Fonctions à trappe avec perte d'information et Applications (d'après Peikert, Waters . . .)

(Lossy Trapdoor Functions and Applications - following Peikert, Waters, . . .)

Damien Vergnaud

École normale supérieure – C.N.R.S. – I.N.R.I.A.

8 avril 2011

Lossy Trapdoor Functions and Their Applications
C. Peikert, B. Waters - STOC 2008, 187-196
(Full version to appear in SIAM Journal on Computing)

- new general *primitive*
- realizations under various computational assumptions (DDH, LWE, QR, Paillier ...)
- used in more than 50 subsequent papers (Crypto'08, TCC'09, Eurocrypt'09, TCC'10, Eurocrypt'10, Crypto'10, ...)

  $\rightsquigarrow$ trapdoor functions, collision-resistant hash functions, oblivious transfer, encryption scheme with strong security properties, ...

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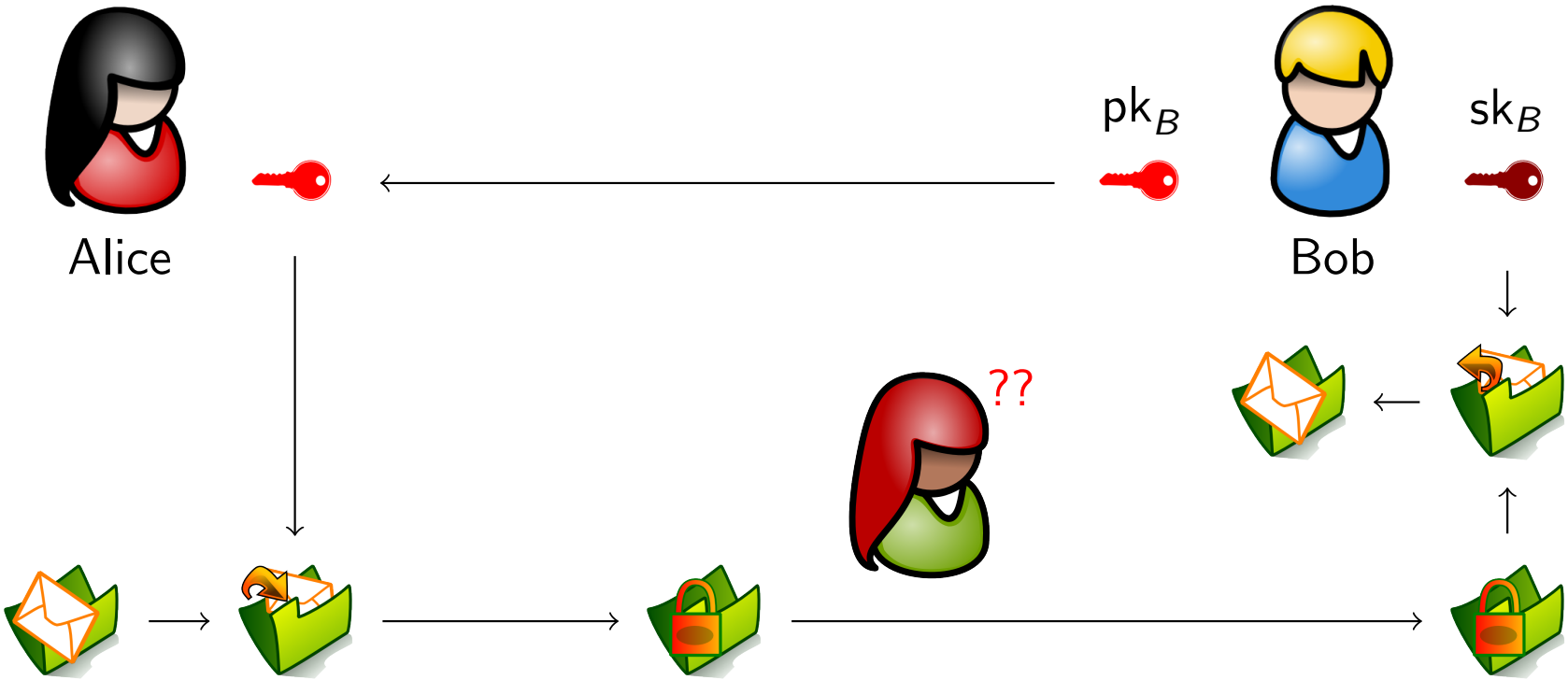
Contents

Public Key Cryptosystems

Asymmetric encryption: Bob owns two "keys"

- a public key — known by everybody (including Alice)
- a secret key — known by Bob only



Public-Key Encryption: Security Notions

Encryption is supposed to provide confidentiality of the data.

But what exactly does this mean?

Security goal	But . . .
Recovery of secret key is infeasible	

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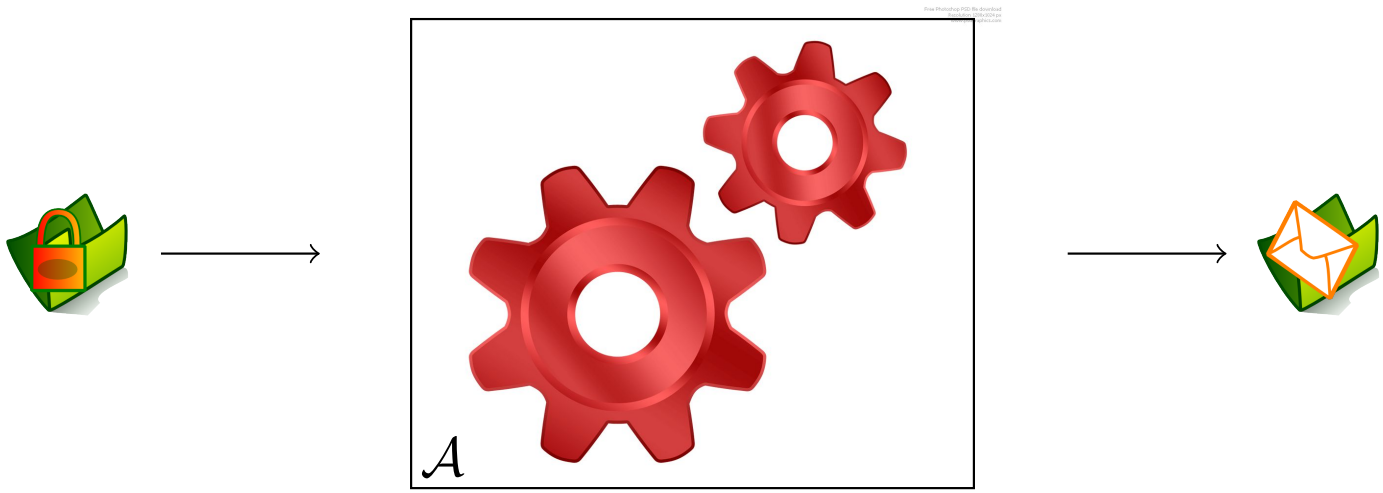
So what is a **secure** encryption scheme ?

Not an easy question to answer . . .

The Methodology of "Provable Security"

1. Define goal of adversary
2. Define security model
3. Define complexity assumptions
4. Provide a proof by reduction
5. Interpret proof

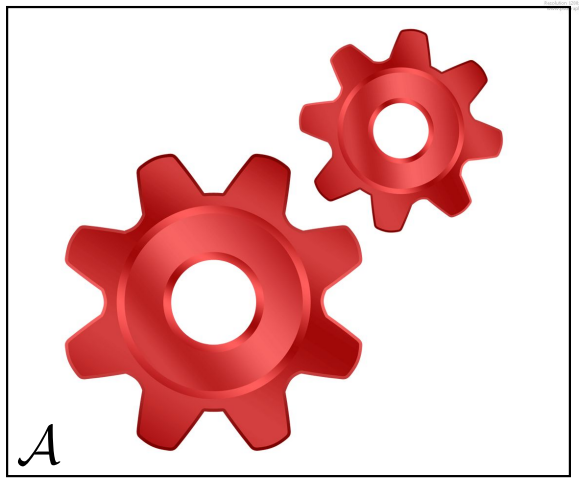
Proof by reduction



$\mathcal{A}$ adversary against e.g. one-wayness

Proof by reduction

Instance $\mathcal{I}$ of a problem $\mathcal{P}$



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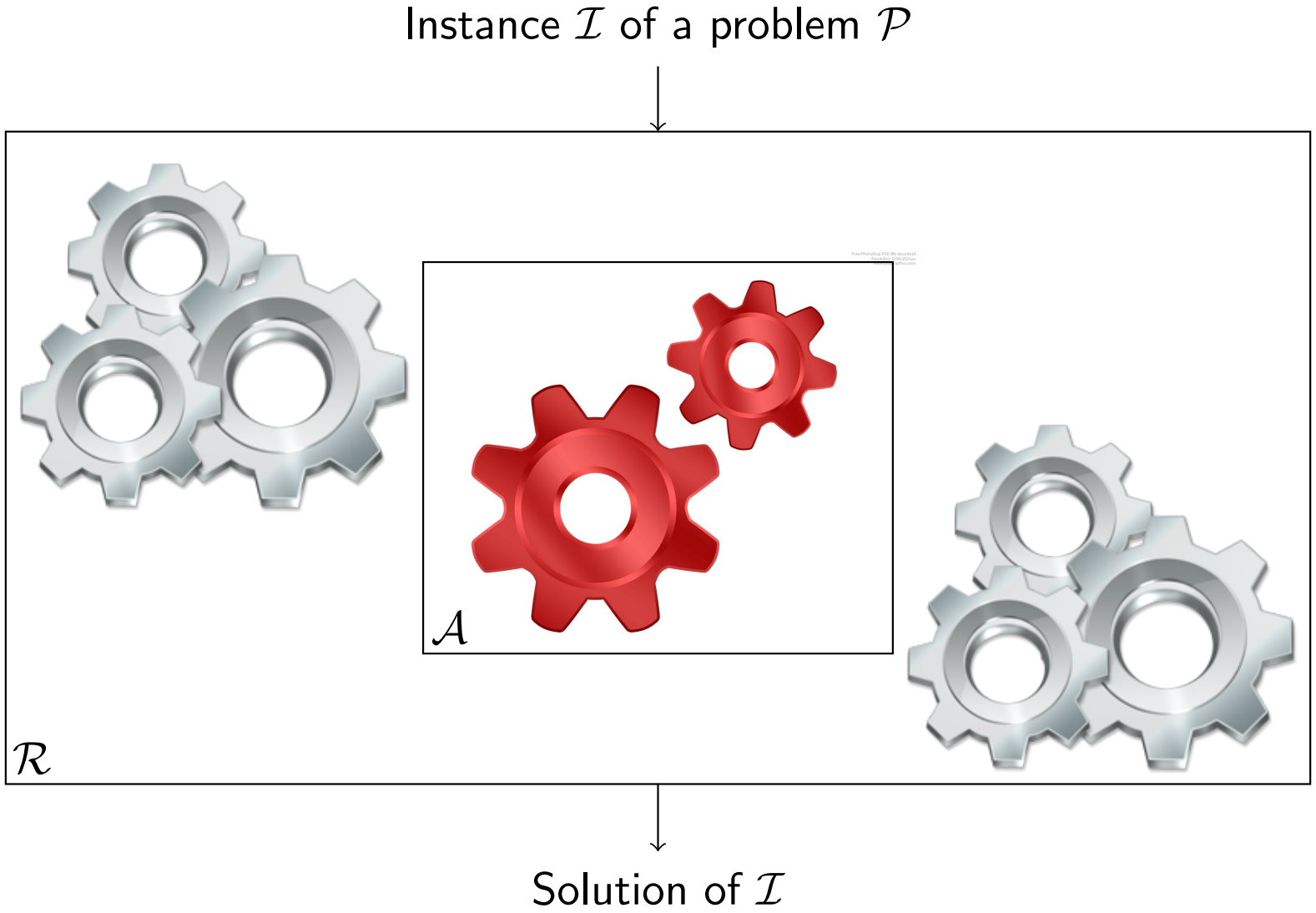

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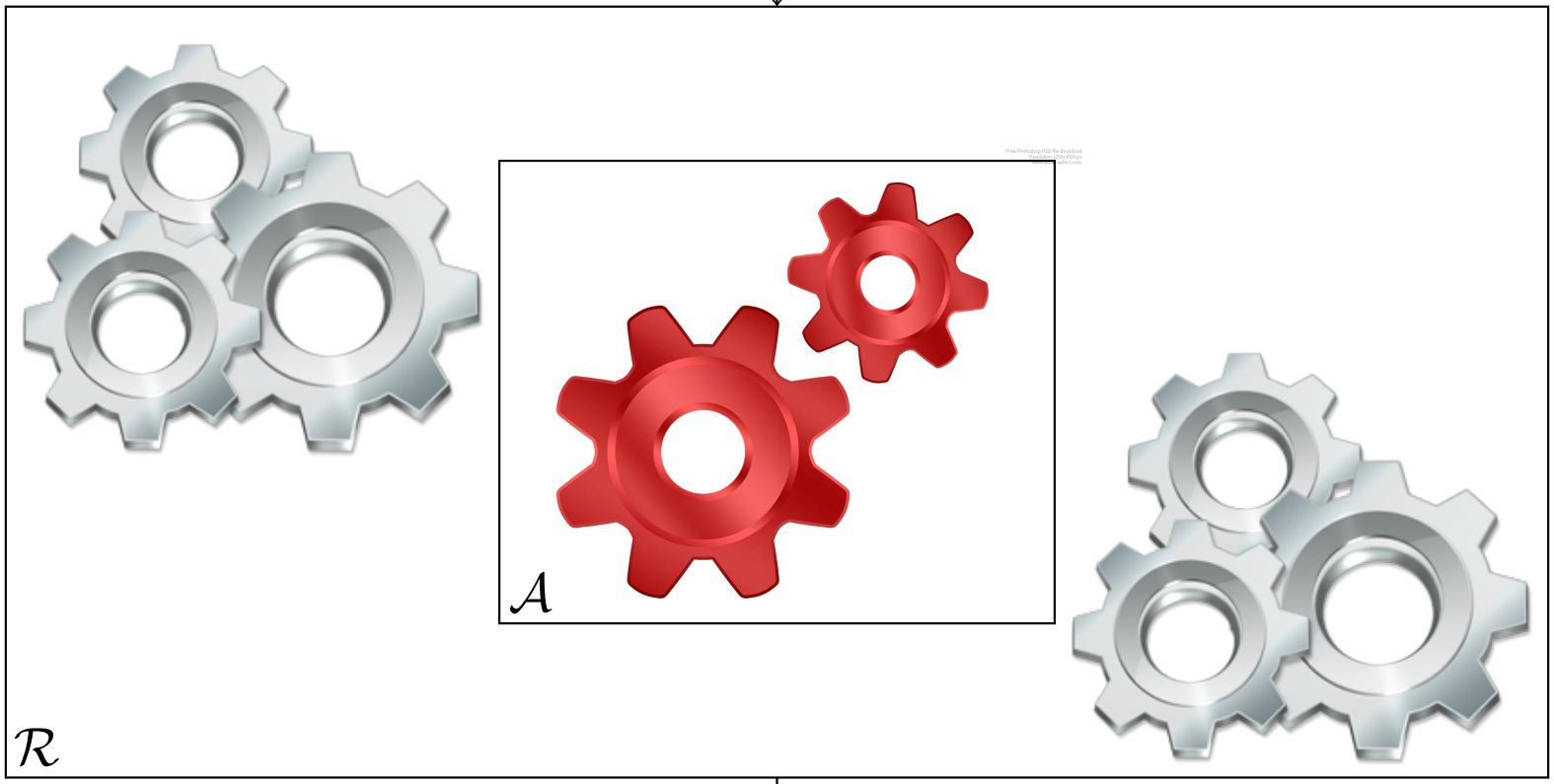

Solution of $\mathcal{I}$

$\mathcal{P}$ intractable $\rightarrow$ contradiction

Public-Key Encryption: Definition

(KeyGen, Encrypt, Decrypt) where:

- KeyGen: $\kappa \longrightarrow (\mathsf{sk}, \mathsf{pk})$
- Encrypt: $\mathsf{pk}, m, \varpi \longrightarrow c$,
- Decrypt: $\mathsf{sk}, c \longrightarrow m$ or $\perp$.

$$(\mathsf{sk}, \mathsf{pk}) \leftarrow \mathsf{KeyGen}(\kappa) \Rightarrow \mathsf{Decrypt}_{\mathsf{sk}}\left(\mathsf{Encrypt}_{\mathsf{pk}}(m, \varpi)\right) = m$$

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History of Security Goals

- infeasible to compute sk from pk
  $\rightsquigarrow$ Unbreakability. (Implicitely appeared with public-key crypto)

- infeasible to invert Encrypt(pk, $\cdot$)
  $\rightsquigarrow$ One-wayness. (Diffie and Hellman, late 70's)

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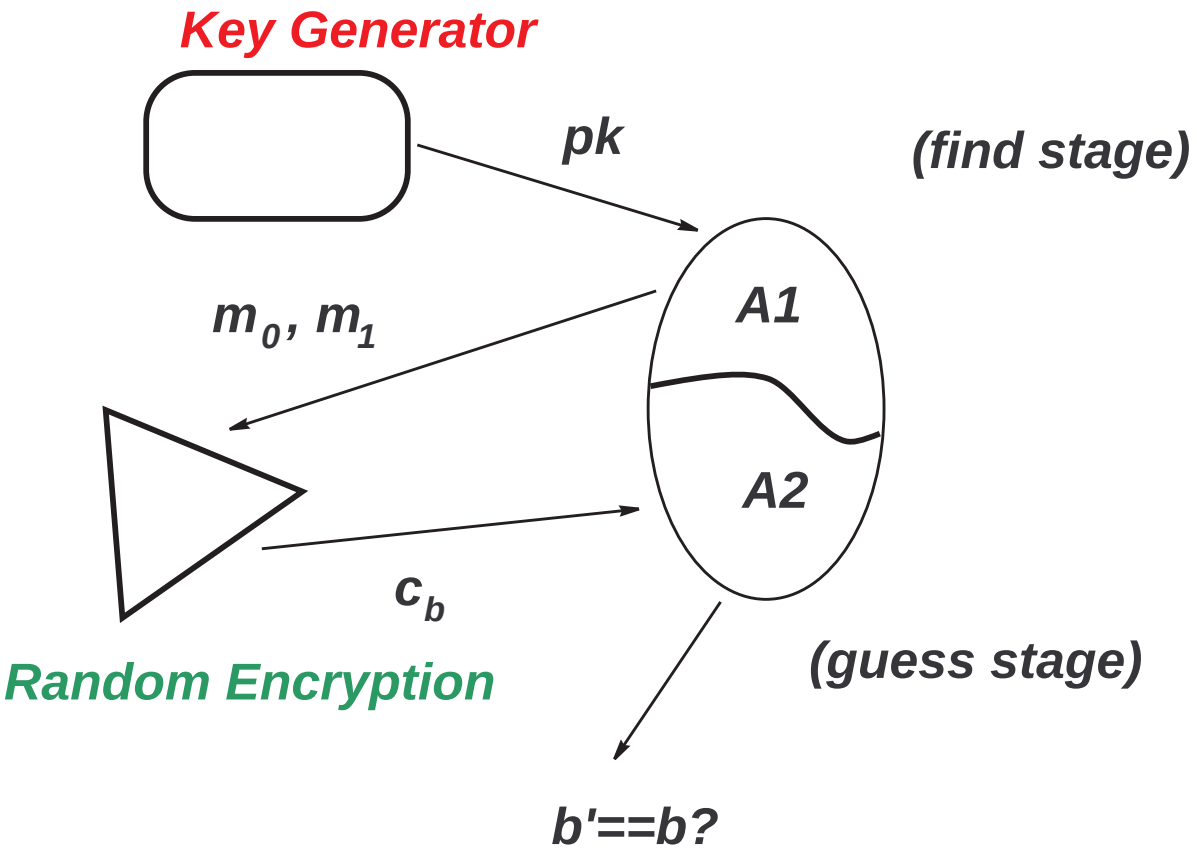
Indistinguishability of encryptions (IND)



History of Adversarial Models

Several types of computational resources an adversary has access to have been considered:

- **chosen-plaintext attacks** (**CPA**), unavoidable scenario.
- **non-adaptive chosen-ciphertext attacks** (**CCA1**),
  the adversary gets access to a decryption oracle before being given the challenge ciphertext. (Naor and Yung, 1990).
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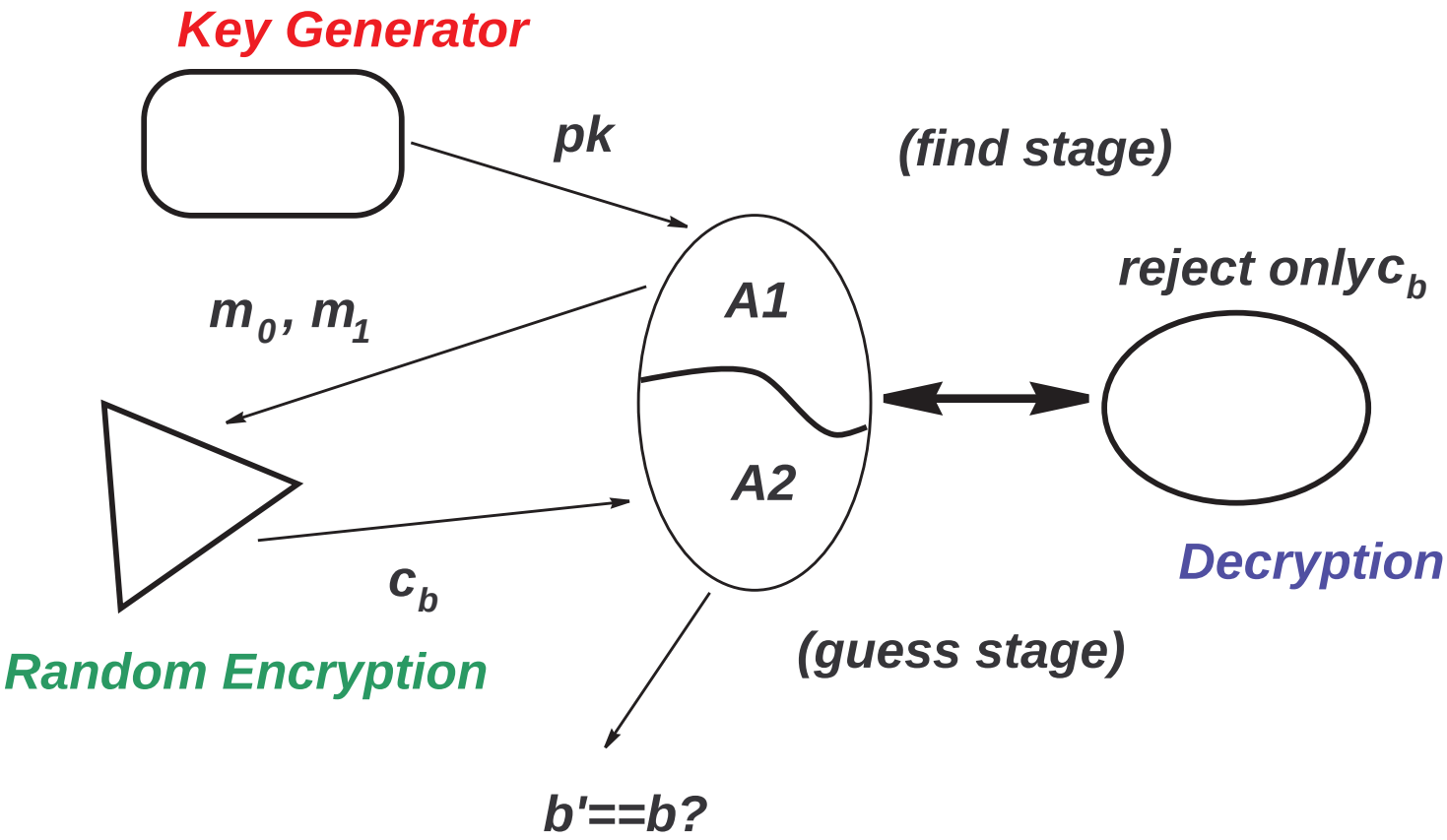
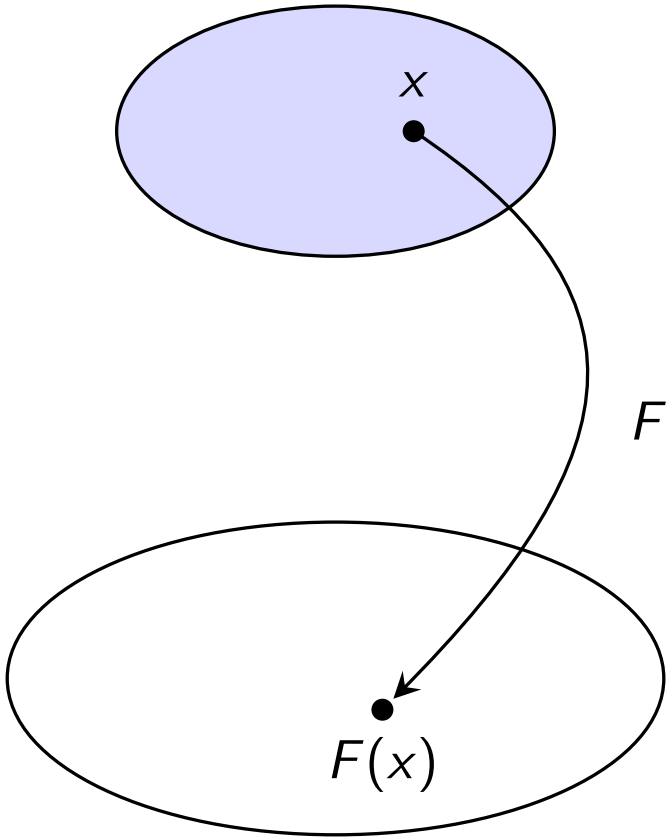
IND-CCA: Playing the Game



Trapdoor One-Way Functions



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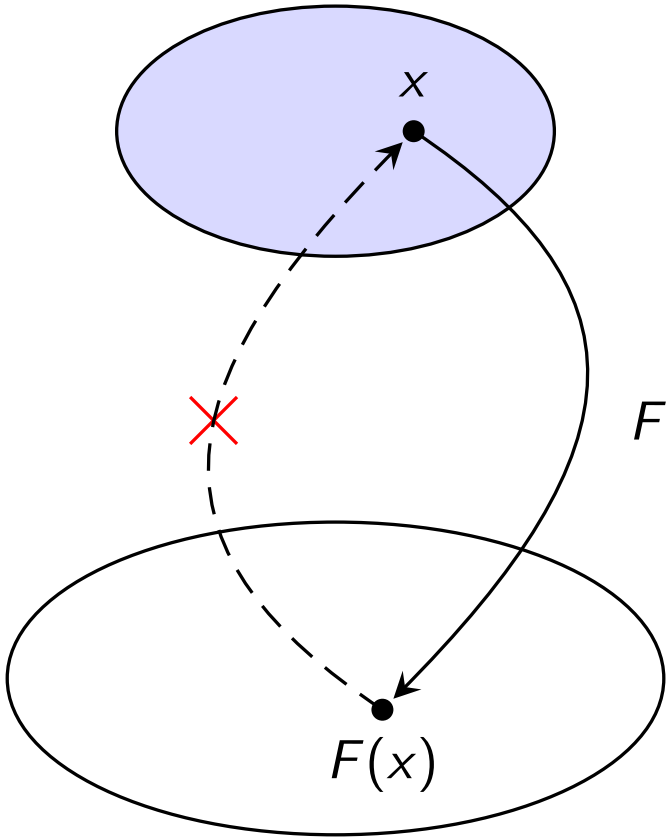

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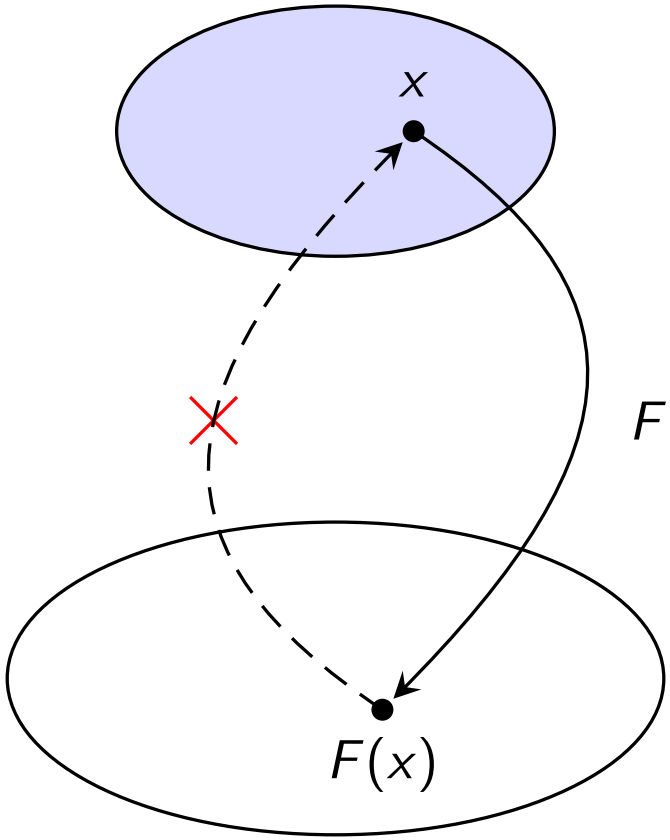

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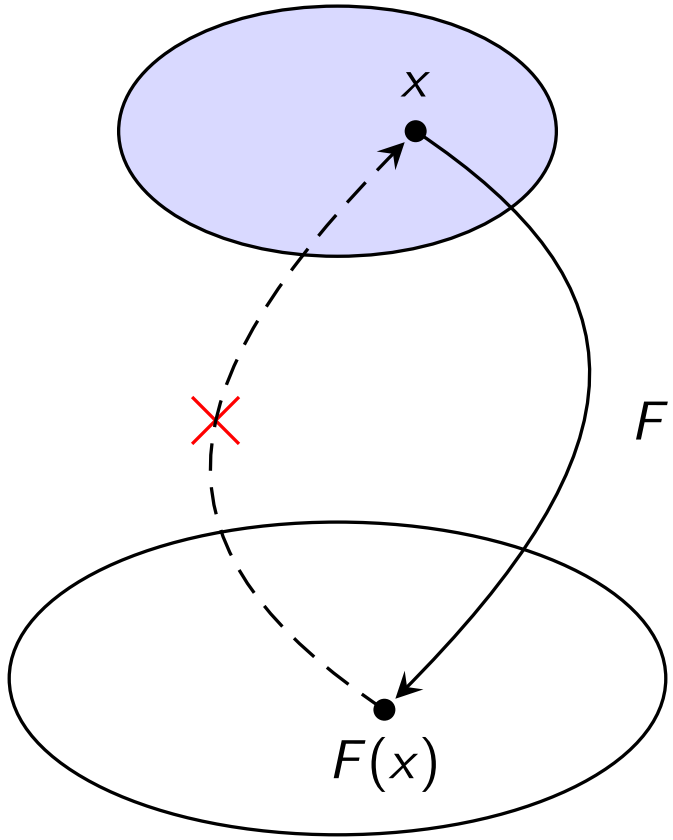

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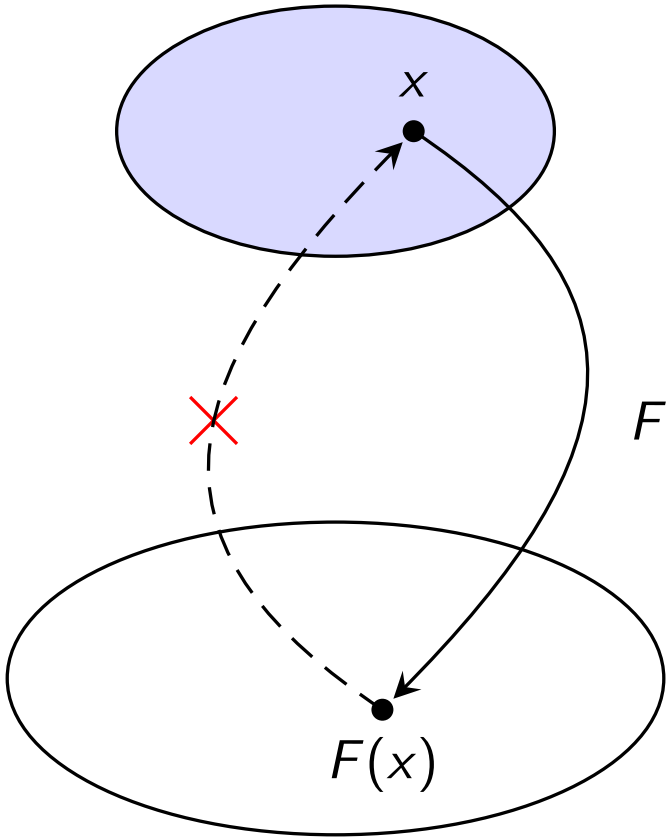

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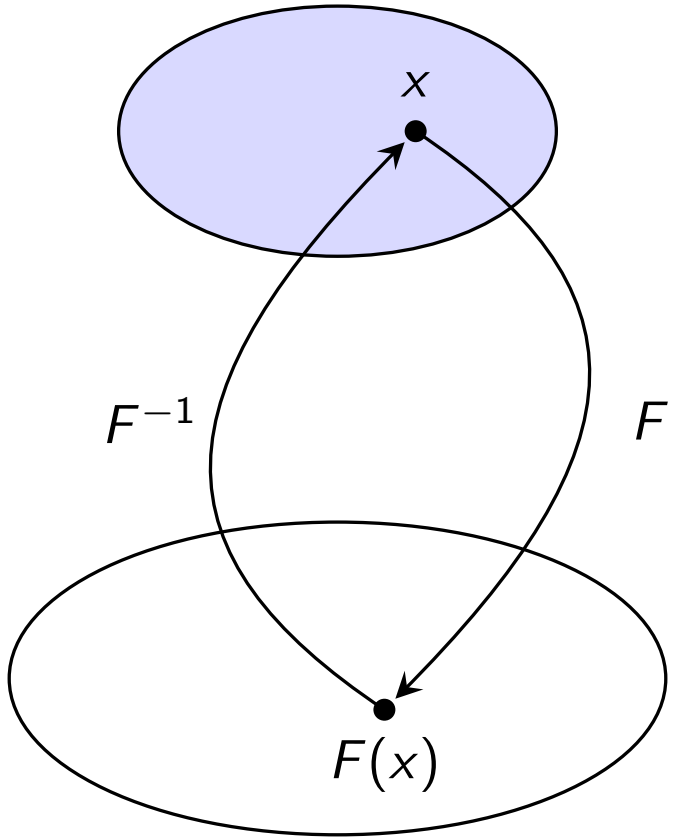

Trapdoor One-Way Function

Extracting Randomness

F = (trapdoor) one-way function

- **Hard-core bit** $B : \mathrm{Dom}_F \longrightarrow \{0,1\}$ of F:

$$x \longmapsto B(x) \qquad \text{easy}$$
$$F(x) \longmapsto B(x) \qquad \text{difficult !}$$

  - F = RSA, LSB $\rightsquigarrow$ hard-core bit
  - $G : (x, r) \longmapsto (F(x), r)$ with $|x| = |r|$
    $\rightsquigarrow B = (x, r) \longmapsto \langle F(x), r \rangle$ hard-core bit for G. (Goldreich-Levin 1989)

- **Randomness extractor** (Universal pairwise independent hash function)

DO NOT READ !

X, Y random variables in $\{0,1\}^n$ with $H_\infty(X|Y) \geq k$
$\mathcal{H} : \{0,1\}^n \longrightarrow \{0,1\}^\ell$ s.t.

$$\Delta\left[(Y, \mathcal{H}(X)), (Y, U_\ell)\right] \leq \epsilon \text{ for } \ell \leq k - 2\log(1/\epsilon)$$

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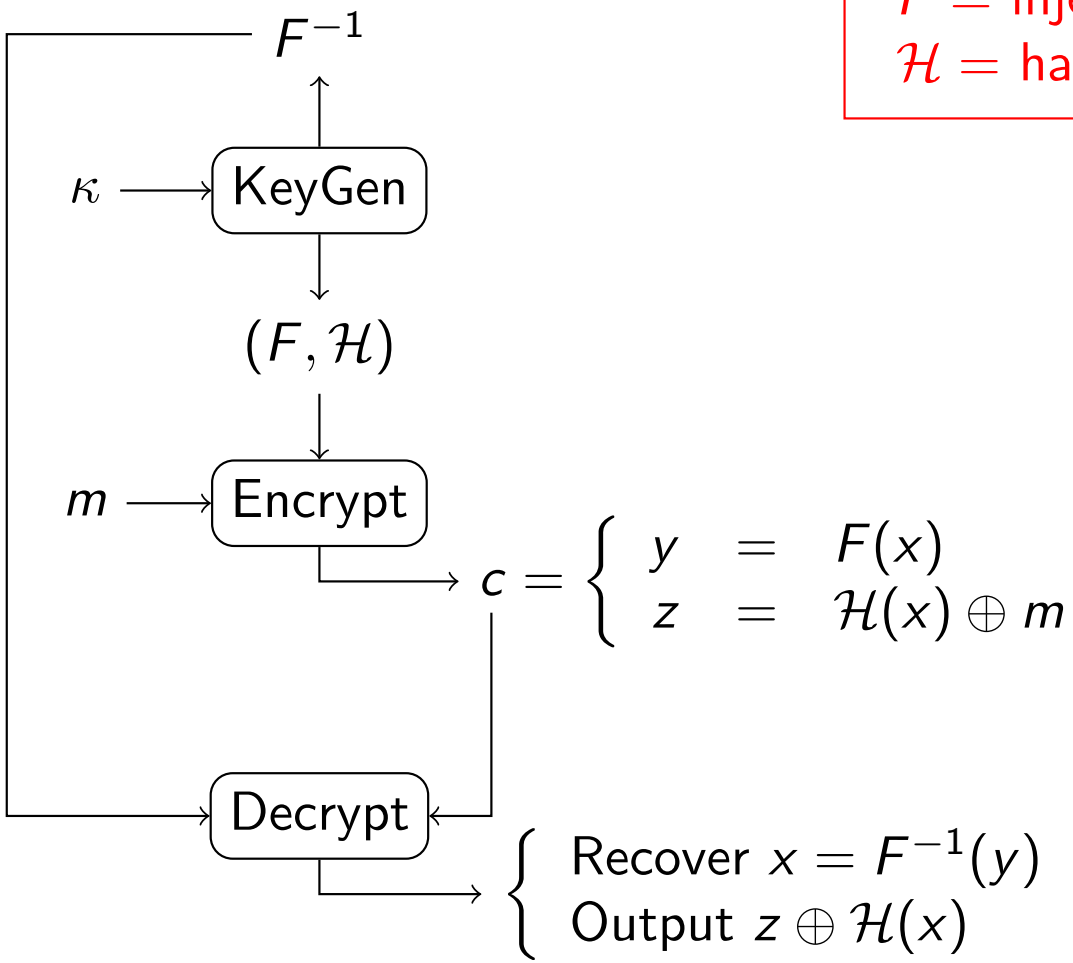
IND-CPA Encryption from TDF

F = injective trapdoor function
$\mathcal{H}$ = hard-core bit for F



How to achieve IND-CCA Security ?

Verify ciphertext is "well-formed"

- **Basic idea:**
  - Add redundancy to the ciphertext
  - Decryption is only performed for ciphertexts with the right redundancy

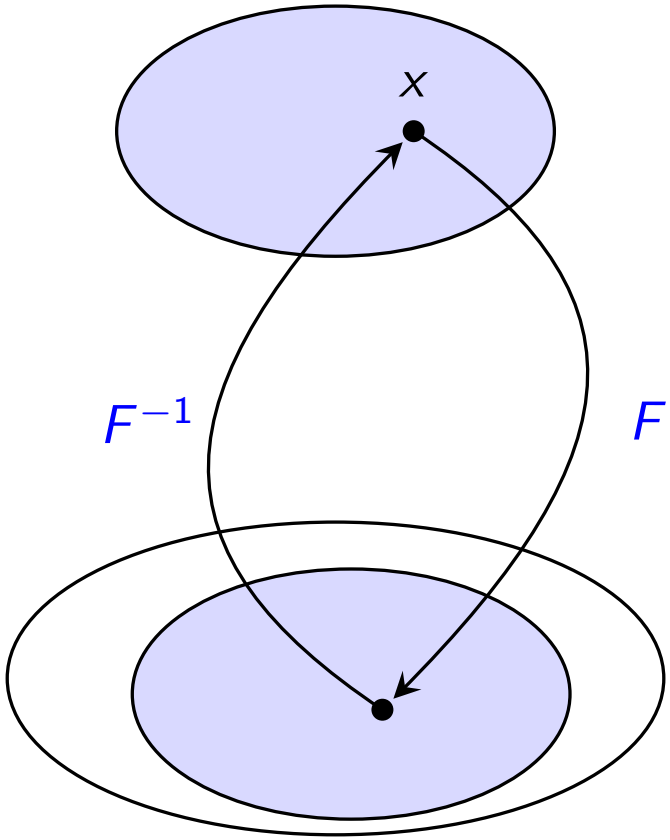
  $\rightsquigarrow \mathcal{A}$ effectively knows the plaintext anyway, the decryption oracle is useless!

- Usually via *zero-knowledge* proof ($\rightsquigarrow$ inefficient)

- **New idea:** recover randomness

Contents

Lossy Trapdoor One-Way Functions



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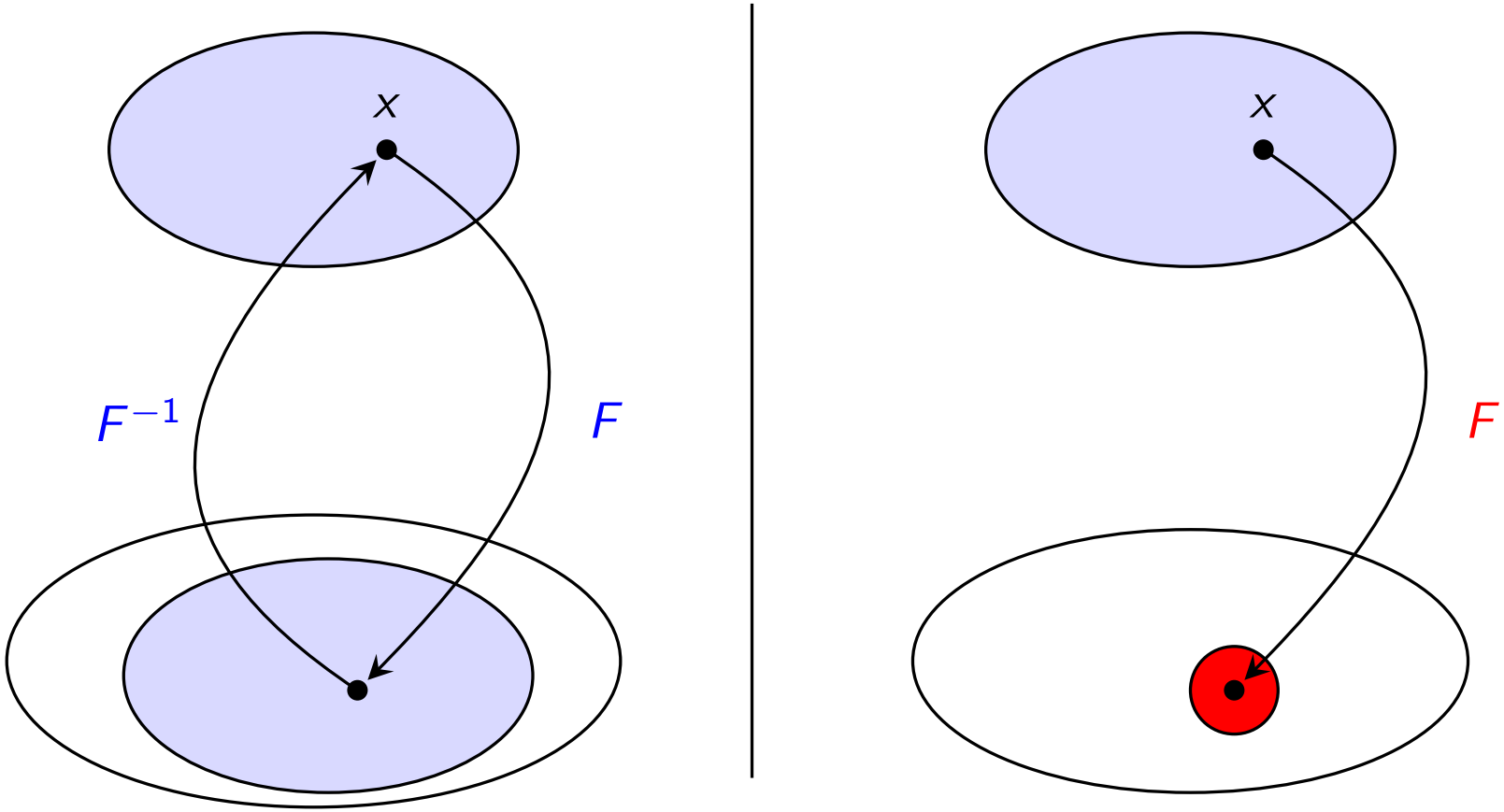

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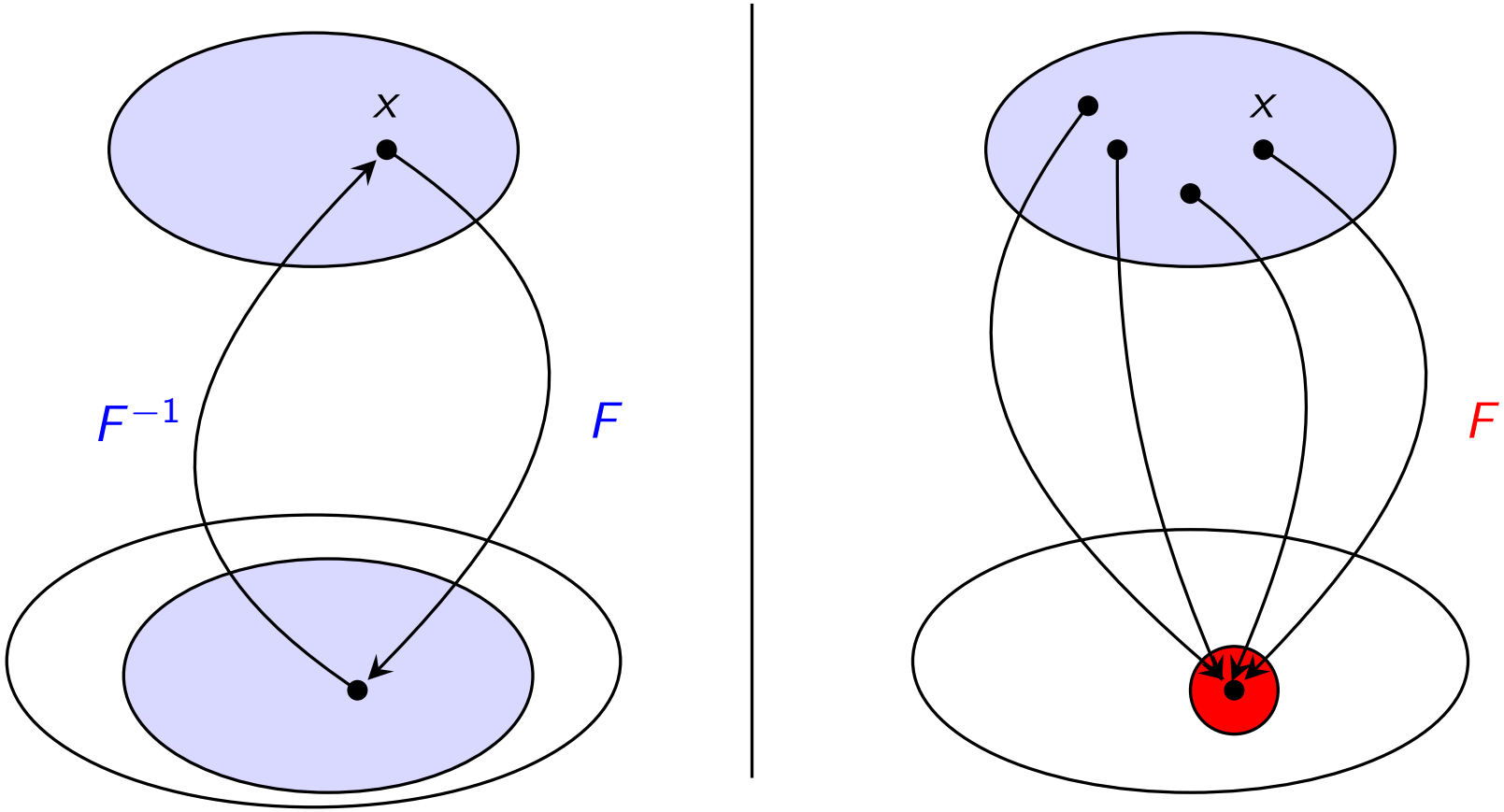

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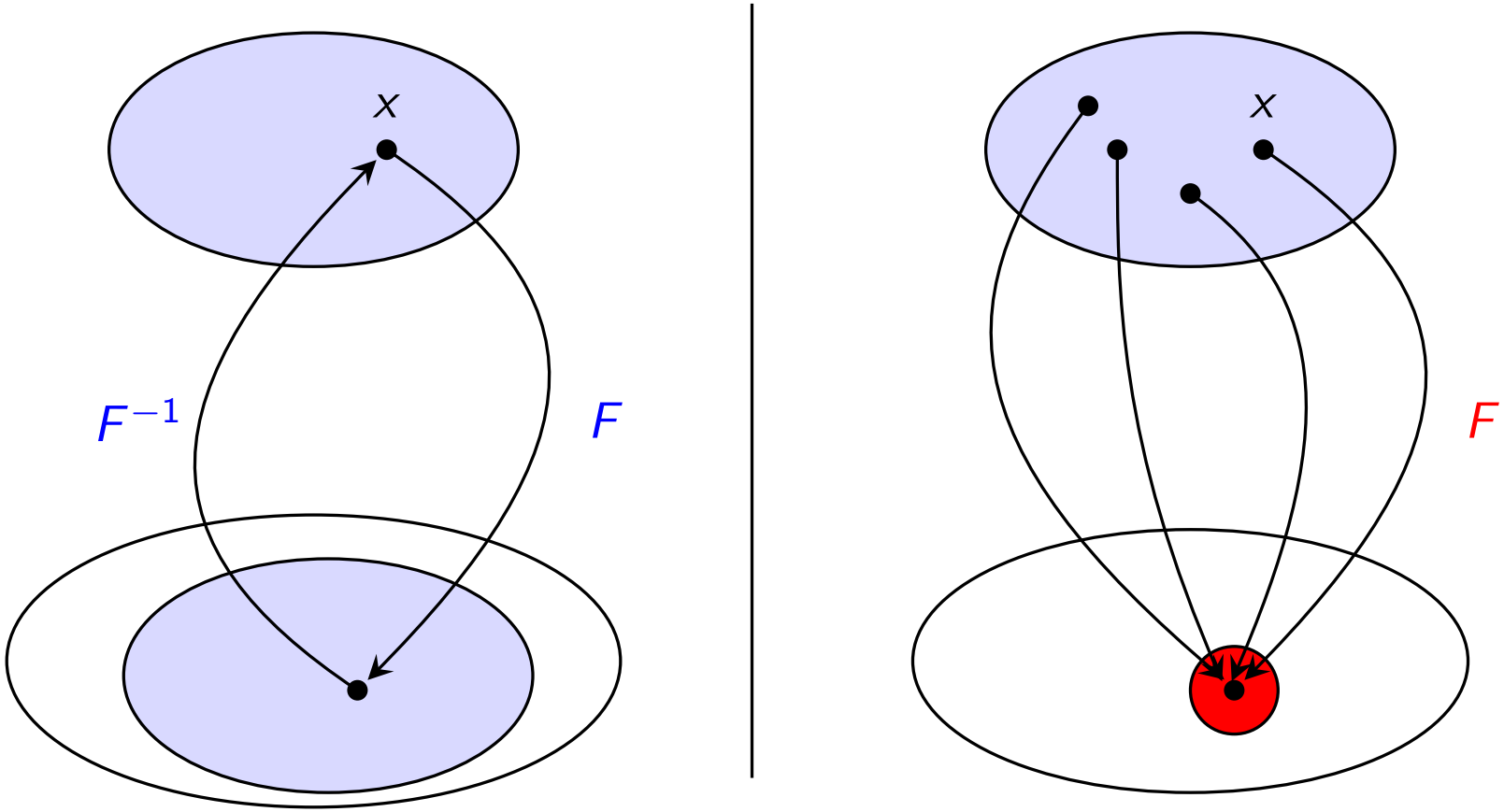

$$F \simeq_c F$$

Lossy Trapdoor One-Way Function

Lossy TDF: Definition

A Lossy TDF with lossiness k is a tuple $(S_{\text{inj}}, F_{\text{inj}}, F_{\text{inj}}^{-1}, S_{\text{loss}}, F_{\text{loss}})$ where:

- **Easy to sample an injective trapdoor function:**
  - $S_{\text{inj}}(\kappa) \longmapsto (n, s, t)$
  - $F_{\text{inj}}(s) : \{0,1\}^n \longrightarrow \{0,1\}^n, x \longmapsto F_{\text{inj}}(s, x)$
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  $$s \leftarrow S_{\text{inj}}(\kappa) \simeq_c s \leftarrow S_{\text{loss}}(\kappa)$$

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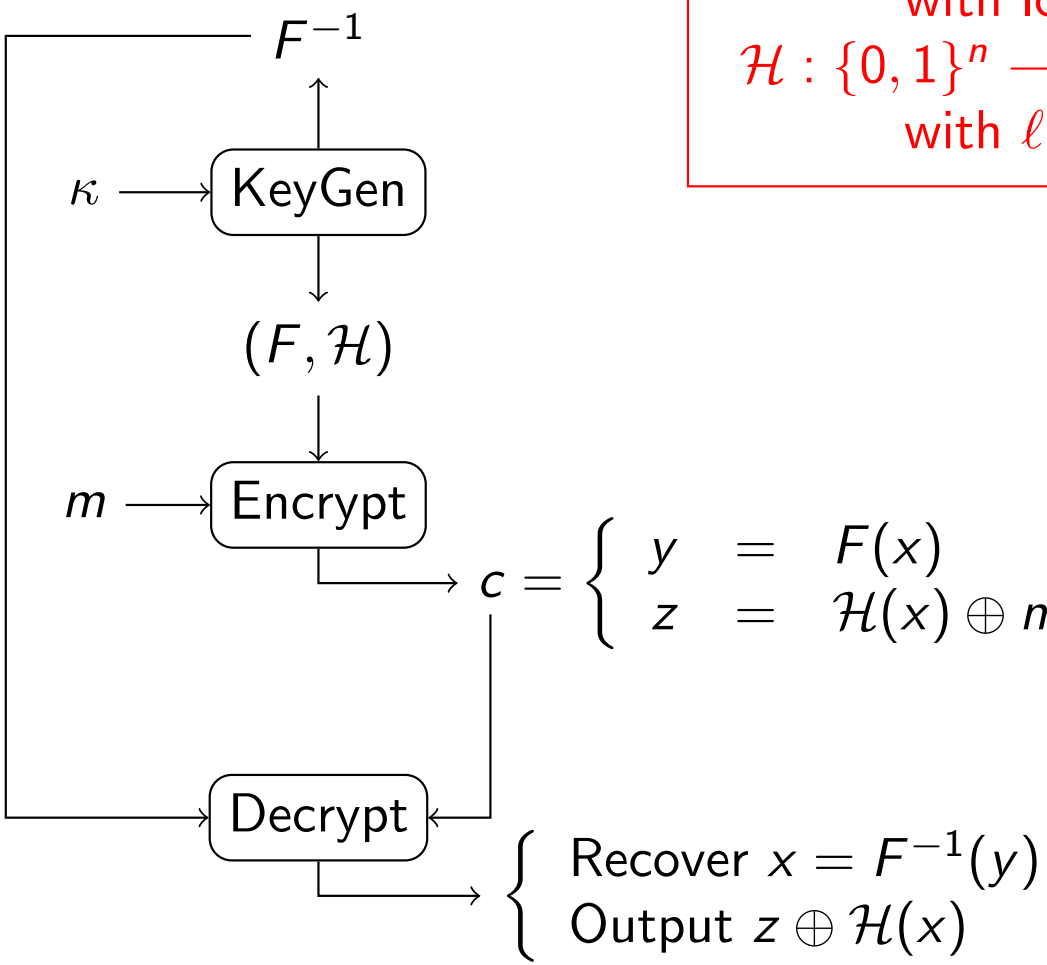
IND-CPA Encryption from Lossy TDF

$F : \{0,1\}^n \longrightarrow \{0,1\}^n$ lossy TDF
with lossiness k
$\mathcal{H} : \{0,1\}^n \longrightarrow \{0,1\}^\ell$ randomness extractor
with $\ell < k$



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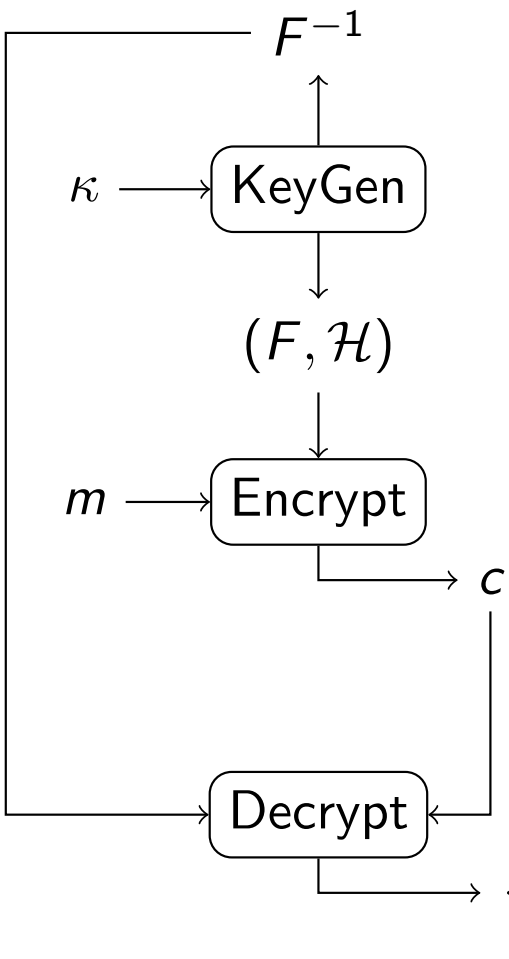

$$c = \begin{cases} y & = & F(x) \\ z & = & \mathcal{H}(x) \oplus m \end{cases}$$

$$\begin{cases} \text{Recover } x = F^{-1}(y) \\ \text{Output } z \oplus \mathcal{H}(x) \end{cases}$$

$\rightsquigarrow$ Oblivious transfer

All-But-One TDF

- $G(b, x)$ has an extra parameter: branch $b \in \{0,1\}^n$
- Generate (G, G^{-1}) with hidden lossy branch ℓ
- Lossy TDF $\Longleftrightarrow$ All-But-One TDF

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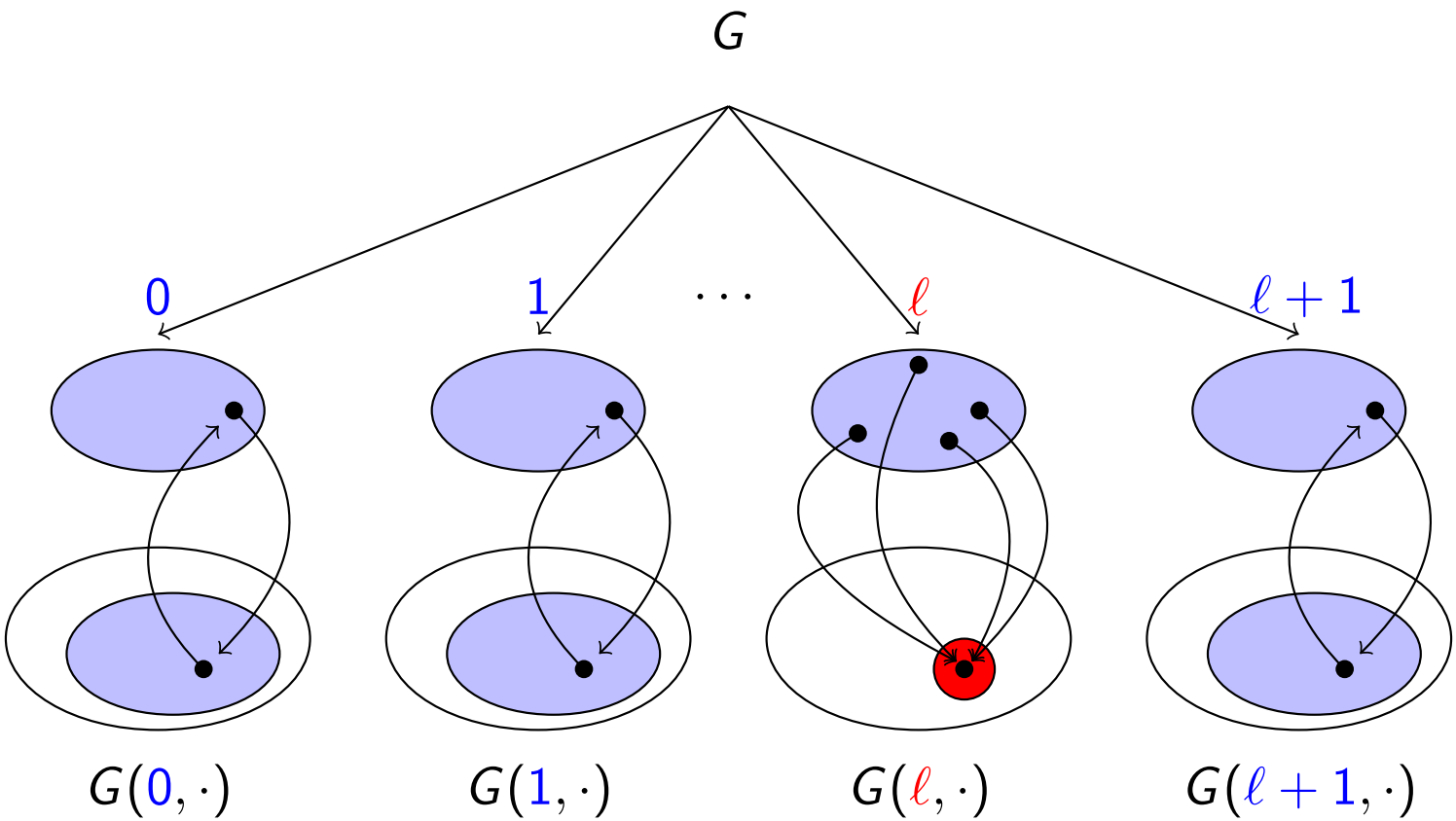

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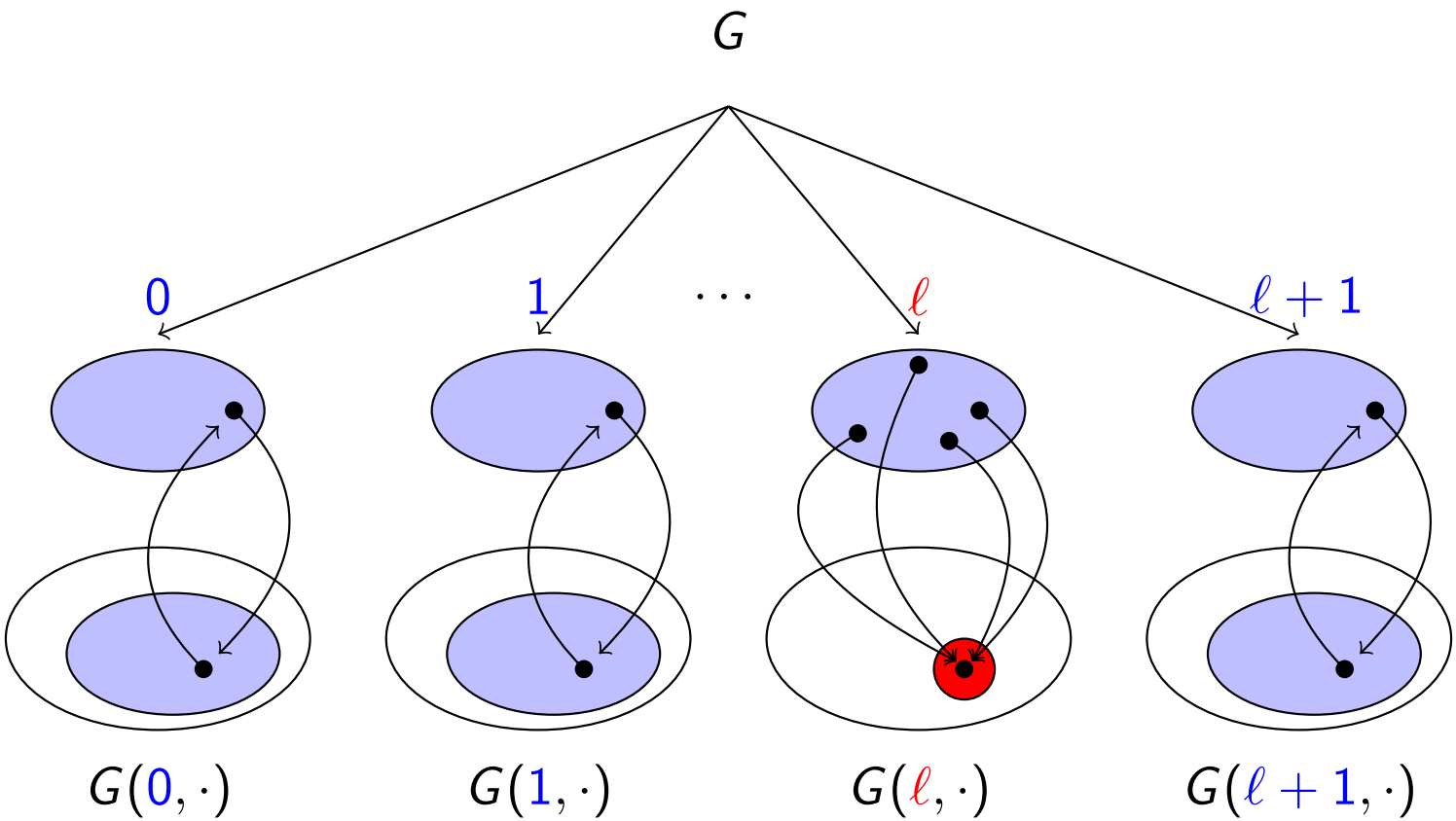
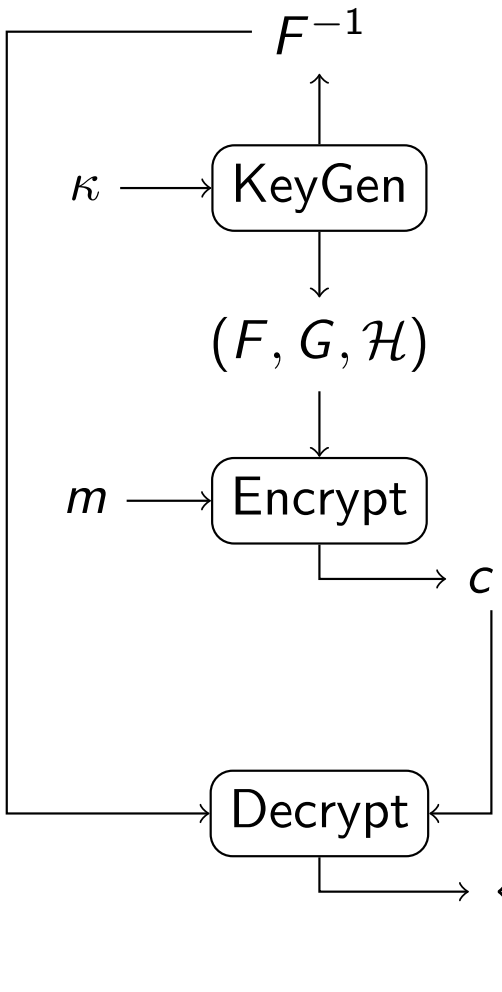


- Lossy TDF $\iff$ All-But-One TDF

IND-CCA Encryption from LTDF: Construction

F = lossy TDF (lossiness k)
G = all-but-one TDF (lossiness k')
$\mathcal{H} : \{0,1\}^n \longrightarrow \{0,1\}^\ell$ randomness extractor for F
$(\ell < k + k')$



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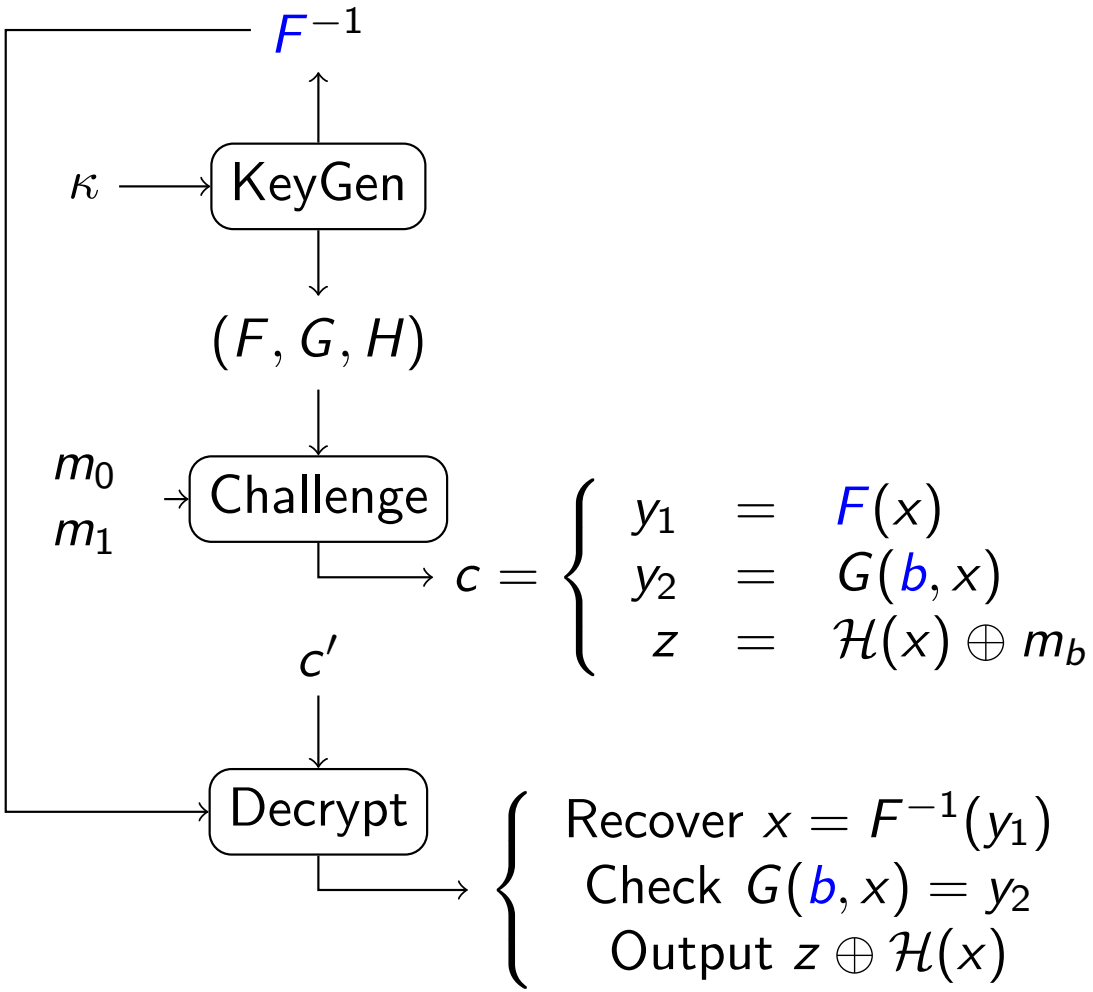
$$\begin{cases} \text{Recover } x = F^{-1}(y_1) \\ \text{Check } G(b, x) = y_2 \\ \text{Output } z \oplus \mathcal{H}(x) \end{cases}$$

IND-CCA Encryption from LTDF: Proof



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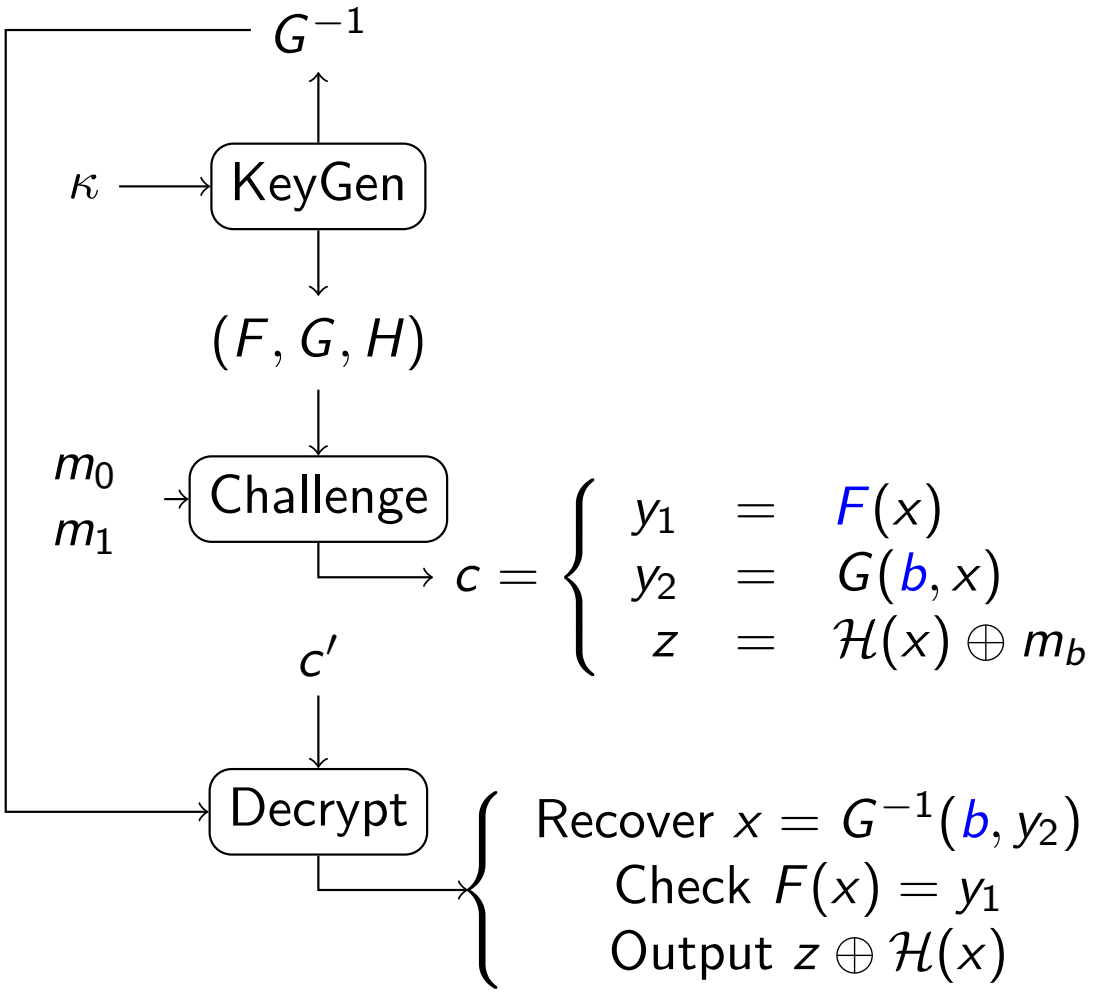

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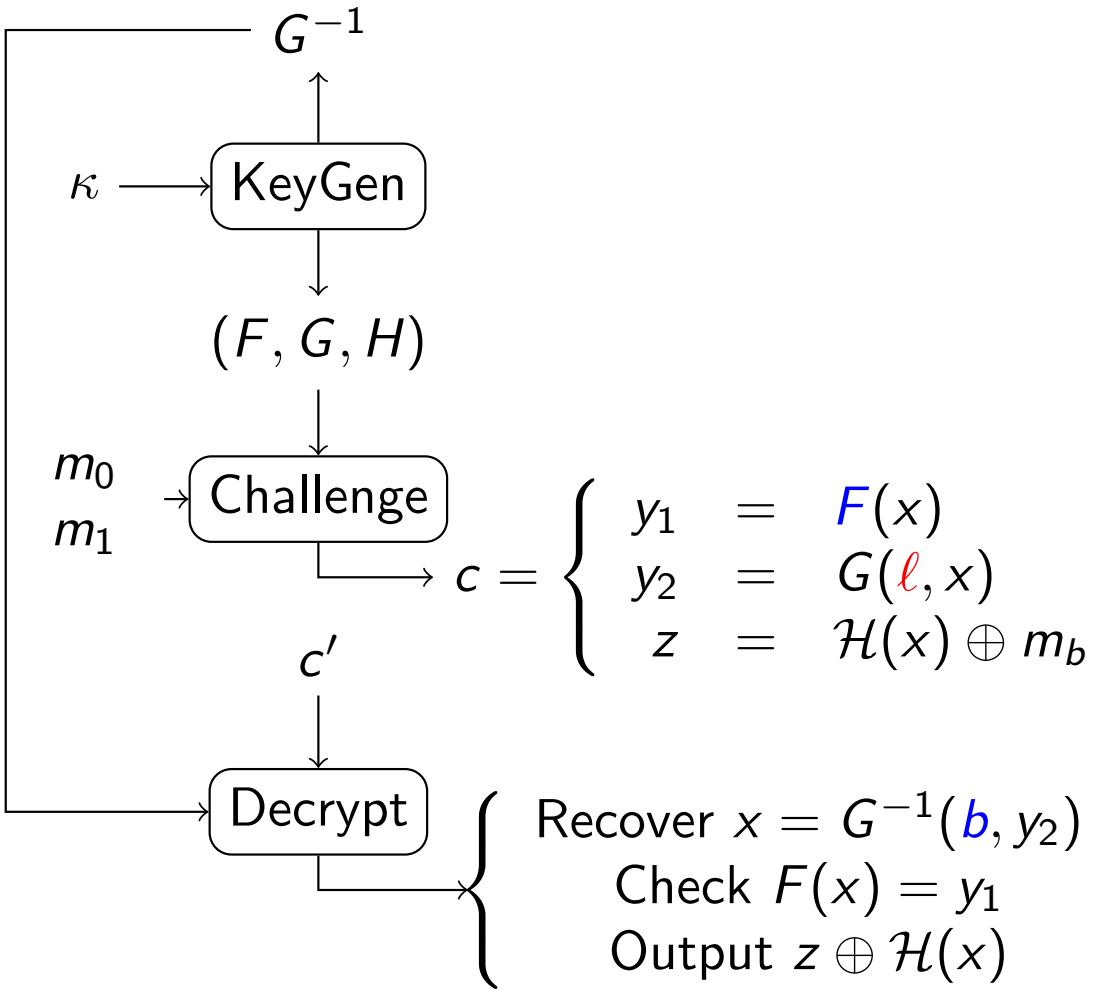

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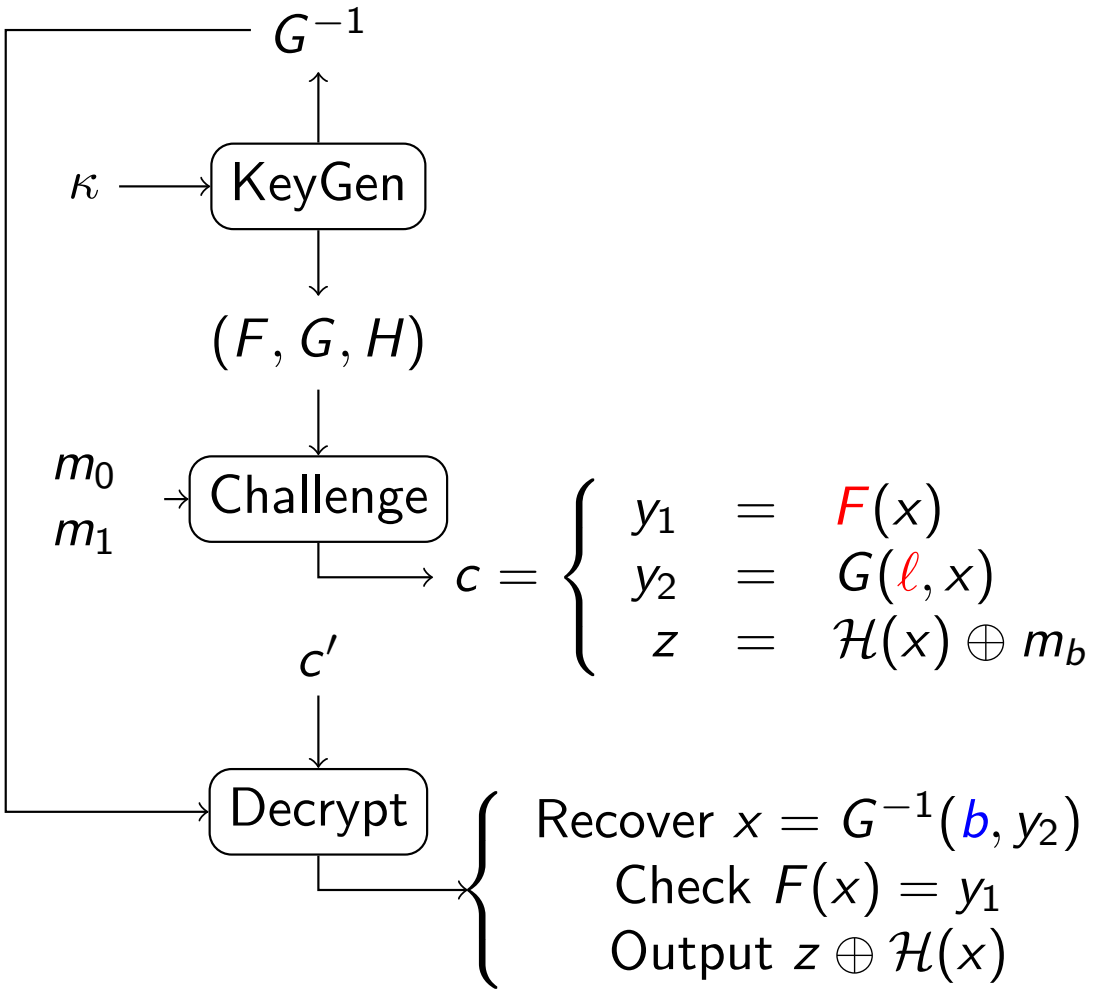

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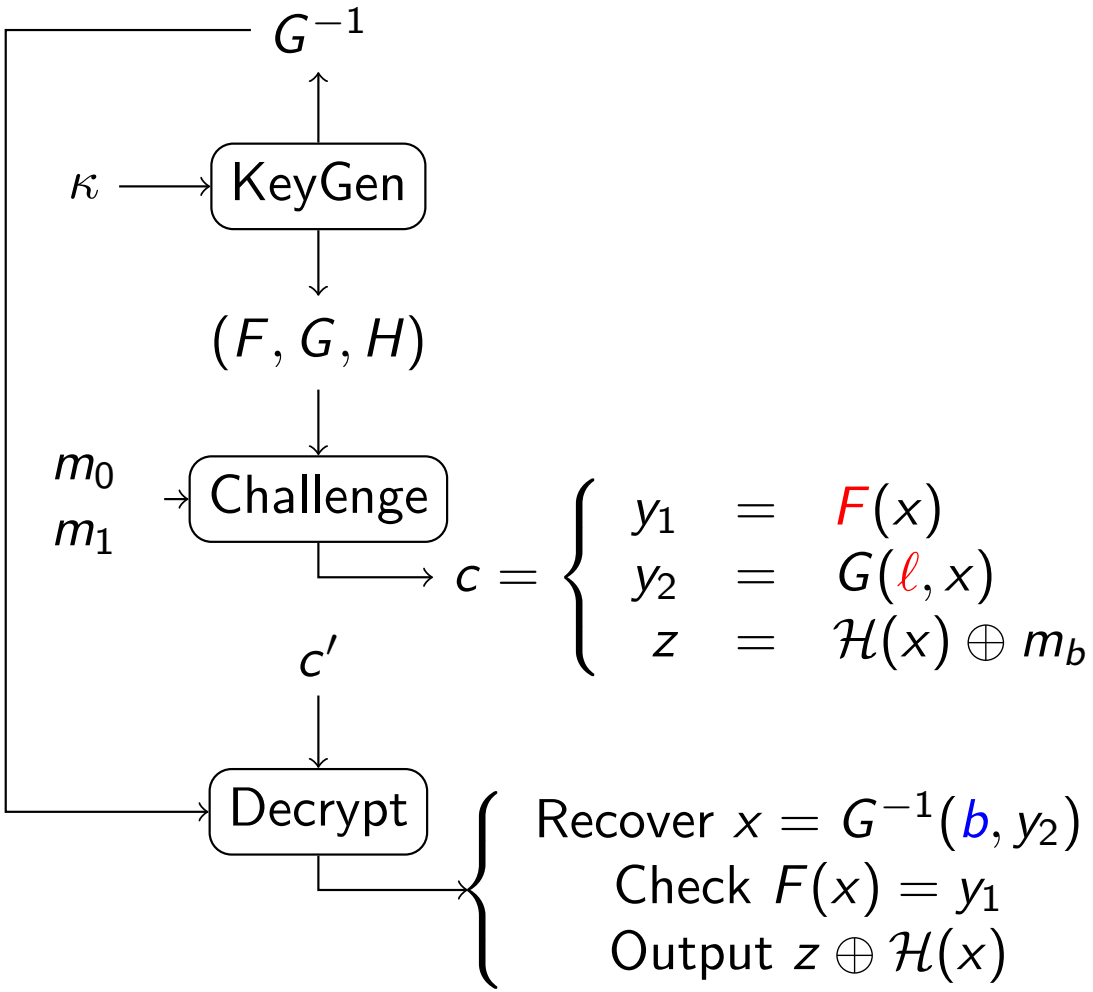

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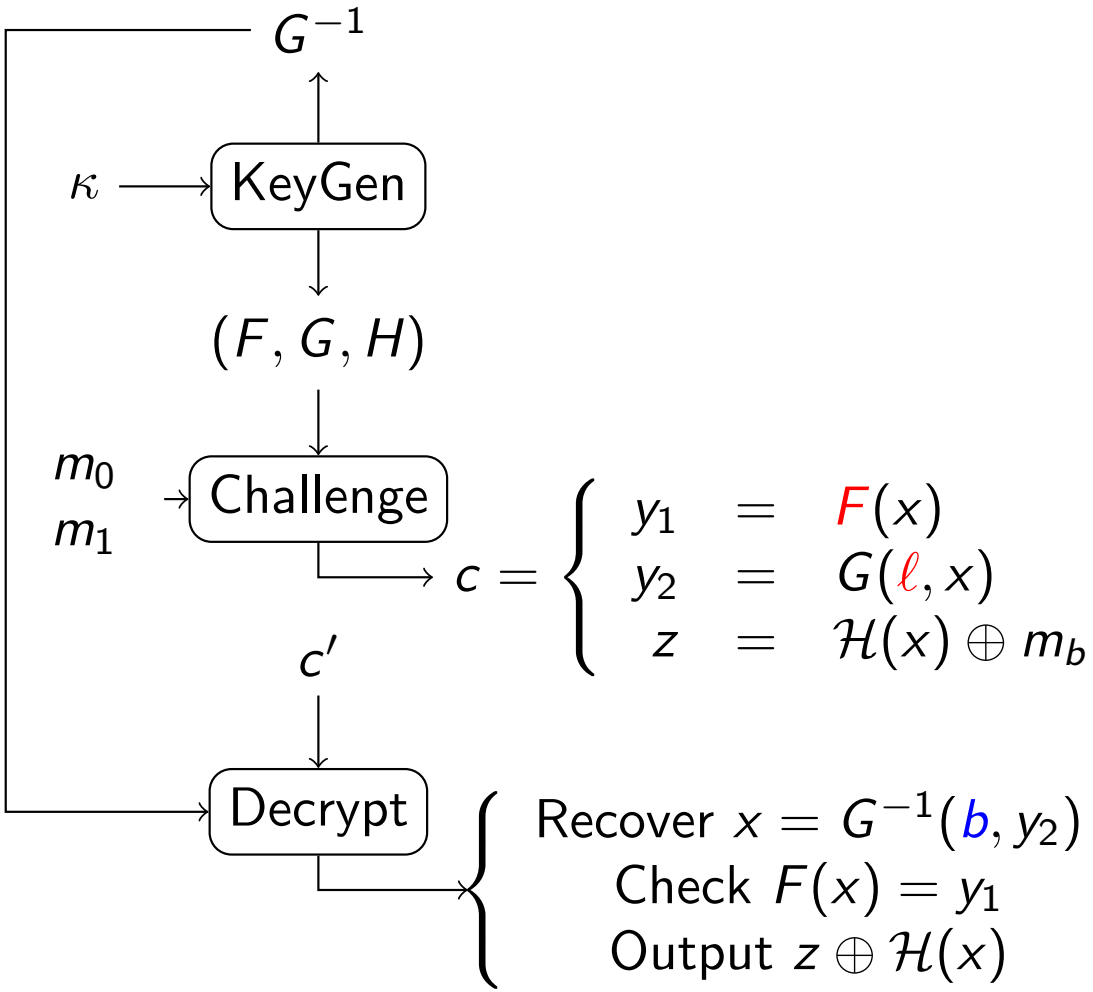

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- (One-time) signature for CCA2 Security (Dolev-Dwork-Naor's trick)

Other Applications

- **Deterministic encryption**
  - A. Boldyreva, S. Fehr, A. O'Neill - Crypto 2008
- **Selective-opening secure encryption**
  - M. Bellare, D. Hofheinz, S. Yilek - Eurocrypt 2009
  - S. Fehr, D. Hofheinz, E. Kiltz, H. Wee - Eurocrypt 2010
  - B. Hemenway, B. Libert, R. Ostrovsky, D. V. - 2011+
- **Encryption with weak randomness**
  - M. Bellare, Z. Brakerski, M. Naor, T. Ristenpart, G. Segev, H. Shacham, S. Yilek - Asiacrypt 2009
  - G. Castagnos, M. Izabachène, B. Libert, D. V. - 2011+
- **IND-CPA Security of OAEP**
  - E. Kiltz, A. O'Neill, A. Smith - Crypto 2010
- ...

Contents

Realizing Lossy TDF: Rabin-Williams

$$N = pq, \qquad p, q \equiv 3 \bmod 4$$

- Squaring function $x \longmapsto x^2 \bmod N$ is lossy
  - 4-to-1 map on $\mathbb{Z}_N^* \rightsquigarrow$ 2 bits of lossiness

- but, inversion is possible if we know its
  - **Jacobi symbol**: $\left(\frac{x}{N}\right)$
  - **"Sign"**: $(x \text{ rem } N) \in [\![-N/2, \ldots, N/2]\!]$

$\rightsquigarrow$ a square $y \in \mathbb{Z}_N^*$ has four square roots $(\pm x_0, \pm x_1)$ with

$$\left(\frac{x_0}{N}\right) = \left(\frac{-x_0}{N}\right) = -\left(\frac{x_1}{N}\right) = -\left(\frac{-x_1}{N}\right)$$

Realizing Lossy TDF: Rabin-Williams

- How to make squaring **injective** ?
  $\rightsquigarrow$ encode two extra bits in the output
- How to encode these bits in a **computationally indistinguishable way** ?
  $\rightsquigarrow$ put them in the exponent of quadratic non-residues !

$$r, s \in \mathbb{Z}_N^* \text{ with } \left(\frac{r}{N}\right) = -1 \text{ and } \left(\frac{s}{N}\right) = 1 \text{ but quadratic non residue}$$

$$F : x \longmapsto y = x^2 \cdot r^{j(x)} \cdot s^{h(x)}$$

where

$$j(x) = 1 \iff \left(\frac{x}{N}\right) = -1 \text{ and } h(x) = 1 \iff (x \text{ rem } N) \in [\![-N/2, \ldots, -1]\!]$$

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- lossy ! (1 bit of lossiness)
- indistinguishable from injective under quadratic residuosity assumption

Realizing Lossy TDF: Homomorphic encryption

- Use an IND-CPA (additively) homomorphic cryptosystem

$$\text{Encrypt}(m) \odot \text{Encrypt}(m') \simeq \text{Encrypt}(m + m')$$

- **Idea:**
  - Encrypted $n \times n$ matrix: I_n for F and 0_n for F
  - $F(x)$ is computed by encrypted linear algebra

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$$N = pq, \qquad \psi(m, r) = (1 + N)^m r^N \bmod N^2$$

(can be inverted given $\lambda(N) = \mathrm{lcm}(P - 1, Q - 1)$)

- $S_{\mathrm{inj}} \longrightarrow s = (N = pq, c = (1 + N)r^N \bmod N^2), t = \lambda(N)$
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Lossy ! ($n/2$ bit of lossiness)

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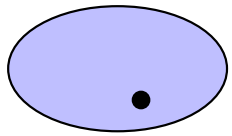

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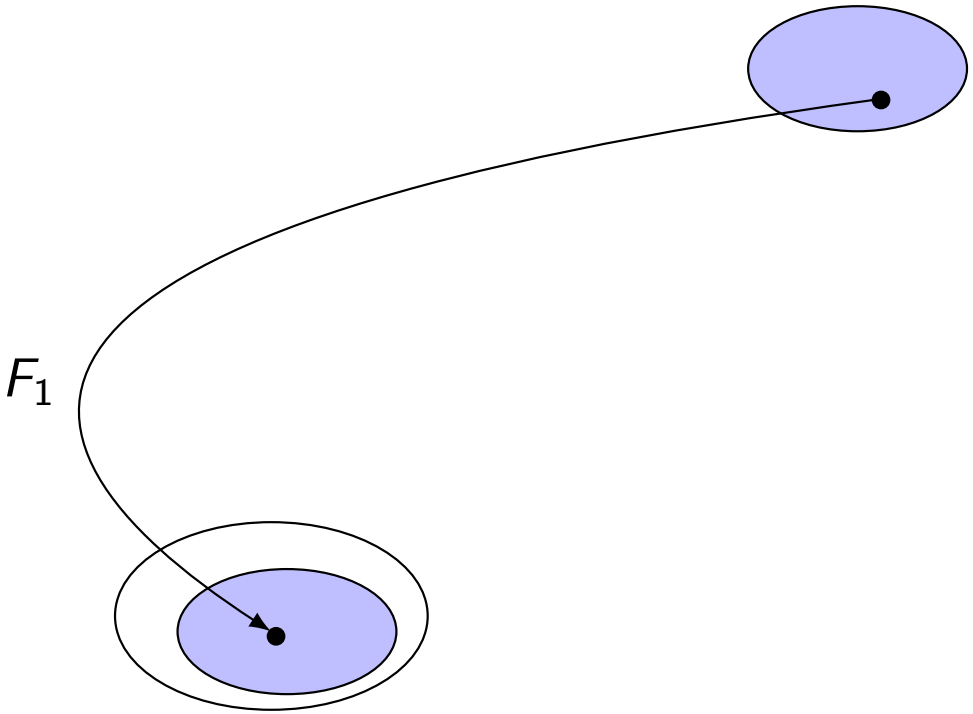

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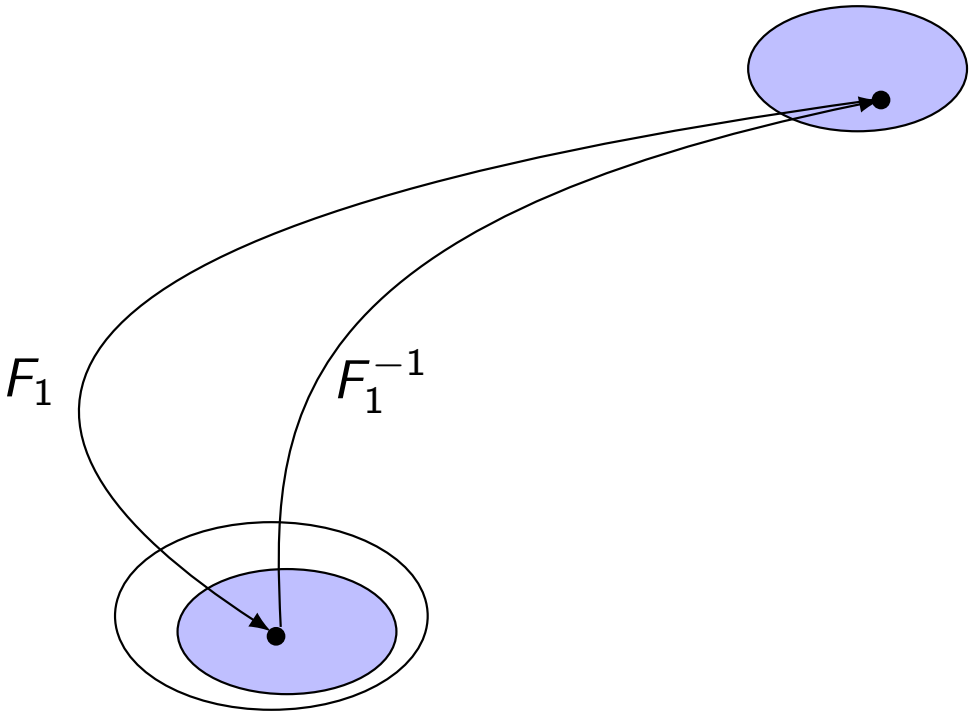

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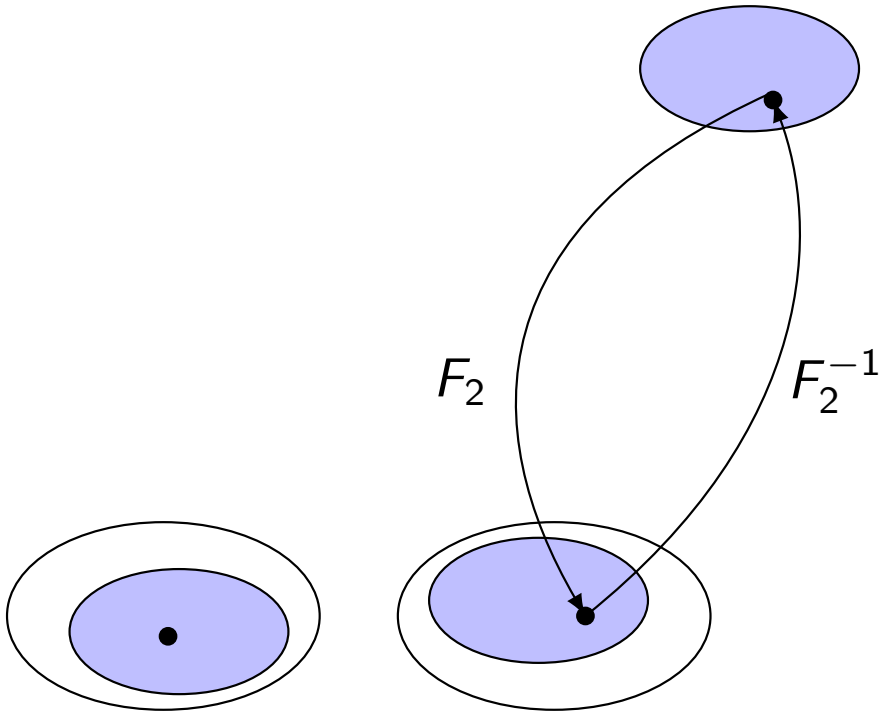

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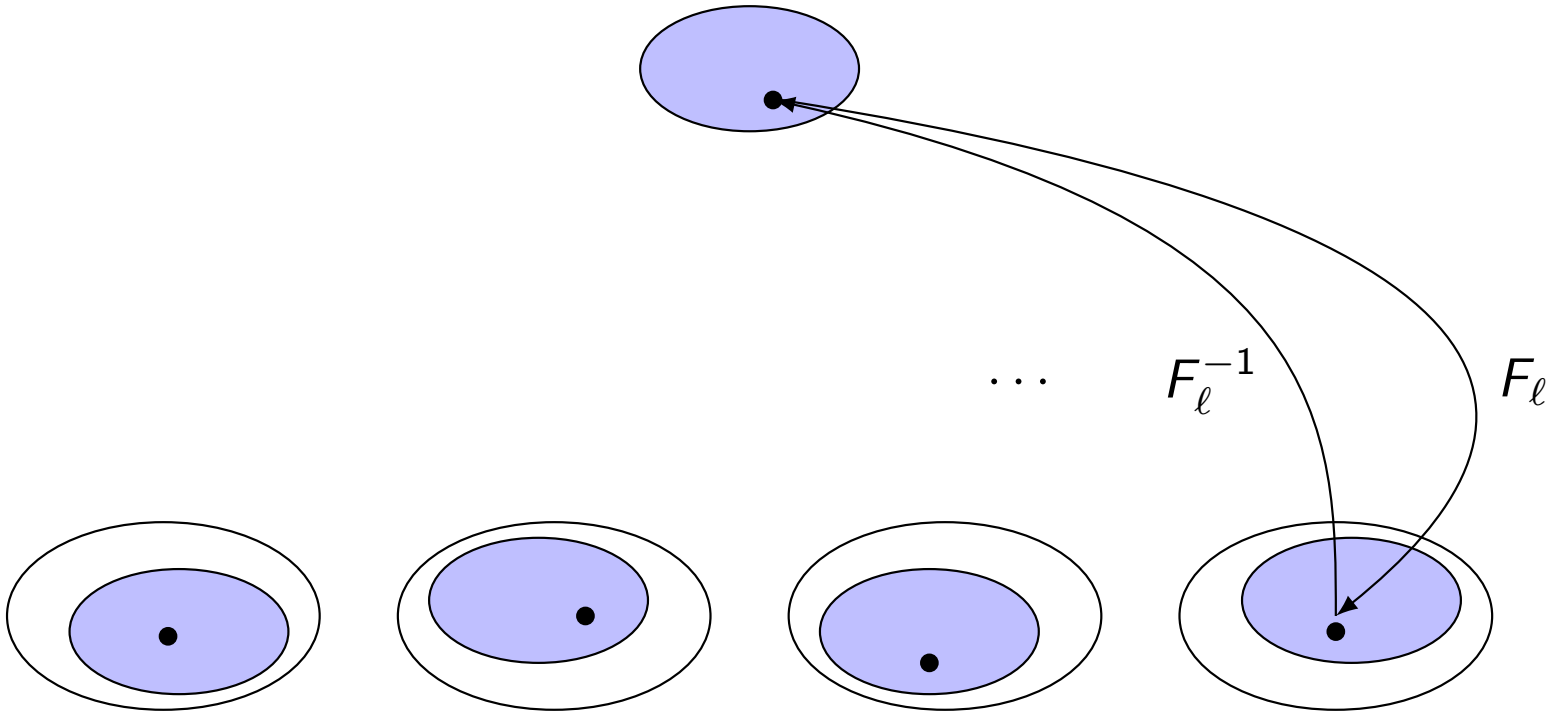

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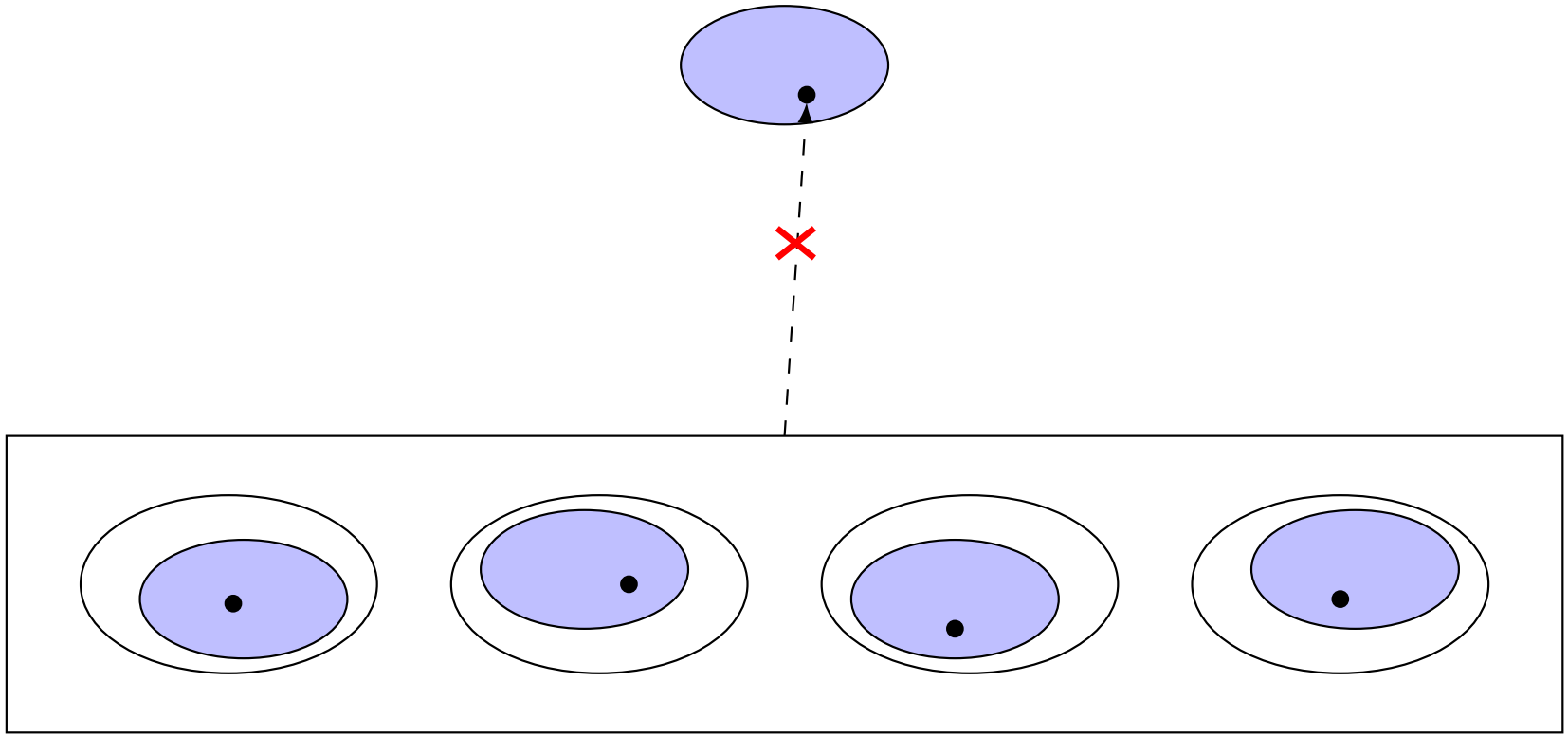

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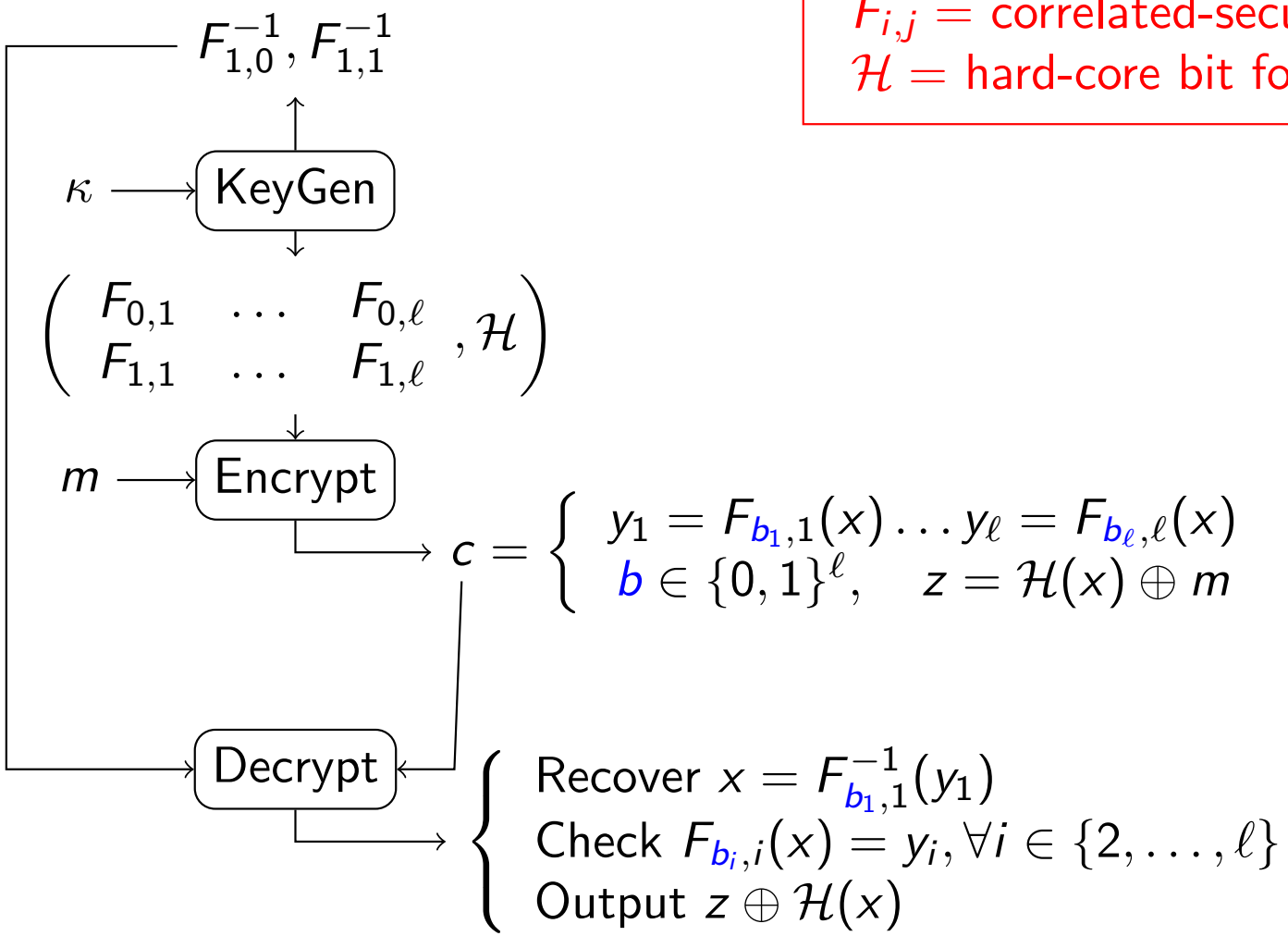
IND-CCA Security from Correlation-Secure TDF: Construction

$F_{i,j}$ = correlated-secure TDF
$\mathcal{H}$ = hard-core bit for $(F_{i,j})_{i,j}$



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Generic constructions (Rosen and Segev, 2009)

- Lossy TDF : $\{0,1\}^n \longrightarrow \{0,1\}^n$ with $n(1-\rho)$ lossiness
  $\Rightarrow$ Correlated Secure TDF under $1/\rho$-repetition

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