Theorem. There is no injective continuous map $S^1 \to \mathbb{R}^2$.

Configuration space: $\Omega = S^1 \times S^1 \setminus \{(x, x) \mid x \in S^1\}$

Test map: $\tau: \Omega \to \mathbb{R}^2, \ (x, y) \mapsto (f(x), f(y))$

Test space: $T = \{(a, b) \mid a = b \in \mathbb{R}\} \subset \mathbb{R}^2$

If there is a continuous injective map $S^1 \to \mathbb{R}$, then there is a $\mathbb{Z}_2$-map $\Omega \to \mathbb{R}^2 \setminus T \cong S^0$.

Every continuous map $\Omega \to S^0$ must be constant. Since $\mathbb{Z}_2$-action on $S^0$ is free, and $\Omega$ is connected, there is no $\mathbb{Z}_2$-map $\Omega \to S^0$.

Theorem. For any continuous map $f: \Delta^{2n-k-2} \to \mathbb{R}$, there are $n$-wisely disjoint faces of the simplex $\Delta^{2n-k-2}$ whose images have a common point.

- $\Delta^{2n-k-2} = \text{conv} \{ \mathbf{v}_0, \ldots, \mathbf{v}_{2n-k-3} \}$
- There is a linear ordering $f(\mathbf{v}_0) < f(\mathbf{v}_1) < \ldots < f(\mathbf{v}_{2n-2}) < f(\mathbf{v}_{2n-1})$
- Required $n$-wisely disjoint faces are: $[v_0, v_{2n-1}, v_{2n-2}, \ldots, v_{2n-k-1}, v_{2n-k}], [v_1, v_{2n-2}, \ldots, v_{2n-k-1}, v_{2n-k}]$

Ham Sandwich Theorem. Let $\mu_1, \ldots, \mu_d$ be finite Borel measures on $\mathbb{R}^d$, such that the measure of every hyperplane is zero. There exists a hyperplane $H \subset \mathbb{R}^d$ such that $\mu_i(H^+) = \mu_i(H^-)$ for every $i \in \{1, \ldots, d\}$.

Configuration space: $\Omega = S^d$, oriented “affine hyperplanes in $\mathbb{R}^d$”

Test map: $\tau: S^d \to \mathbb{R}^d, \ H \mapsto (\mu_i(H^+) - \mu_i(H^-))_{i = 1, \ldots, d}$

Test space: $T = \{(0, \ldots, 0)\} \subset \mathbb{R}^d$

Acting group: $\mathbb{Z}_2$ acts antipodally on $S^d$ and $\mathbb{R}^d$, and $\tau$ is a $\mathbb{Z}_2$-map.

If there is a $\mathbb{Z}_2$-map $S^d \to \mathbb{R}^d \setminus T \to S^d$, then there is an equi-parting hyperplane, where the action is antipodal on both spheres.
Small Kuratowski Theorem  \( K_{3,3} \) is not a planar graph.

\[ K^{*N}_{\Delta(z)} \overset{\text{def}}{=} \{ F_1 \cup \ldots \cup F_n \in K^{n+1} \mid F_1, \ldots, F_n \text{ are pairwise disjoint} \} \]

**Example:** \( [2]^{*2}_{\Delta(z)} \)

**Example:** \( [3]^{*2}_{\Delta(z)} \approx S^1 \)

**Example:** \( I^{*2}_{\Delta(z)} \approx S^1 \) where \( I \) is interval.

**Lemma** \((K \times I)^{*p}_{\Delta(z)} \overset{\text{def}}{=} K^{*p}_{\Delta(z)} \times I^{*p}_{\Delta(z)}\)

Small Kuratowski Theorem  There is no injective map \([3]^{*2} \rightarrow IR^2\).

Configuration space: \( S = ([3]^{*2})^{*2}_{\Delta(z)} \equiv ([3]^{*2})^{*2}_{\Delta(\infty)} \equiv (S^1)^{*2} \approx S^3 \)

Test map: \( \tau: S \rightarrow IR^3 \)

\( \lambda x + (1-\lambda) y \in F \cup G \) where \( F \cap G = \emptyset \) \( \tau \rightarrow (\lambda, \lambda f(x)) \oplus (1-\lambda, (1-\lambda) f(y)) \)

Test space: \( T = \{ (a, b, c) + (a, b, c) \mid (a, b, c) \in IR^3 \} \)

Acting group: \( \mathbb{Z}_2 \) permutes copies and \( \tau \) is a \( \mathbb{Z}_2 \)-map

If there is an injective map \([3]^{*2} \rightarrow IR^2\), then there is a \( \mathbb{Z}_2 \)-map \( S^3 \rightarrow IR^3 \rightarrow IR^3 \rightarrow S^2 \).

**Sperber Theorem** (1966) \( d, k \geq 1 \) and \( N = (d+1)(k-1) \)

For every affine map \( \Lambda N \rightarrow IR^d \), there exist \( k \) disjoint faces \( F_1, \ldots, F_k \) of \( \Lambda N \) such that \( \bigcap_{i=1}^{k} f(F_i) \neq \emptyset \)

Remark: \( N \) optimal; \( d = 2, k = 4, N = 3 \cdot 3 = 9 \)
**Topological Sperner Conjecture** For every continuous map $f : \Delta^N \to \mathbb{R}^d$, there exist $k$ disjoint faces $F_1, \ldots, F_k$ of $\Delta^N$ such that

$$\bigcap_{i=1}^k f(F_i) \neq \emptyset.$$

- $d = 1$ and any $k$: we already proved
- any $d$ and $k$ prime: Bárány, Shlosman, Szücs, 1981
- any $d$ and $k$ prime power: Özaydin, 1987

**Configuration space:** $S_1 = (\Delta^N)^{(N-1)} = (\Delta^N)^{(N-1)}\Delta_1 = (\Delta^N)^{(N-1)}\Delta_1 = \mathbb{R}^d \times \mathbb{R}^{N-1}$

Since $\mathbb{R}^d$ is $(d+1)$-connected and $\text{Conn } X \times Y = \text{Conn } X + \text{Conn } Y + 2$, then

$S_1$ is $(N-1)$-connected.

**Test map:**

$$\tau : S_1 \to (\mathbb{R}^{d+1}) \times \mathbb{R}^k$$

$$A_1, x_1 + \ldots + A_k x_k \in F_1, \ldots, F_k \mapsto (\alpha_1, \alpha_2 + x_1, \ldots, \alpha_k + x_k)$$

$F_1, \ldots, F_k$ pairwise disjoint; $\sum A_i = 1$

**Test space:**

$$T = \{(x_1, \ldots, x_k) \in (\mathbb{R}^{d+1}) \times \mathbb{R}^k | x_1 = \ldots = x_k \}$$

**Acting group:** Symmetric group $S_k$ permuting copies of join and direct sum

$\tau$ is an $S_k$-map

If there is a continuous map $f : \Delta^N \to \mathbb{R}^d$ that fails Sperner conjecture, then there exists an $S_k$-map

$$S_1 = [k]^{(N-1)} \to \mathbb{R}^{(d+1)k} \to T \to S^{N-1}$$

**Gold's Theorem** Let $G$ be a finite group with $|G| > 1$. Let $X$ be an $n$-connected $G$-space, and $Y$ be a free $G$-space of dimension at most $n$. Then there is no $G$-map $X \to Y$. 

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*Equiariant Methods in Geometric Combinatorics*

"Paul V. N. Blagojevic" page 3.
Proof of topological Eberlein theorem for \( k \) prime: If \( k \) is a prime, then cyclic permutation induces a subgroup \( \mathbb{Z}_k \subseteq S_k \) that acts freely on \( \Omega \) and on \( S^{n-1} \).

\[ \mathbb{Z}_k \text{ acts freely on } \Omega \text{ and on } S^{n-1} \]

- If \( k \) is not a prime then there is no subgroup of \( S_k \) that acts freely on \( S^{n-1} \).
- \( k \) divides \( S^{n-1} \), then there is an elementary abelian subgroup \( \mathbb{Z}_k \) that acts fixed point free on \( S^{n-1} \).

Next objective: Prove that there is no \( (\mathbb{Z}_p^n) \)-map \( \Omega \rightarrow S^{n-1} \).

Group cohomology: \( G \) a finite group

\[ E_G = \text{ a free contractible } G\text{-complex} \]

\[ = \bigoplus_{i \geq 0} \mathbb{Z} \cdot t_i g_i \quad \text{where } g_i \in G, t_i \in \mathbb{Z}, \text{ only finitely many } t_i \neq 0, \sum t_i = 1. \]

- \( G \)-action: \( (\sum t_i g_i) \cdot g = \sum t_i (g g_i) \)
- Topology: \( t_i : E_G \rightarrow [0,1] \) and \( g_i : t_i^{-1} (e,1) \rightarrow G \) continuous maps
- Functoriality: \( f : H \rightarrow G \) group homomorphism induces a continuous map \( BH \rightarrow BG \)

\[ BG \overset{def}{=} E_G/G \quad \text{classifying space of group } G \]

Cohomology groups of \( G \) with coefficients in \( \mathbb{Z}[G] \)-module

\[ H^*(G,M) \overset{def}{=} H^*(BG,M) \quad \text{singular (or cellular) cohomology with local coefficients:} \]

The local coefficient system is derived from the given action of \( G = \pi_1(BG) \) on \( M \).

\[ = \text{ cohomology group of the cochain complex } \text{Hom}_{\mathbb{Z}[G]}(S^*,E_G,M) \].
A ring with trivial $G$-action \( \Rightarrow \) $\text{H}^*(G, R)$ has a natural graded multiplicative structure

A field with trivial $G$-action \( \Rightarrow \text{H}^*(G, F_p)$ has an action of the Steenrod algebra $\mathcal{A}_p$

If $R$ is a ring with trivial $G$-action, then $\text{res}^G_H$ is a map of graded $R$-algebras.

### Equivariant Cohomology

Given a finite group $G$, a $G$-space $X$, and a ring $R$ with trivial $G$-action, we define the equivariant cohomology of $X$ as follows:

\[ H^*_G(X, R) \overset{def}{=} H^*(EG \times_G X) \]

**Equivariant Cohomology**

**Borel construction fibration:**

\[ X \rightarrow EG \times X \quad \text{mod out} \quad \text{G-action} \rightarrow \quad EG \rightarrow \text{mod out} \quad \text{G-action} \rightarrow \quad EG/G = BG \]

- $H^*_G(pt, R) = H^*(BG; R) = H^*(G; R)$
- $H^*_G(X, R)$ is a module over $H^*_G(pt, R)$; the module structure is induced by projection $\pi : X \rightarrow pt; i.e.

\[ \Pi^* : H^*_G(pt, R) \rightarrow H^*_G(X, R) \]

- If $X$ is a free $G$-space $\Rightarrow X/G = EG \times_G X \Rightarrow H^*_G(X; R) \cong H^*(X/G; R)$

- **Serre spectral sequence of Borel construction fibration**

\[
\begin{align*}
E_2^{p,q} & = H^p(BG, H^q(X; R)) \\
& \cong H^p(G, H^q(X; R)) \\
\text{if } G \text{ is finite } & \Rightarrow H^*_G(X, R)
\end{align*}
\]

Converging to the appropriate filtered graded group $Gr(H^*_G(X, R))$ associated with $H^*_G(X, R)$. [100]
- If \( f: X \to Y \) is a \( G \)-map, then there is a morphism of fibrations

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow \mathbb{E}_G \times_G X & & \downarrow \mathbb{E}_G \times_G Y \\
\mathbb{B}G & \xrightarrow{id} & \mathbb{B}G
\end{array}
\]

which induces a morphism of equivariant cohomologies associated Serre spectral sequences.

- \( K \leq G, X \) a \( G \)-space, there is a morphism of fibrations

\[
\begin{array}{ccc}
X & \xrightarrow{f} & X \\
\downarrow \mathbb{E}_G \times_G X & & \downarrow \mathbb{E}_G \times_G X \\
\mathbb{B}G = \mathbb{E}_G / G & \xrightarrow{Bf} & \mathbb{B}K = \mathbb{E}_G / K
\end{array}
\]

\[\text{Lyndon-Hochschild-Serre spectral sequence } H \circ G, G/H \text{ acts freely on } (EG)/H = BH\]

and associated Borel construction is

\[
(EG)/H \xrightarrow{\mathbb{B}H} \mathbb{E}(G/H) \times_{G/H} (EG)/H \cong (EG/H)/G/H = BG
\]

The Serre spectral sequence of this Borel construction fibration is Lyndon-Hochschild-Serre spectral sequence and \( E_2 \)-term is given by

\[
E_{2}^{pq} = H^{p}(G/H, H^{q}(H, M)) \Rightarrow H^{p+q}(G, M)
\]

Local coefficients determined by the action of \( G/H \) on \( H^{q}(H, M) \).

Theorem \( \text{Let } G = (S^1)^n \) or \( G = \mathbb{Z}_p^n \), and \( X \) be a compact \( G \)-space \( p \) is a prime

\[X^G \neq \emptyset \iff \prod_{p: H^n_p(pt, R)} \to H^n_G(X, R) \text{ is monomorphism} \]

set of fixed points where \( R = \begin{cases} \mathbb{Q} & \text{for } G = (S^1)^n \\ \mathbb{F}_p & \text{for } G = (\mathbb{Z}_p)^n \end{cases} \)
Proof of topological Freudenthal theorem for a prime power: We prove that there is no $\mathbb{Z}_p^n$-map

$$\Omega = [k]^{*(N + 1)} \rightarrow S^{N-1}$$

Assume that $f: [k]^{*(N + 1)} \rightarrow S^{N-1}$ is a $\mathbb{Z}_p^n$-map. Then there is an induced map in the equivariant cohomology and between Serre spectral sequences of Borel constructions:

$$f^*: H^*_{\mathbb{Z}_p^n}(S^{N-1}; F_p) \rightarrow H^*_{\mathbb{Z}_p^n}([k]^{*(N + 1)}; F_p)$$

$$E_{l, 0}^*(f): (E')_{l,*} \rightarrow E_{l,*}^*$$

Here

$$E_{l,*}^* = H^*(\mathbb{Z}_p^n; H^*([k]^{*(N + 1)}; F_p))$$
$$E_{l,*} = H^*(\mathbb{Z}_p^n; H^*(S^{N-1}; F_p))$$

and $E_{l,*}^*; H^*(\mathbb{Z}_p^n; F_p) \rightarrow H^*(\mathbb{Z}_p^n; F_p)$ is identity map!

\[ Z_p^n \text{ acts trivially on } H^*(S^{N-1}; F_p) \]

\[ (E')_{l,*} = H^*(\mathbb{Z}_p^n; F_p) \otimes H^*(S^{N-1}; F_p) \]

$\mathbb{Z}_p^n$ acts fixed point free on $S^{N-1}$

$$\text{Th. } \Rightarrow \tilde{H}^*: H^*_p(pt; F_p) \rightarrow H^*_p(S^{N-1}; F_p) \text{ is not a monomorphism}$$

$$\Rightarrow \text{Some } s_*(E')_* \rightarrow (E')_* \text{ is not zero}$$

$$\Rightarrow d_N: (E')_N \rightarrow (E')_N \text{ is not zero}$$

$$\Rightarrow d_N: (E')_N^{(N-1)} \rightarrow (E')_N^{(N-1)}$$

$$\Rightarrow \text{For } 1 \in (E')_N^{(N-1)}, d_N(1) = \infty \neq 0$$

Thus, $E_{N, 0}^*(f)(x) = \infty \neq 0$ and $E_{N, 0}^{(N)}(f)(0) = x \neq 0$. Contradiction!
Colored Tverberg problems

Báróny, Füredi, Lovász: On number of halving planes, 1990
Báróny, Karman: A colored version of Tverberg's theorem, 1992

Colored Tverberg problem: Determine the least number \(N(k,d)\) such that, for every collection
\[X = C_0 \cup \ldots \cup C_d \subset \mathbb{R}^d\]
with \(|X| > N(k,d)\) and \(|C_i| > k\), there exist \(r\) disjoint
subcollections \(F_1, \ldots, F_r\) of \(X\) satisfying:
\[|F_i \cap C_j| = 1\] for every \(i \in \{1, \ldots, k\}\), \(j \in \{0, \ldots, d\}\) and \(\bigcap_{i=1}^{k} \text{conv } F_i \neq \emptyset\).

- \(N(k,1) = 2k\)
- \(N(k,2) = 3k\)
- \(N(k,3) = 2(k+1)\) \(\quad\) \(\text{Lovász}\)

Báróny--Karman conjecture \(N(k,d) = k(d+k)\)

Vrečica, Zivaljević: The colored Tverberg's problem of injective functions, 1992
Colored Tverberg problem*: Determine the least number \(t(d,k)\) such that, for every
collection \(X = C_0 \cup \ldots \cup C_d \subset \mathbb{R}^d\) with \(|C_i| > t(d,k)\), there are \(r\) disjoint subcollections
\(F_1, \ldots, F_r\) of \(X\) satisfying
\[|F_i \cap C_j| = 1\] for every \(i \in \{1, \ldots, k\}\), \(j \in \{0, \ldots, d\}\) and \(\bigcap_{i=1}^{k} \text{conv } F_i \neq \emptyset\).

- for \(k\) prime \(t(d,k) \leq 2k-1\) \(\quad\) Vrečica, Zivaljević 1992
- for any \(k\) \(t(d,k) \leq 4k-3\)
- for \(k\) prime power \(t(d,k) \leq 2k-1\) \(\quad\) Zivaljević 1998

Blagojević, Matschke, Ziegler: Optimal bounds for a colorful Tverberg--Vrečica type problem

Optimal colorful Tverberg's theorem: Let \(k > 1\) and \(N = (dk)(k-1)\). Let \(\Delta^n\)
be a \(N\)-dimensional simplex with a partition of the vertices into parts ("colored classes")
\(C = C_0 \cup \ldots \cup C_d\) with \(|C_i| > k-1\) for all \(i\). Then for every continuous map \(f: \Delta^n \to \mathbb{R}^d\) there
are \(k\) disjoint facets \(F_1, \ldots, F_k\) of \(\Delta^n\) such that
\[|F_i \cap C_j| = 1\] for every \(i \in \{1, \ldots, k\}\), \(j \in \{0, \ldots, m\}\) and \(\bigcap_{i=1}^{k} f(F_i) \neq \emptyset\).

Remarks: \(|C_i| < k-1 \Rightarrow\) there are at least \(d+1\) non-empty colored classes
- Theorem is tight; counter example \(|C_0| = k, |C_1| = \ldots = |C_m|\)
Corollary [ Bárány - Larman Conjecture ] If \( k+1 \) is a prime, then
\[
e(d, k) = \#(d, k) = k
\]

Corollary [ Optimal Bound ] For all \( d \geq 1 \) and \( r \geq 2 \)
\[
r \leq \#(d, r) \leq \#(d, r) \leq 2r-2
\]

Corollary [ Optimal Bounds in Computational Geometry ] New bounds for:
- The number of hyperplanes that bisect the set \( S \) and are spanned by the elements of the set \( S \)
- The constant \( c_d \) in the second selection lemma
- The number of halving facets of an \( n \) element set \( X \) \( \leq 8R^d \)