial u}{\partial t} = -\Delta \left( \epsilon^2 \Delta u + u - u^3 \right) \\
+ \text{no flux boundary conditions}
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Cahn-Hilliard Equation

\[ \frac{\partial u}{\partial t} = -\Delta \left( \epsilon^2 \Delta u + u - u^3 \right) + \text{no flux boundary conditions} \]
Cahn-Hilliard Equation

\[ \frac{\partial u}{\partial t} = -\Delta \left( \varepsilon^2 \Delta u + u - u^3 \right) \]
\[ + \text{ no flux boundary conditions} \]

\[ H_k \approx \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}^{1701} & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases} \]

Fe-Cr alloy
Miller et. al. (1995)
Applications

Fluid Dynamics
Rayleigh-Benard Convection
Rayleigh-Benard Convection
Rayleigh-Benard Convection

Raw intensity image captured using 12-bit digital camera and filtering
Rayleigh-Benard Convection

Raw intensity image captured using 12-bit digital camera and filtering

Binary image obtained by thresholding at the median value of intensity
Count connected components
Count connected components  34
Count connected components: 34

Count holes:
Count connected components 34

Count holes 14
Count connected components 34

\((\beta_0^{\text{cold}}, \beta_1^{\text{cold}}) = (34, 14)\)

Count holes 14
These statistics distinguish experimental states.
These statistics distinguish experimental states.
These statistics distinguish experimental states.
Statistics can be compared against numerical simulations
Statistics can be compared against numerical simulations

Non-Boussinesq terms off
Statistics can be compared against numerical simulations.

Non-Boussinesq terms off

Non-Boussinesq terms on
Statistics can be compared against numerical simulations.

Non-Boussinesq terms off

Non-Boussinesq terms on

\[
\begin{align*}
\beta^\text{hot}_1 & \\ \beta^\text{cold}_1 \\
\beta^\text{hot}_0 & \\
\beta^\text{cold}_0 \\
\end{align*}
\]
Applications

Cahn-Hilliard Model Identification

M. Gameiro, K. M., and T. Wanner 04
Cahn-Hilliard Equation

\[
\frac{\partial u}{\partial t} = -\Delta \left( \varepsilon^2 \Delta u + u - u^3 \right)
\]

No-flux Boundary Conditions
Cahn-Hilliard Equation

\[ \frac{\partial u}{\partial t} = -\Delta \left( \epsilon^2 \Delta u + u - u^3 \right) \]

No-flux Boundary Conditions

On a one dimensional domain we see the following dynamics
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\[
\frac{\partial u}{\partial t} = -\Delta \left( \epsilon^2 \Delta u + u - u^3 \right)
\]

No-flux Boundary Conditions

Spinodal decomposition followed by

Coarsening

\[
\frac{\partial u}{\partial t} = - (\epsilon^2 u_{xx} + u - u^3)_{xx} \quad x \in (0, 1)
\]
Now consider the domain \( \Omega = (0, 1) \times (0, 1) \)
Now consider the domain $\Omega = (0, 1) \times (0, 1)$.
Finally, let's compare the Cahn-Hilliard equation

$$\frac{\partial u}{\partial t} = -\Delta (\epsilon^2 \Delta u + u - u^3)$$

with the Cahn-Hilliard-Cook equation

$$\frac{\partial u}{\partial t} = -\Delta (\epsilon^2 \Delta u + u - u^3) + \sigma \cdot \xi$$

where

$$\langle \xi(t, x) \rangle = 0, \quad \langle \xi(t_1, x_1) \xi(t_2, x_2) \rangle = \delta(t_1 - t_2) \delta(x_1 - x_2)$$

(The noise conserves the alloy composition and is uncorrelated in time.)
Evolution of patterns in 2-D space

\[ \sigma = 0 \]

\[ \sigma = 0.01 \]

\( t = 0.0004 \)

\( t = 0.0012 \)

\( t = 0.0036 \)

\( u \geq 0 \quad u \leq 0 \)
Betti numbers as a function of time 2d Cahn–Hilliard
$H_0$, positive, 100 samples

$\varepsilon = 0.005$, $\sigma = 0.0$
$\varepsilon = 0.005$, $\sigma = 0.01$

$H_0$, negative, 100 samples

$\varepsilon = 0.005$, $\sigma = 0.0$
$\varepsilon = 0.005$, $\sigma = 0.01$

$H_1$, positive, 100 samples

$\varepsilon = 0.005$, $\sigma = 0.0$
$\varepsilon = 0.005$, $\sigma = 0.01$

$H_1$, negative, 100 samples

$\varepsilon = 0.005$, $\sigma = 0.0$
$\varepsilon = 0.005$, $\sigma = 0.01$
Is the nonmonotonicity a bulk or boundary effect?
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\# (+) boundary components + \# (+) interior components = \# (+) components
Is the nonmonotonicity a bulk or boundary effect?

\[ \#(+) \text{ boundary components} + \#(+) \text{ interior components} = \#(+) \text{ components} \]

\[ \beta_0^+_{\text{int}} = \beta_1^- \]
Is the nonmonotonicity a bulk or boundary effect?

\[
\# (+) \text{ boundary components} + \# (+) \text{ interior components} = \# (+) \text{ components}
\]

\[
\beta^+_{\text{int}} = \beta^-_1
\]

\[
\# (+) \text{ boundary components} = \# (+) \text{ components} - \# (-) \text{ loops}
\]
Is the nonmonotonicity a bulk or boundary effect?

# (+) boundary components + # (+) interior components = # (+) components

\[ \beta_{0 \text{ int}}^{+} = \beta_{1}^{-} \]

# (+) boundary components = # (+) components - # (-) loops

\[ \beta_{0 \text{ bdy}}^{+} = \beta_{0}^{+} - \beta_{1}^{-} \]
Morse Theory

The classical theory
INGREDIENTS:

Finite dimensional compact Riemannian manifold
Smooth Morse function $V : T^2 \to \mathbb{R}$
Gradient flow \[ \dot{x} = f(x) := -\nabla V(x) \]
INGREDIENTS: Finite dimensional compact Riemannian manifold
Smooth Morse function $V : T^2 \to \mathbb{R}$

Gradient flow $\dot{x} = f(x) := -\nabla V(x)$

Critical Points $\{x \in T^2 \mid f(x) = 0\}$
INGREDIENTS:

- Finite dimensional compact Riemannian manifold
- Smooth Morse function $V : T^2 \rightarrow \mathbb{R}$
- Gradient flow $\dot{x} = f(x) := -\nabla V(x)$

Critical Points $\{x \in T^2 \mid f(x) = 0\}$

Morse Index $\mu(x) = \# \text{ positive eigenvalues of } Df(x)$
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\[
d_k(x) := \sum \# \{x \rightarrow y\} \ y \quad \text{with orientation}
\]
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$d_k \circ d_{k+1} = 0$

$d_0 = 0$
$d_1 = 0$
$d_2 = 0$

$H_k(T^2) \cong \begin{cases} 
\mathbb{Z} & \text{if } k = 0, 2 \\
\mathbb{Z}^2 & \text{if } k = 1 \\
0 & \text{otherwise}
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\end{cases} \]
Computing Homology

Ace-King-Queen Decomposition

S. Harker, K. M., M. Mrozek, V. Nanda
King

Queen

(pair)
Aces

King

Queen

(pair)

(degree)
<table>
<thead>
<tr>
<th>King</th>
<th>(pair)</th>
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</thead>
<tbody>
<tr>
<td>Queen</td>
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</table>
Aces

(degree)

0

King

Queen

(pair)
Aces (degree)

- 0
- 1
- 2

King
Queen (pair)
Aces

King

Queen

(pair)

\[ d_1 = [ \quad ] \]

(degree)

0

1

2
Aces (degree)

0
1
1
2

\[ d_1 = \begin{bmatrix} 0 \end{bmatrix} \]
\[ d_1 = \begin{bmatrix} 0 & 0 \end{bmatrix} \]
Aces

(degree)

0
1
1
2

\[ d_1 = \begin{bmatrix} 0 & 0 \end{bmatrix} \]

\[ d_2 = \begin{bmatrix} \end{bmatrix} \]
Aces

Queen

King

(degree)

\[
d_1 = \begin{bmatrix} 0 & 0 \end{bmatrix}
\]

\[
d_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]
\[ H_k(T^2) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0, 2 \\ \mathbb{Z}^2 & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases} \]
Does it work?
Does it work? 7-dimensional Torus \((\Box)^7\)
Does it work? 7-dimensional Torus \((□)^7\)

\[ H_k(T^7) \cong \begin{cases} 
\mathbb{Z} & \text{if } k = 0, 7 \\
\mathbb{Z}^7 & \text{if } k = 1, 6 \\
\mathbb{Z}^{21} & \text{if } k = 2, 5 \\
\mathbb{Z}^{35} & \text{if } k = 3, 4 
\end{cases} \]

<table>
<thead>
<tr>
<th># Cells</th>
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<tbody>
<tr>
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<td>7</td>
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<tr>
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<td>21</td>
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</tr>
<tr>
<td>2,097,152</td>
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Thursday, March 4, 2010
Does it work? 7-dimensional Torus \((\square)^7\)

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<tr>
<th>Operation</th>
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<tbody>
<tr>
<td>AKQ decomposition</td>
<td>93.69 sec</td>
</tr>
<tr>
<td>Morse boundary map</td>
<td>0.21 sec</td>
</tr>
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<td>Total Time</td>
<td>93.91 sec</td>
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Thursday, March 4, 2010
Does it work?
Does it work?  Random 5-d cubical complex
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$$H_k(X) \cong \begin{cases} 
\mathbb{Z} & \text{if } k = 0 \\
\mathbb{Z}^{2706} & \text{if } k = 1 \\
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<td>Morse boundary map</td>
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<tr>
<td>Total Time</td>
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</table>
Does it work?  Random Complexes

Morally the gap between blue and red is

\[ \left( \#\text{Aces} - \sum \text{Betti numbers} \right)^{3.376...} \]
R. Forman [98]: Discrete Morse Function for Cell Complexes

Partially ordered Ace-King-Queen decomposition
Corollary: We are constructing discrete Morse functions for cell complexes
Corollary: We are constructing discrete Morse functions for cell complexes

Morse functions $\sim$ Lyapunov functions
Corollary: We are constructing discrete Morse functions for cell complexes

Morse functions \sim Lyapunov functions

Philosophy: minimal Morse functions provide “optimal obtainable” control.
Perfect Morse function:

\[ \frac{\# \text{Aces}}{\sum \text{Betti numbers}} = 1 \]
Perfect Morse function:
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\frac{\#\text{Aces}}{\sum \text{Betti numbers}} = 1
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Q1: For triangulated smooth manifolds will the AKQ algorithm produce perfect Morse functions if they exist?
Perfect Morse function:
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**Q1:** For triangulated smooth manifolds will the AKQ algorithm produce perfect Morse functions if they exist?

**Q2:** Are there obstructions to non-manifold complexes having perfect Morse functions? (Bing’s house)
Perfect Morse function:
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\]

**Q1:** For triangulated smooth manifolds will the AKQ algorithm produce perfect Morse functions if they exist?

**Q2:** Are there obstructions to non-manifold complexes having perfect Morse functions? (Bing’s house)

**Q3:** If so what are the implications for control theory?
Why does Morse Theory compute Homology?
$S^1 \quad H_k(S^1) = \begin{cases} \mathbb{Z} & k = 0, 1 \\ 0 & \text{otherwise} \end{cases}$
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Aces degree

$\bullet$ 0
$S^1 \quad H_k(S^1) = \begin{cases} \mathbb{Z} & k = 0, 1 \\ 0 & \text{otherwise} \end{cases}$

\begin{tikzpicture}
  \draw[thick, red] (0,0) -- (0,1) -- (1,1) -- (1,0) -- cycle;
  \fill[red] (0.5,0.5) circle (0.1); % Aces
  % Degree
  \node at (0.5,0.5) {0};
\end{tikzpicture}
$S^1$ \quad $H_k(S^1) = \begin{cases} \mathbb{Z} & k = 0, 1 \\ 0 & \text{otherwise} \end{cases}$
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Aces \hspace{1cm} \text{degree} \hspace{1cm} \text{Morse Theory:}

\begin{align*}
\bullet & \quad 0 \\
\_ & \quad 1 \\
\partial_1( \_ ) & = 0
\end{align*}
$S^1 \quad H_k(S^1) = \left\{ \begin{array}{ll} \mathbb{Z} & k = 0, 1 \\ 0 & \text{otherwise} \end{array} \right.$

Think of this as two complexes

Morse Theory:

Aces \quad degree \quad Morse Theory:

\[ \partial_1(\ ) = 0 \]

Think of this as two complexes
$S^1$ \hspace{1cm} $H_k(S^1) = \begin{cases} \mathbb{Z} & k = 0, 1 \\ 0 & \text{otherwise} \end{cases}$

Aces \hspace{1cm} degree \hspace{1cm} Morse Theory:

\begin{align*}
\cdot & \quad 0 \\
\rule{3cm}{0.3mm} & \quad 1 \\
\partial_1 (\rule{3cm}{0.3mm}) & = 0
\end{align*}

Think of this as two complexes
\[ S^1 \quad H_k(S^1) = \begin{cases} \mathbb{Z} & k = 0, 1 \\ 0 & \text{otherwise} \end{cases} \]

\[ \text{Aces} \quad \text{degree} \quad \text{Morse Theory:} \]

\[ \bullet \quad 0 \]

\[ \quad \quad 1 \quad \partial_1(\quad \quad ) = 0 \]

Think of this as two complexes

\[ C_k(S^1) = \begin{cases} \mathbb{Z}^{12} & k = 0, 1 \\ 0 & \text{otherwise} \end{cases} \]

\[ C_k(A) = \begin{cases} \mathbb{Z}^{12} & k = 0 \\ \mathbb{Z}^{11} & k = 1 \\ 0 & \text{otherwise} \end{cases} \]
$S^1 \quad H_k(S^1) = \begin{cases} \mathbb{Z} & k = 0, 1 \\ 0 & \text{otherwise} \end{cases}$

Aces degree Morse Theory:

\[
\begin{array}{ccc}
\bullet & 0 & \partial_1 ( \quad ) = 0 \\
\hline \\
\end{array}
\]

Think of this as two complexes

\[
C_k(S^1) = \begin{cases} \mathbb{Z}^{12} & k = 0, 1 \\ 0 & \text{otherwise} \end{cases} \quad C_k(A) = \begin{cases} \mathbb{Z}^{12} & k = 0 \\ \mathbb{Z}^{11} & k = 1 \\ 0 & \text{otherwise} \end{cases}
\]

\[
C_k(S^1, A) := \frac{C_k(S^1)}{C_k(A)} = \begin{cases} \mathbb{Z} & k = 1 \\ 0 & \text{otherwise} \end{cases}
\]
\[ S^1 \quad H_k(S^1) = \begin{cases} \mathbb{Z} & k = 0, 1 \\ 0 & \text{otherwise} \end{cases} \]

Aces degree Morse Theory:

- 0
- 1 \[ \partial_1( ~ ) = 0 \]

Think of this as two complexes

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\[ Z_k(S^1, A) = \begin{cases} \mathbb{Z} & k = 1 \\ 0 & \text{otherwise} \end{cases} \]
$S^1$  \[ H_k(S^1) = \begin{cases} \mathbb{Z} & k = 0, 1 \\ 0 & \text{otherwise} \end{cases} \]

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$Z_k(S^1, A) = \begin{cases} \mathbb{Z} & k = 1 \\ 0 & \text{otherwise} \end{cases}$

$B_k(S^1, A) = 0$

\[ \partial_1( ) = 0 \]
\[ S^1 \quad H_k(S^1) = \begin{cases} \mathbb{Z} & k = 0, 1 \\ 0 & \text{otherwise} \end{cases} \]

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$B_k(S^1, A) = 0$

This is the element generated by

$Z_k(S^1, A) = \begin{cases} \mathbb{Z} & k = 1 \\ 0 & \text{otherwise} \end{cases}$

$H_k(S^1, A) = \begin{cases} \mathbb{Z} & k = 1 \\ 0 & \text{otherwise} \end{cases}$

Aces \hspace{1cm} \text{degree} \hspace{1cm} \text{Morse Theory:}

\[ \bullet \quad 0 \]

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Formal Algebra of Relative Homology
$P_0 \subset P_1$
$P_0 \subset P_1$

**Relative Chains:** $C_k(P_1, P_0)$
Formal Algebra of Relative Homology

\[ P_0 \subset P_1 \]

**Relative Chains:** \( C_k(P_1, P_0) \)

**Relative Cycles:** kernel \( \partial_k : C_k(P_1, P_0) \to C_{k-1}(P_1, P_0) \)
Formal Algebra of Relative Homology

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Relative Chains: \( C_k(P_1, P_0) \)

Relative Cycles: kernel \( \partial_k: C_k(P_1, P_0) \rightarrow C_{k-1}(P_1, P_0) \)

Relative Boundaries: image \( \partial_k: C_{k+1}(P_1, P_0) \rightarrow C_k(P_1, P_0) \)
Formal Algebra of Relative Homology

$P_0 \subset P_1$

**Relative Chains:** $C_k(P_1, P_0)$

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**Relative Homology:** $H_k(P_1, P_0) := \frac{Z_k(P_1, P_0)}{B_k(P_1, P_0)}$
Induced Maps on Homology
Consider a continuous function \( f : X \rightarrow Y \)

We want this to “naturally” induce a group homomorphism

\[
f_* : H_*(X) \rightarrow H_*(Y)
\]
Consider a continuous function \( f : X \to Y \)

We want this to “naturally” induce a group homomorphism

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f_* : H_*(X) \to H_*(Y)
\]

Starting point for the algebra are the chains. Want to define

\[
\varphi_k : C_k(X) \to C_k(Y)
\]

such that

\[
f_* : \frac{Z_k(X)}{B_k(X)} \to \frac{Z_k(Y)}{B_k(Y)}
\]
The collection of maps $\varphi_k : C_k(X) \to C_k(Y)$ is a chain map if

$$\partial^Y_k \circ f_k = f_{k-1} \circ \partial^X_k$$
The collection of maps $\varphi_k : C_k(X) \to C_k(Y)$ is a chain map if

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If $z \in Z_k(X)$ then

$$\partial^Y_k(\varphi_k(z)) = \varphi_{k-1}(\partial^X_k(z)) = \varphi_{k-1}(0) = 0$$
The collection of maps $\varphi_k: C_k(X) \to C_k(Y)$ is a chain map if
\[ \partial^Y_k \circ f_k = f_{k-1} \circ \partial^X_k \]
If $z \in Z_k(X)$ then
\[ \partial^Y_k (\varphi_k(z)) = \varphi_{k-1}(\partial^X_k(z)) = \varphi_{k-1}(0) = 0 \]
If $z \in B_k(X)$ then there exists $c \in C_{k+1}(X)$ such that
\[ \partial^X_{k+1}(c) = z. \] Then
\[ \varphi_k(z) = \varphi_k(\partial^X_{k+1}(c)) = \partial^Y_k(\varphi_{k+1}(c)) \]
The collection of maps \( \varphi_k: C_k(X) \to C_k(Y) \) is a chain map if
\[
\partial_k^Y \circ f_k = f_{k-1} \circ \partial_k^X
\]
If \( z \in Z_k(X) \) then
\[
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\]
If \( z \in B_k(X) \) then there exists \( c \in C_{k+1}(X) \) such that
\[
\partial_{k+1}^X(c) = z. \text{ Then}
\]
\[
\varphi_k(z) = \varphi_k(\partial_{k+1}^X(c)) = \partial_k^Y(\varphi_{k+1}(c))
\]
Let \( [z] \in H_k(X) \). Then \( z \in Z_k(X) \). Define
\[
\varphi_*([z]) := [\varphi_k(z)] \in H_k(Y)
\]
Exercises:
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Consider \( \text{id}_X : C_k(X) \to C_k(X) \). Then

\[
\text{id}_{X^*} : H_*(X) \to H_*(X)
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is the identity map.
Exercises:

Consider $\text{id}_X : C_k(X) \to C_k(X)$. Then

$$\text{id}_X^* : H_*(X) \to H_*(X)$$

is the identity map.

Consider $\varphi_k : C_k(X) \to C_k(Y)$ and $\psi_k : C_k(Y) \to C_k(W)$. Then

$$(\psi \circ \varphi)^*_* = \psi^*_* \circ \varphi^*_*$$
Approximating Continuous Functions
Restrict our attention to $f: X \rightarrow Y$ where $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$
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f(x) = f(x_1, \ldots, x_n) = (f_1(x), \ldots, f_m(x))
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Restrict our attention to $f : X \to Y$ where $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$

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$\epsilon$ and $\delta$ need not be small.
Cubical Homology

A Brief Interlude

T. Kaczynski, K. M., M. Mrozek, 04
An **elementary interval** is a closed interval $I \subset \mathbb{R}$ of the form

$$I = [l, l + 1] \quad \text{or} \quad I = [l, l]$$

for some $l \in \mathbb{Z}$. 
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I = [l, l + 1] \quad \text{or} \quad I = [l, l]
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for some \( l \in \mathbb{Z} \).

An *elementary cube* is a finite product of elementary intervals

\[
Q = I_1 \times I_2 \times \cdots \times I_d \subset \mathbb{R}^d,
\]

where each \( I_i \) is an elementary interval.
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$$\mathcal{K}_k^d := \{k\text{-dimensional cubes in } \mathbb{R}^d\}$$
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$$\mathcal{K}_k^d := \{k\text{-dimensional cubes in } \mathbb{R}^d \}$$

A **cubical set** is a finite union of elementary cubes
The elementary cell associated to an elementary interval is

\[ I := \begin{cases} (l, l + 1) & \text{if } I = [l, l + 1], \\
[l] & \text{if } I = [l, l]. \end{cases} \]
The elementary cell associated to an elementary interval is

\[
\circ \ I := \begin{cases} 
(l, l + 1) & \text{if } I = [l, l + 1], \\
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\end{cases}
\]

The elementary cell associated to an elementary cube

\[
Q = I_1 \times I_2 \times \ldots \times I_d \subset \mathbb{R}^d
\]

is

\[
\circ \ Q := \circ \ I_1 \times \circ \ I_2 \times \ldots \times \circ \ I_d \subset \mathbb{R}^d
\]
Cubical Maps

Intuition and Important Examples
Example:

Let $A \subset X \subset \mathbb{R}^d$ be cubical sets.

Consider the inclusion map $\iota : A \hookrightarrow X$

For each $Q \in \mathcal{K}_k(A)$

$$\iota(Q) = Q \in \mathcal{K}_k(X)$$

This induces the chain map $\iota_k : C_k(A) \to C_k(X)$ given by

$$\iota_k(Q) = Q$$
\[ H_k(S^1) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 1 \\ 0 & \text{otherwise} \end{cases} \]
\[ H_k(S^1) = \begin{cases} \mathbb{Z} & k = 0, 1 \\ 0 & \text{otherwise} \end{cases} \]

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$$H_k(Z) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}^2 & k = 1 \\ 0 & \text{otherwise} \end{cases}$$
\[ H_k(S^1) = \begin{cases} \mathbb{Z} & k = 0, 1 \\ 0 & \text{otherwise} \end{cases} \]

\[ H_k(X) = \begin{cases} \mathbb{Z} & k = 0 \\ 0 & \text{otherwise} \end{cases} \]

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A variation of this example:

\[ H_k(X) = \begin{cases} \mathbb{Z} & k = 0, 1 \\ 0 & \text{otherwise} \end{cases} \]

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When do different chain maps produce the same map on homology?
Let $\varphi, \psi : C \to C'$ be chain maps. A collection of group homomorphisms

$$D_k : C_k \to C'_{k+1}$$

is a chain homotopy between $\varphi$ and $\psi$ if, for all $k \in \mathbb{Z}$,

$$\partial'_{k+1} D_k + D_{k-1} \partial_k = \psi_k - \varphi_k.$$

Prop: If $\varphi$ and $\psi$ are chain homotopic, then $\varphi_* = \psi_*$. 
Let \( \pi : \mathbb{R}^d \rightarrow \mathbb{R}^{d-1} \) be the projection map

\[
\pi(x_1, \ldots, x_{d-1}, x_d) = (x_1, \ldots, x_{d-1})
\]
Let $\pi : \mathbb{R}^d \to \mathbb{R}^{d-1}$ be the projection map

$$\pi(x_1, \ldots, x_{d-1}, x_d) = (x_1, \ldots, x_{d-1})$$

Let $X \subset \mathbb{R}^d$, $Y \subset \mathbb{R}^{d-1}$ be cubical sets with $\pi(X) \subset Y$
Let $\pi : \mathbb{R}^d \to \mathbb{R}^{d-1}$ be the projection map

$$\pi(x_1, \ldots, x_{d-1}, x_d) = (x_1, \ldots, x_{d-1})$$

Let $X \subset \mathbb{R}^d$, $Y \subset \mathbb{R}^{d-1}$ be cubical sets with $\pi(X) \subset Y$

Each $Q \subset \mathcal{K}(X)$ decomposes into a product of elementary intervals

$$Q = I_1 \times I_2 \times \cdots \times I_{d-1} \times I_d$$
Let $\pi : \mathbb{R}^d \to \mathbb{R}^{d-1}$ be the projection map

$$\pi(x_1, \ldots, x_{d-1}, x_d) = (x_1, \ldots, x_{d-1})$$

Let $X \subset \mathbb{R}^d$, $Y \subset \mathbb{R}^{d-1}$ be cubical sets with $\pi(X) \subset Y$

Each $Q \subset \mathcal{K}(X)$ decomposes into a product of elementary intervals

$$Q = I_1 \times I_2 \times \cdots \times I_{d-1} \times I_d$$

Observe that $P = \pi(Q) \subset \mathcal{K}(Y)$ has the form

$$P = I_1 \times I_2 \times \cdots I_{d-1}$$
Let $\pi : \mathbb{R}^d \to \mathbb{R}^{d-1}$ be the projection map

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Observe that $P = \pi(Q) \subset \mathcal{K}(Y)$ has the form

$$P = I_1 \times I_2 \times \cdots I_{d-1}$$

This induces the chain map $\iota_k : C_k(X) \to C_k(Y)$ given by

$$\iota_k(Q) = \begin{cases} P & \text{if } I_d = [a, a] \\ 0 & \text{if } I_d = [a, a + 1] \end{cases}$$
Approximating Continuous Functions (Part II)
\[ f : X \rightarrow Y \]
\[ f : X \rightarrow Y \]
Need to approximate in an efficient manner taking into account errors.
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Need to approximate in an efficient manner taking into account errors.
Let $X$ and $Y$ be cubical sets. A multivalued map $F: X \Rightarrow Y$ is \textbf{cubical} if

1. For every $x \in X$, $F(x)$ is a cubical set.

2. For every $Q \in \mathcal{K}(X)$, $F\mid_Q$ is constant.
Let $X$ and $Y$ be cubical sets. A multivalued map $F : X \Rightarrow Y$ is **cubical** if

1. For every $x \in X$, $F(x)$ is a cubical set.

2. For every $Q \in \mathcal{K}(X)$, $F|_Q$ is constant.
Let $X$ and $Y$ be cubical sets. A multivalued map $F: X \multimap Y$ is **cubical** if

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Let $X$ and $Y$ be cubical sets. A multivalued map $F : X \rightrightarrows Y$ is **cubical** if

1. For every $x \in X$, $F(x)$ is a cubical set.

2. For every $Q \in \mathcal{K}(X)$, $F|_Q$ is constant.

A cubical map $F$ is **lower semicontinuous** if and only if the following property is satisfied:

If $P$ is a face of $Q$, then $F(\overset{\circ}{P}) \subset F(\overset{\circ}{Q})$. 

"
The **support** of a $k$-chain $c = \sum_{Q \in \mathcal{K}_k(X)} \alpha_Q Q$ is

$$|c| := \bigcup_{\alpha_Q \neq 0} Q$$
The support of a $k$-chain $c = \sum_{Q \in \mathcal{K}_k(X)} \alpha_Q Q$ is

$$|c| := \bigcup_{\alpha_Q \neq 0} Q$$

Theorem: Assume $F : X \Rightarrow Y$ is a lower semicontinuous acyclic-valued cubical map. Then there exists a chain map $\varphi : C(X) \rightarrow C(Y)$ satisfying the following two conditions:

$$|\varphi(Q)| \subset F(\hat{Q}) \text{ for all } Q \in \mathcal{K}(X),$$

$$\varphi(Q) \in \mathcal{K}_0(F(Q)) \text{ for any vertex } Q \in \mathcal{K}_0(X).$$
The support of a $k$-chain $c = \sum_{Q \in \mathcal{K}_k(X)} \alpha_Q Q$ is

$$|c| := \bigcup_{\alpha_Q \neq 0} Q$$

**Theorem:** Assume $F : X \Rightarrow Y$ is a lower semicontinuous acyclic-valued cubical map. Then there exists a chain map $\varphi : C(X) \to C(Y)$ satisfying the following two conditions:

$$|\varphi(Q)| \subset F(\tilde{Q}) \text{ for all } Q \in \mathcal{K}(X),$$

$$\varphi(Q) \in \mathcal{K}_0(F(Q)) \text{ for any vertex } Q \in \mathcal{K}_0(X).$$

The above mentioned $\varphi$ is called a **chain selector** for $F$. 
To construct chain selector we need to solve

$$\partial_k x = c$$

where $c$ is a cycle over every elementary cell of the domain. Theoretically this can be done because the images of elementary cells are acyclic.
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**Problem:** Computational cost of solving many moderately sized linear algebra problems
Theorem: Let \( \varphi, \psi : C(X) \to C(Y) \) be chain selectors for the lower semicontinuous acyclic-valued cubical map \( F : X \Rightarrow Y \). Then \( \varphi \) is chain homotopic to \( \psi \) and hence they induce the same homomorphism in homology.
Theorem: Let $\phi, \psi : C(X) \to C(Y)$ be chain selectors for the lower semicontinuous acyclic-valued cubical map $F : X \Rightarrow Y$. Then $\phi$ is chain homotopic to $\psi$ and hence they induce the same homomorphism in homology.

Let $F : X \Rightarrow Y$ be a lower semicontinuous acyclic-valued cubical map. Let $\phi : C(X) \to C(Y)$ be a chain selector of $F$. The induced map on homology of $F$, $F_* : H_*(X) \to H_*(Y)$ is defined by

$$F_* := \phi_*.$$
\[ f: X \rightarrow Y \]
$f: X \rightarrow Y$

$g: X \rightarrow Y$
From the perspective of homology there is no difference between $f$ and $g$. Either model produces the same results. Thus we can as well use the yellow multivalued map as a model.
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Efficiently producing chain maps

\[ f : X \rightarrow Y \]
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Efficiently producing chain maps

\[ f: X \rightarrow Y \quad \text{such that} \quad f = \Pi_Y \circ \Pi_X^{-1} \]
Efficiently producing chain maps

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\[ K. M., M. Mrozek, P. Pilarczyk \]
Efficiently producing chain maps

\[ f : X \to Y \quad f = \Pi_Y \circ \Pi_X^{-1} \]

\[ F \text{ acyclic implies} \]

\[ \Pi_{X*} : H_*(\Gamma_F) \to H_*(X) \]

is an isomorphism
Efficiently producing chain maps

\[ f : X \rightarrow Y \quad f = \Pi_Y \circ \Pi_X^{-1} \]

\[ \Pi_Y \]

\[ \Pi_X \]

\[ \Gamma_F \]

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is an isomorphism

\[ f_* = \Pi_{Y*} \circ \Pi_{X*}^{-1} \]
Dynamical Systems

Motivation
The Logistic Map

\[ f(x, r) = r x (1 - x) \quad x \in [0, 1] \]
\[ r \geq 0 \]

Population Level
reproduction rate
The Logistic Map

\[ f(x, r) = rx(1 - x) \quad x \in [0, 1] \]
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Population Level
reproduction rate

Assume the logistic map is a perfect model for population growth of an insect.
The Logistic Map

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Assume the logistic map is a perfect model for population growth of an insect.

Assume we can perform perfect numerical simulations of the model.
The Logistic Map

\[ f(x, r) = rx(1 - x) \quad \text{where} \quad x \in [0, 1] \]

Population Level

\[ r \geq 0 \]

reproduction rate

**Assume** the logistic map is a perfect model for population growth of an insect.

**Assume** we can perform perfect numerical simulations of the model.

It is still possible that with high probability the conclusions will be wrong!
\[ f(x) = rx(1 - x) \]
Assume the field biologist can measure birth rate to within one decimal place.

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Any computation probably suggests the wrong dynamics!
Consider a continuous parameterized family of maps

\[ f: X \times \Lambda \rightarrow X \]

\[ (x, \lambda) \mapsto f_\lambda(x) := f(x, \lambda) \]

- \( X \) locally compact metric space (Phase space)
- \( \Lambda \) locally connected compact (Parameter space)
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Ideas are applicable to ODEs, PDEs, FDEs

Computational issues are more challenging
Consider a continuous parameterized family of maps

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Ideas are applicable to ODEs, PDEs, FDEs

Computational issues are more challenging

To simplify discussion will assume \(f\) is a homeomorphism, \(X \subset \mathbb{R}^n\), and \(\Lambda \subset \mathbb{R}^m\).
$S \subset X$ is an **invariant set** for $f_\lambda$ if

$$f_\lambda(S) = S$$
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$$f_\lambda(S) = S$$

Example: $\bar{x} \in X$ is a fixed point if $f_\lambda(\bar{x}) = \bar{x}$

Solve $f(x, r) = rx(1 - x) = x$, $x \in [0, 1]$, $r > 0$

$$\bar{x}(r) = \begin{cases} 
0 & r > 0 \\
\frac{r-1}{r} & r > 1 
\end{cases}$$
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![Graph showing the behavior of $f(x, r)$ for different values of $r$.]
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Observe: We don’t see all the fixed points
\( S \subset X \) is an **invariant set** for \( f_\lambda \) if

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\[ \text{Thursday, March 4, 2010} \]
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\[ \bar{x}(r) = \begin{cases} 
0 & r > 0 \\
\frac{r-1}{r} & r > 1 
\end{cases} \]

Observe: We don’t see all the fixed points

We don’t see “most” invariant sets.
The dynamics at $\lambda_0$ and $\lambda_1$ are \textbf{equivalent} if there exists a homeomorphism $h: X \to X$ such that

$$f_{\lambda_0} = h^{-1} \circ f_{\lambda_1} \circ h$$
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$\lambda_0$ is a **bifurcation** point if on any neighborhood $\lambda_0 \in U \subset \Lambda$ there are parameter values at which the dynamics is not equivalent.
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$\lambda_0$ is a bifurcation point if on any neighborhood $\lambda_0 \in U \subset \Lambda$ there are parameter values at which the dynamics is not equivalent.

**Fact:** In nonlinear systems the set of parameter values at which bifurcations occur can consist of Cantor sets of positive measure.
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**Fact:** In nonlinear systems the set of parameter values at which bifurcations occur can consist of Cantor sets of positive measure.

**Conclusion:** The topological structure of invariant sets is too fine an invariant for many applications.
Why is it easy to see the fixed point for low values of \( r \)?
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$$f_\lambda(N) \subset \text{int}(N)$$
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Remarks: $A = \bigcap_{n=1}^{\infty} f_\lambda^n(N) \neq \emptyset$
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$f_\lambda(A) = A$  We have existence of invariant sets!
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If \( N \) is an attracting neighborhood for \( f_{\lambda_0} \) then \( N \) is an attracting neighborhood for all \( \lambda \) close to \( \lambda_0 \).  Robust
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$$A = \bigcap_{n=1}^{\infty} f^n_\lambda(N) \neq \emptyset$$

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If $N$ is an attracting neighborhood for $f_{\lambda_0}$ then $N$ is an attracting neighborhood for all $\lambda$ close to $\lambda_0$. **Robust**

**WARNING:** $f_{\lambda_0}(A) = A \not\Rightarrow f_\lambda(A) = A$
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WARNING: $f_{\lambda_0}(A) = A \quad \not\Rightarrow \quad f_\lambda(A) = A$

Let $N \subset X$. The omega and alpha limit sets are

$$\omega(N, f_\lambda) := \bigcap_{k=0}^{\infty} \text{cl} \left( \bigcup_{n=k}^{\infty} f_\lambda^n(N) \right) \quad \text{and} \quad \alpha(N, f_\lambda) := \bigcap_{k=0}^{\infty} \text{cl} \left( \bigcup_{n=-k}^{-\infty} f_\lambda^n(N) \right)$$

Thursday, March 4, 2010
A compact set $N \subset X$ is an **isolating neighborhood** for $f_\lambda$ if

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Isolating neighborhoods are Robust.
A compact set $N \subset X$ is an *isolating neighborhood* for $f_\lambda$ if

$$\text{Inv}(N, f_\lambda) \subset \text{int}(N)$$

Isolating neighborhoods are Robust.

Only useful if we can find isolating neighborhoods.
Invariant Sets

Decompositions
Let $S$ be an isolated invariant set for $f_\lambda$. 
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A (compact) invariant set $A \subset S$ is an attractor in $S$ if there exists a neighborhood $U$ of $A$ such that

$$\omega(U \cap S, f_\lambda) = A.$$
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The dual repeller of $A$ in $S$ is

$$A^* := \{ x \in S \mid \omega(x, f_\lambda) \cap A = \emptyset \}.$$
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$$A^* := \{x \in S \mid \omega(x, f_\lambda) \cap A = \emptyset\}.$$ 

The pair $(A, A^*)$ is called an **attractor-repeller pair decomposition** of $S$. 
Theorem: (Conley) Let \((A, A^*)\) be an attractor-repeller pair decomposition of \(S\) under \(f_\lambda\). Then there exists a continuous Lyapunov function

\[ V: S \rightarrow [0,1] \]

such that

- \(A = V^{-1}(0)\) and \(A^* = V^{-1}(1)\)
- if \(x \in S \setminus (A \cup A^*)\) then \(V(f(x)) < V(x)\).
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**Observe:** $S \setminus (A \cup A^*)$ consists of gradient like-dynamics
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Recurrent dynamics occurs in \(A \cup A^*\)
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Recurrent dynamics occurs in \(A \cup A^*\)

Basic decomposition of global dynamics:

Gradient-like vs. Recurrent
Attractors and dual repellers can be fractal sets.
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What are the attractor repeller pairs?
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In general there are many attractor-repeller pairs.
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In general there are many attractor-repeller pairs.

Observe: The intersection of the unions of all the attractor repeller pairs gives the critical points

$$\left\{ a, b, c, d \right\} = \bigcap_{\text{attractors}} (A \cup A^*)$$
A Morse decomposition of $S$ is a finite collection of disjoint isolated invariant subsets of $S$, called Morse sets,

$$M(S) := \{ M(p) \subset S \mid p \in \mathcal{P} \}$$

for which there exists a strict partial order, called an admissible order, on the indexing set $\mathcal{P}$ such that for every $x \in S \setminus \bigcup_{p \in \mathcal{P}}$ there exists $p, q \in \mathcal{P}$ such that $p > q$ and

$$\alpha(x, f_\lambda) \subset M(p) \quad \text{and} \quad \omega(x, f_\lambda) \subset M(q)$$
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Since \( \mathcal{P} \) is a partially ordered set, a Morse decomposition can be represented as an acyclic directed graph called the Morse graph.
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Since $\mathcal{P}$ is a partially ordered set, a Morse decomposition can be represented as an acyclic directed graph called the Morse graph.
Approximating Dynamical Systems
Some Notation

Recall we are studying $f : X \times \Lambda \to X$
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Parameterized Dynamical System

\[
F : X \times \Lambda \rightarrow X \times \Lambda \\
F(x, \lambda) = (f_\lambda(x), \lambda) = (f(x, \lambda), \lambda)
\]
Some Notation

Recall we are studying \( f : X \times \Lambda \to X \)

Parameterized Dynamical System

\[ F : X \times \Lambda \to X \times \Lambda \]

\[ F(x, \lambda) = (f_\lambda(x), \lambda) = (f(x, \lambda), \lambda) \]

Given \( Q \subset \Lambda \) denote the restriction of \( F \) to \( X \times Q \) by

\[ F_Q : X \times Q \to X \times Q \]
Some Notation

Recall we are studying \( f : X \times \Lambda \to X \)

Parameterized Dynamical System

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Parameterized Dynamical System

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Given \( Q \subset \Lambda \) denote the restriction of \( F \) to \( X \times Q \) by

\[
F_Q : X \times Q \to X \times Q
\]

Observe: \( f_\lambda \iff F_{\{\lambda\}} \)
Choose a compact region in parameter space: \( Q \subset \Lambda \)
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Choose a (cubical) grid \( \mathcal{X} \) that covers \( X \)
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Choose a (cubical) grid \( \mathcal{X} \) that covers \( X \)
Define a multivalued map \( F_Q : \mathcal{X} \rightrightarrows \mathcal{X} \)
Choose a compact region in parameter space: \( Q \subset \Lambda \)

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Define a multivalued map \( \mathcal{F}_Q : \mathcal{X} \rightrightarrows \mathcal{X} \)
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Choose a (cubical) grid $\mathcal{X}$ that covers $X$

Define a multivalued map $\mathcal{F}_Q : \mathcal{X} \rightarrow \mathcal{X}$

\[ \mathcal{F}_Q(G) \]
Choose a compact region in parameter space: $Q \subset \Lambda$

Choose a (cubical) grid $\mathcal{X}$ that covers $X$

Define a multivalued map $\mathcal{F}_Q : \mathcal{X} \Rightarrow \mathcal{X}$

$\mathcal{F}_Q$ is an outer approximation if for every $G \in \mathcal{X}$

$$f(G, Q) \subset \text{int}(||\mathcal{F}_Q(G)||)$$
Think of $\mathcal{F}_Q$ as a directed graph:
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*vertices:* $G \in \mathcal{X}$
Think of $\mathcal{F}_Q$ as a directed graph:

**vertices:** $G \in \mathcal{X}$

**edges:** $H \in \mathcal{F}_Q(G) \Rightarrow G \rightarrow H$
Think of $\mathcal{F}_Q$ as a directed graph:

**vertices:** \( G \in \mathcal{X} \)

**edges:** \( H \in \mathcal{F}_Q(G) \Rightarrow G \rightarrow H \)

Basic decomposition of global dynamics:
Gradient-like vs. Recurrent
Think of $\mathcal{F}_Q$ as a directed graph:

vertices: $G \in \mathcal{X}$

edges: $H \in \mathcal{F}_Q(G) \Rightarrow G \rightarrow H$

Basic decomposition of global dynamics: Gradient-like vs. Recurrent

A strongly connected path component in a directed graph is a maximal collection of vertices such that there exists a nontrivial path between any two pair of vertices.
Fact: There exists an algorithm $O(|\mathcal{X}| + |\mathcal{F}_Q|)$ that produces a function $V: \mathcal{X} \rightarrow \mathbb{Z}$ such that for all $H \in \mathcal{F}_Q(G)$

1. $V(G) = V(H)$ if $G, H$ are in the same strongly connected path component.

2. $V(G) > V(H)$ otherwise.
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1. $V(G) = V(H)$ if $G, H$ are in the same strongly connected path component.
2. $V(G) > V(H)$ otherwise.

Theorem: This algorithm produces a Morse graph for $F_Q: X \times Q \to X \times Q$

and hence for $f_\lambda$ for all $\lambda \in Q$. 
Idea of proof:
Idea of proof: \[ \bigcup_{i=1}^{n} G_i \subset X \]
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Because \( \mathcal{F}_Q(\bigcup_{i=1}^{n} G_i) = \bigcup_{i=1}^{n} G_i \).
Idea of proof: \[ \bigcup_{i=1}^{n} G_i \subset \mathcal{X} \]

Because \( F \) is minimal, \( \mathcal{F}_Q \left( \bigcup_{i=1}^{n} G_i \right) = \bigcup_{i=1}^{n} G_i \).

\( \mathcal{F}_Q \) is an outer approximation, thus

\[ f \left( \bigcup_{i=1}^{n} G_i, Q \right) \subset \text{int} \left( \bigcup_{i=1}^{n} G_i \right). \]
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Thus \( \text{Inv}(\bigcup_{i=1}^{n} G_i, F_Q) \) is an attractor (isolated invariant set).
Conley Index (part 1)

Definitions
Recall the Notation

Recall we are studying $f : X \times \Lambda \to X$

Parameterized Dynamical System

$F : X \times \Lambda \to X \times \Lambda$

$F(x, \lambda) = (f_\lambda(x), \lambda) = (f(x, \lambda), \lambda)$

Given $Q \subset \Lambda$ denote the restriction of $F$ to $X \times Q$ by

$F_Q : X \times Q \to X \times Q$

Observe: $f_\lambda \iff F_{\{\lambda\}}$
Let $P = (P_1, P_0)$ be a pair of compact sets in $X \times Q$ with $P_0 \subset P_1$. 
Let $P = (P_1, P_0)$ be a pair of compact sets in $X \times Q$ with $P_0 \subset P_1$.

Define $F_{Q,P} : P_1/P_0 \to P_1/P_0$ by

$$F_{Q,P}(x, \lambda) := \begin{cases} 
F_Q(x, \lambda) & \text{if } (x, \lambda), F_Q(x, \lambda) \in P_1 \setminus P_0 \\
[P_0] & \text{otherwise}
\end{cases}$$
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$P$ is an index pair for $F_Q$ if

- $F_{Q,P}$ is continuous
- $\text{cl}(P_1 \setminus P_0)$ is an isolating maxinmal invariant set.
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$P$ is an index pair for $F_Q$ if

- $F_{Q,P}$ is continuous
- $\text{cl}(P_1 \setminus P_0)$ is an isolating set

To be useful we need to be able to find Index Pairs.
\[ F_{Q,P}(x, \lambda) := \begin{cases} F_Q(x, \lambda) & \text{if } (x, \lambda), F_Q(x, \lambda) \in P_1 \setminus P_0 \\ [P_0] & \text{otherwise} \end{cases} \]
Theorem: If no iterate of $F_{Q,P}$ is homotopic to the trivial map, then

$$\text{Inv}(\text{cl}(P_1 \setminus P_0), F_\lambda) \neq \emptyset, \quad \forall \lambda \in Q$$
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Corollary: If $F_{Q,P \ast}: H_\ast(P_1/P_0, [P_0]) \rightarrow H_\ast(P_1/P_0, [P_0])$ is not nilpotent, then

$$\text{Inv}(\text{cl}(P_1 \setminus P_0), F_\lambda) \neq \emptyset, \quad \forall \lambda \in Q$$
Examples:
Examples:

\( \mathbb{R}^{n-k} \)

\( \mathbb{R}^k \)

hyperbolic
fixed
point
Examples:

\[ P = (P_1, P_0) \]

hyperbolic fixed point

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If $k = 1$

$P = (P_1, P_0)$

Examples:
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If \( k = 1 \)

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H_j(P_1/P_0, [P_0]) \cong \begin{cases} 
\mathbb{Z} & j = 1 \\
0 & \text{otherwise}
\end{cases}
\]
Examples:

If \( k = 1 \)

\[ \mathbb{P} = (P_1, P_0) \]

\[ \mathbb{R}^{n-k} \]

\[ \mathbb{R}^k \]

Hyperbolic fixed point

\[ F_{Q,P} \]

\[ H_j(P_1/P_0, [P_0]) \cong \begin{cases} \mathbb{Z} & j = 1 \\ 0 & \text{otherwise} \end{cases} \]

\[ F_{Q,P} = \pm 1 \]
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F_{Q,P} = \pm 1
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\[
P' = (P'_1, P'_0)
\]
Examples:

If $k = 1$

$$P = (P_1, P_0)$$

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$F_{Q,P}$

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$H_j(P_1/P_0, [P_0]) \cong \begin{cases} 
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Let $S$ be an isolated invariant set
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Let $P = (P_1, P_0)$ and $P' = (P'_1, P'_0)$ be index pairs such that

$$S = \text{Inv}(\text{cl}(P_1 \setminus P_0)) = \text{Inv}(\text{cl}(P'_1 \setminus P'_0))$$
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**Theorem:** (Szymczak, Franks-Richeson) $F_{Q,P^*}$ and $F_{Q,P'^*}$ are shift equivalent.
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The shift equivalence class is the **Conley Index** of $S$. 
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**Theorem:** Let $S_Q = \text{Inv}(\text{cl}(P_1 \setminus P_0), F_Q)$. If $Q$ is simply connected then the Conley index of $S_\lambda$ is constant for all $\lambda \in Q$. 

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**Conley Index is Robust**
Finding Index Pairs
Recall:

\[ \mathcal{F}_Q(G) \]

\( \mathcal{F}_Q \) an outer approximation:

\[ f(G, Q) \subset \text{int}(|\mathcal{F}_Q(G)|) \]
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\mathcal{F}_Q(G)
\]

\[f(G, Q) \subset \text{int}(|\mathcal{F}_Q(G)|)\]

\[\mathcal{F}_Q\text{ an outer approximation:}\]

\[
\mathcal{S} \subset \mathcal{X}\text{ is an invariant set for }\mathcal{F}_Q\text{ if}\]

\[
\mathcal{S} \subset \mathcal{F}_Q \quad \text{and} \quad \mathcal{S} \subset \mathcal{F}_Q^{-1}
\]
Recall: $\mathcal{F}_Q$ an outer approximation: 
$f(G, Q) \subset \text{int}(\mathcal{F}_Q(G))$

Assume: $S \subset \mathcal{X}$ is an invariant set for $\mathcal{F}_Q$ such that $|S|$ is an isolating neighborhood for $F_Q$

$S \subset \mathcal{F}_Q$ and $S \subset \mathcal{F}_Q^{-1}$
Recall: $S \subset \mathcal{X}$ is an **invariant set** for $\mathcal{F}_Q$ if

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Example: $S \subset \mathcal{X}$ is a strongly connected path component for $\mathcal{F}_Q$
Finding Index Pairs

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**Example:** $S \subset X$ is a strongly connected path component for $F_Q$

**Theorem:** Define $P_i = |P_i|$ where

$$P_1 := S \cup F_Q(S) \quad \text{and} \quad P_0 := P_1 \setminus S$$

then, $(P_1, P_0)$ is an index pair for $F_Q$. 

Recall:

$F_Q$ an outer approximation:

$$f(G, Q) \subset \text{int}(|F_Q(G)|)$$

$G$
Conley Index (part 2)
Question: Given the index pair $P$ and the Conley index

$$F_{P^*} = \begin{cases} 
\pm 1: \mathbb{Z} \to \mathbb{Z} & j = k \\
0 & \text{otherwise}
\end{cases}$$

what can be said about the structure of

$$S = \text{Inv}(\text{cl}(P_1 \setminus P_0), F)$$
Question: Given the index pair $P$ and the Conley index $F_{P^*}$, what can be said about the structure of $S = \text{Inv} (\text{cl}(P_1 \setminus P_0), F)$?

**Diagram:**

$$
\mathbb{R}^n \rightarrow \mathbb{R}^k
$$

$$
\mathbb{R}^n \rightarrow \mathbb{R}^k
$$

$P = (P_1, P_0)$
Question: Given the index pair $P$ and the Conley index $F_{P*} = \begin{cases} \pm 1: \mathbb{Z} \to \mathbb{Z} & j = k \\ 0 & \text{otherwise} \end{cases}$, what can be said about the structure of $S = \text{Inv} (\text{cl}(P_1 \setminus P_0), F)$?

$Lefschetz fixed point theorem$ $S$ contains a fixed point.

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$N_1$

$N_0$

$f(N_0)$ $f(N_1)$

Chaotic Dynamics
Consider $f : \mathbb{R}^2 \to \mathbb{R}^2$

$N_0 \cup N_1$ is an isolating neighborhood

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Consider $f : \mathbb{R}^2 \to \mathbb{R}^2$.

$N_0 \cup N_1$ is an isolating neighborhood

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\[ S = \text{Inv}(N_0 \cup N_1, f) \]

Define $\rho : S \to \Sigma_2$ by

\[ \rho(x)_n := \begin{cases} 
0 & \text{if } f^n(x) \in N_0 \\
1 & \text{if } f^n(x) \in N_1 
\end{cases} \]

\[ \Sigma_2 := \prod_{n=0}^{\infty} \{0, 1\} \]
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**Theorem:** $\rho$ is onto.
Consider $f : \mathbb{R}^2 \to \mathbb{R}^2$

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Theorem: $\rho$ is onto.
A Much Cruder Result
A Much Cruder Result

Let $f: X \to X$ be continuous. A nonempty isolated invariant set $S \subset X$ is a $T$-cycle set if there exist $T$ disjoint compact regions $N_1, \ldots, N_T$ such that $N = \bigcup_{i=1}^{T} N_i$ is an isolating neighborhood for $S$ and

$$f(N_i) \cap N \subset N_{i+1}, \quad i = 0, \ldots, T \quad (N_0 = N_T)$$
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\[ f(N_i) \cap N \subset N_{i+1}, \quad i = 0, \ldots, T \quad (N_0 = N_T) \]

$S$ is an attracting $T$-cycle set if $f(N_i) \subset N_{i+1}, i = 0, \ldots, T$
Consider a Morse graph defined by the strongly connected path components of an outer approximation $\mathcal{F}_Q$ of

$$F_Q : N \rightarrow N$$

where $N$ is an attracting neighborhood.
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$$F_Q : N \rightarrow N$$

where $N$ is an attracting neighborhood.

**Theorem:** At a minimal node in the graph the nonzero eigenvalues of the index map on the 0-th level of homology is either

$$\emptyset \text{ or } \left\{ e^{2\pi i \frac{k}{T}} | k = 0, \ldots, T - 1 \right\}.$$

In the latter case the associated invariant set $S$ is an attracting $T$-cycle set.
Describing the gradient-like dynamics
Describing the gradient-like dynamics

unstable equilibrium (origin)

gradient-like dynamics

stable equilibrium
Gradient-like Structure:
directed graph
Morse Graph

Describing the gradient-like dynamics
Describing the gradient-like dynamics

Gradient-like Structure:
directed graph
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unstable equilibrium (origin)

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gradient-like dynamics

unstable equilibrium (origin)
gradient-like dynamics

stable period 2 orbit

gradient-like dynamics
Describing the gradient-like dynamics

**Gradient-like Structure:**
- directed graph
- Morse Graph

**Recurrent Dynamics:**
- eigenvalues of induced maps on homology
- Conley Index

**Unstable Equilibrium (origin)**

**Stable Equilibrium**

**Unstable Period 2 Orbit**
Describing the gradient-like dynamics
Describing the gradient-like dynamics
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Impossible to identify every invariant set
Describing the gradient-like dynamics

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Impossible to identify every invariant set

origin
fixed pt
period 2
period 4

origin
fixed pt
Morse Set
Describing the gradient-like dynamics

Impossible to identify every invariant set

Morse Graph

origin
fixed pt
period 2
period 4

period 2
period 4

Isolating Neighborhood
Comparing Dynamics at Different Parameter Values
Comparing Dynamics at Different Parameter Values

![Graph showing dynamics at different parameter values with key points labeled as origin and 1-cycle.](image-url)
Comparing Dynamics at Different Parameter Values

![Diagram showing the origin, 1-cycle, and 2-cycle at different parameter values](image)

- **Origin**: The starting point of the system.
- **1-cycle**: A cycle that repeats after one iteration of the system.
- **2-cycle**: A cycle that repeats after two iterations of the system.

The graph illustrates the system's behavior as the parameter value changes from 2.4 to 4.0, highlighting the transitions between the origin, 1-cycle, and 2-cycle.
Comparing Dynamics at Different Parameter Values

- Origin
- 1-cycle
- 2-cycle
Comparing Dynamics at Different Parameter Values
Comparing Dynamics at Different Parameter Values

Clutching Graphs
Comparing Dynamics at Different Parameter Values

Clutching Graphs

Not Equivalent
Comparing Dynamics at Different Parameter Values

Clutching Graphs

Not Equivalent

Equivalent
Comparing Dynamics at Different Parameter Values

Clutching Graphs

Origin
1-cycle
2-cycle

Origin
1-cycle
2-cycle

Origin
1-cycle
2-cycle

Origin
1-cycle
2-cycle

Not Equivalent
Equivalent
Not Equivalent
Comparing Dynamics at Different Parameter Values

Clutching Graphs

- Origin
- 1-cycle
- 2-cycle
- 4-cycle

Not Equivalent
Equivalent
Not Equivalent

Bifurcation occurs at parameter value 3.4.
Comparing Dynamics at Different Parameter Values

Clutching Graphs

Not Equivalent  Equivalent  Not Equivalent

origin 1-cycle  origin 1-cycle  origin 1-cycle  origin 1-cycle
origin 1-cycle  origin 1-cycle  origin 2-cycle  origin 2-cycle
origin 1-cycle  origin 1-cycle  origin 2-cycle  origin 4-cycle

bifurcation occurs  No Bifurcation

Thursday, March 4, 2010
Building Databases for Global Dynamics

Putting it all together
A Simple Population Model

A density dependent Leslie model:

\[
\begin{bmatrix}
x \\
y
\end{bmatrix} \mapsto \begin{bmatrix}
(\theta_1 x + \theta_2 y)e^{-c(x+y)} \\
(1 - \mu)x
\end{bmatrix}
\]
A Simple Population Model

A density dependent Leslie model:

First year population
Second year population

Mathematically:

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\begin{bmatrix}
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  (1 - \mu)x
\end{bmatrix}
\]

\[ f : \mathbb{R}^2 \times \mathbb{R}^4 \rightarrow \mathbb{R}^2 \]
A Simple Population Model

A density dependent Leslie model:

first year population
second year population

Mathematically: \[ f : \mathbb{R}^2 \times \mathbb{R}^4 \to \mathbb{R}^2 \]

\[ f(x, \theta, \mu, c) = \frac{1}{c} f(cx, \theta, \mu, 1) \]
A Simple Population Model

A density dependent Leslie model:

First year population
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\begin{bmatrix}
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y
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Second year population

Mathematically:
\[
f : \mathbb{R}^2 \times \mathbb{R}^4 \rightarrow \mathbb{R}^2
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\[
f(x, \theta, \mu, c) = \frac{1}{c} f(cx, \theta, \mu, 1)
\]

To communicate I want to show pictures:

\[
\begin{bmatrix}
x \\
y
\end{bmatrix}
\mapsto
\begin{bmatrix}
(\theta_1 x + \theta_2 y)e^{-0.1(x+y)} \\
0.7x
\end{bmatrix}
\]

\[
f : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2
\]
\[
(x, y; \theta_1, \theta_2)
\]
\[ x + y \]

\[ \theta_1 = \theta_2 \]

Ugarcovici & Weiss, Nonlinearity ‘04
Claim: Database provides a mathematically rigorous, queryable description of the global dynamics of

\[
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x \\
y
\end{bmatrix} \mapsto \begin{bmatrix}
(\theta_1 x + \theta_2 y) e^{-0.1(x+y)} \\
0.7x
\end{bmatrix}
\]

for the phase space

\([0, \infty) \times [0, \infty)\)

and all

\[\theta = (\theta_1, \theta_2) \in [8, 37] \times [3, 50]\]
Claim: Database provides a mathematically rigorous, queryable description of the global dynamics of

\[
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x \\
y
\end{bmatrix}
\mapsto
\begin{bmatrix}
(\theta_1 x + \theta_2 y)e^{-0.1(x+y)} \\
0.7x
\end{bmatrix}
\]

for the phase space $[0, \infty) \times [0, \infty)$ and all

\[\theta = (\theta_1, \theta_2) \in [8, 37] \times [3, 50]\]

Input: Nonlinear map, Phase space, Parameter space
Resolution in phase space
Resolution in parameter space
The Data Base

The Continuation Graph
The Data Base

Class 10 [43 boxes]

Class 7 [66 boxes]

Class 9 [50 boxes]

Class 8 [65 boxes]

Class 3 [251 boxes]

Class 4 [196 boxes]

Class 6 [73 boxes]

Class 11 [12 boxes]

Class 12 [2 boxes]

Class 5 [88 boxes]

Class 13 [1 box]

Class 17 [1 box]

Class 16 [1 box]

Class 15 [1 box]

Class 14 [1 box]

The Continuation Graph

Nodes represent Conley–Morse Graphs
The Data Base

The Continuation Graph

Nodes represent Conley–Morse Graphs

Edges indicate connectivity in parameter space
The Data Base

The Continuation Graph

Nodes represent Conley–Morse Graphs

Edges indicate connectivity in parameter space

(Should contain clutching graph)
The Continuation Diagram

Different colors represent different continuation classes.
The Data Base

The Continuation Graph

Nodes represent Conley–Morse Graphs

Edges indicate connectivity in parameter space

(Should contain clutching graph)
Let’s Query the Database!
Are there multiple basins of attraction?

Class 1 [890 boxes]
Class 2 [759 boxes]
Class 3 [251 boxes]
Class 4 [196 boxes]
Class 5 [88 boxes]
Class 6 [73 boxes]
Class 7 [66 boxes]
Class 8 [65 boxes]
Class 9 [50 boxes]
Class 10 [43 boxes]
Class 11 [12 boxes]
Class 12 [2 boxes]
Class 13 [1 box]
Class 14 [1 box]
Class 15 [1 box]
Class 16 [1 box]
Class 17 [1 box]
Are there multiple basins of attraction?

Directed Graph
(gradient structure)
Are there multiple basins of attraction?

Directed Graph (gradient structure)

Classes:
- Class 1: 890 boxes
- Class 2: 759 boxes
- Class 3: 251 boxes
- Class 4: 196 boxes
- Class 5: 88 boxes
- Class 6: 12 boxes
- Class 7: 66 boxes
- Class 8: 65 boxes
- Class 9: 50 boxes
- Class 10: 43 boxes
- Class 11: 2 boxes
- Class 12: 2 boxes
- Class 13: 1 box
- Class 14: 1 box
- Class 15: 1 box
- Class 16: 1 box
- Class 17: 1 box

Points:
- p0 : 0 → \{1\}
- p1 : 0 → \{-0.5-0.866i, -0.5+0.866i, 1\}
- p2 : 1 → \{-0.5-0.866i, -0.5+0.866i, 1\}
- p3 : origin
Are there multiple basins of attraction?

Directed Graph (gradient structure)

2 basins of attraction
Are there multiple basins of attraction?

Directed Graph (gradient structure)

2 basins of attraction

Conley Index (recurrent structure)
Are there multiple basins of attraction?

Directed Graph (gradient structure)
2 basins of attraction

Conley Index (recurrent structure)
"3 cycle"
"1 cycle"
Are there multiple basins of attraction?

Directed Graph
(gradient structure)
2 basins of attraction

Conley Index
(recurrent structure)
“3 cycle”
“1 cycle”
Are there multiple basins of attraction?

Directed Graph (gradient structure)
2 basins of attraction

Conley Index (recurrent structure)
“3 cycle”
“1 cycle”
Are there multiple basins of attraction?

Directed Graph (gradient structure)
2 basins of attraction

Conley Index (recurrent structure)
"3 cycle"
"1 cycle"
Are there multiple basins of attraction?

Directed Graph
(gradient structure)
2 basins of attraction

Conley Index
(recurrent structure)

"3 cycle"
"1 cycle"

Is Not Part of the Database
Can the population collapse?
Can the population collapse?

(Minimal element of graph contains a cube which intersects origin)
Can the population collapse?

(Minimal element of graph contains a cube which intersects origin)
Can the population collapse?

(Minimal element of graph contains a cube which intersects origin)
How does this relate to classical bifurcations?
Period Doubling Bifurcation

stable equilibrium

$0 \rightarrow \{1\}$

unstable equilibrium

$1 \rightarrow \{-1\}$

stable period 2 orbit

$0 \rightarrow \{\pm 1\}$
Period Doubling Bifurcation

stable equilibrium

$0 \rightarrow \{1\}$

unstable equilibrium

$1 \rightarrow \{-1\}$

stable period 2 orbit

$0 \rightarrow \{\pm 1\}$
Period Doubling Bifurcation

stable equilibrium

0 → \{1\}

unstable equilibrium

1 → \{-1\}

stable period 2 orbit

0 → \{±1\}
Period Doubling Bifurcation

stable equilibrium

0 → \{1\}

unstable equilibrium

1 → \{-1\}

stable period 2 orbit

0 → \{±1\}
Period Doubling Bifurcation

stable equilibrium

\[ 0 \rightarrow \{1\} \]

unstable equilibrium

\[ 1 \rightarrow \{-1\} \]

stable period 2 orbit

\[ 0 \rightarrow \{\pm 1\} \]
Period Doubling Bifurcation

- Stable equilibrium
  - $0 \rightarrow \{1\}$
- Unstable equilibrium
  - $1 \rightarrow \{-1\}$
- Stable period 2 orbit
  - $0 \rightarrow \{\pm 1\}$
Period Doubling Bifurcation

stable equilibrium

\[0 \rightarrow \{1\}\]

unstable equilibrium

\[1 \rightarrow \{-1\}\]

stable period 2 orbit

\[0 \rightarrow \{\pm 1\}\]
Period Doubling Bifurcation

stable equilibrium

$0 \rightarrow \{1\}$

unstable equilibrium

$1 \rightarrow \{-1\}$

stable period 2 orbit

$0 \rightarrow \{\pm 1\}$
Possible Period Doubling
Possible Period Doubling

- p2: origin
  - p1: origin
    - p1: 1 → {1}
    - p0: 0 → {1}
  - p0: 0 → {-1, 1}
Possible Period Doubling
Thank-you for your attention

http://chomp.rutgers.edu/

A Database Schema for the Analysis of Global Dynamics of Multiparameter Systems


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