Euler Tour Lock-in Problem in the Rotor-Router Model

I choose pointers and you choose port numbers

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Definitions

Anonymous graphs / networks

- No (used) node labeling
- Local port numbering at node v from 1 to $\deg(v)$

Mobile agent / robot / message / anything

Follows the rotor-router mechanism

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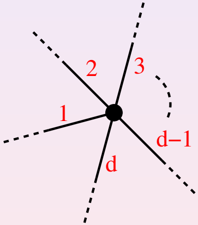

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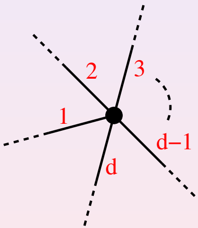

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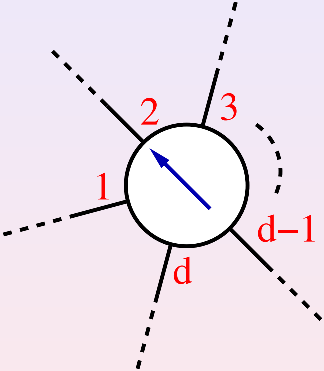
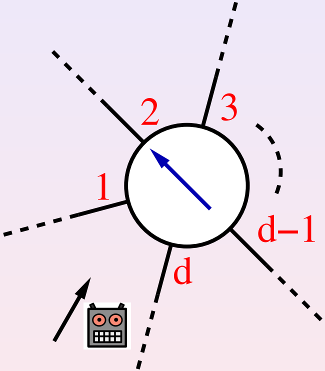


Very simple mechanism:

- Each node has a pointer ↗
- The agent follows the pointer and "increments" it with respect to the cyclic ordering ↻ (induced by the port numbering)

Other names: Propp machine, Next-Port, Edge Ant Walk

Rotor-router mechanism



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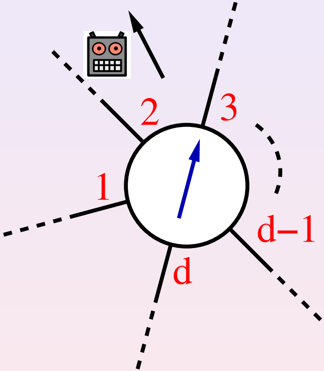

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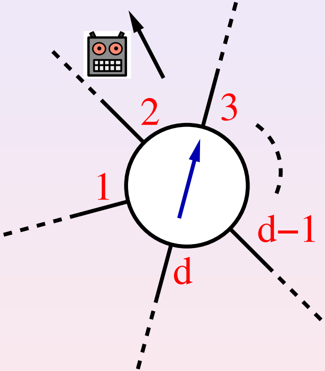

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Known results

- The agent eventually traverses each edge
- The traversal stabilizes into an Euler tour: each edge is traversed once in each direction within a period
- Lock-in time: $\Theta(m \cdot D)$ (m: # edges, D: diameter)

Motivations / Applications

- Graph exploration (by a software agent / robot)
- Mutual exclusion
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- S. Bhatt, S. Even, D. Greenberg, and R. Tayar, *Journal of Graph Algorithms and Applications*, 2002.
- J.N. Cooper and J. Spencer, *Combinatorics, Probability and Computing*, 2006.
- B. Doerr and T. Friedrich, *Combinatorics, Probability and Computing*, 2009.
- A.S. Fraenkel, *Mathematics Magazine*, 1970.
- L. Gasieniec and T. Radzik, *Proc. WG*, 2008.
- V.B. Priezzhev, D. Dhar, A. Dhar, and S. Krishnamurthy, *Physics Review Letters*, 1996.
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Our results

Problem

How does the lock-in time depend on the initial configuration of the ports $\circlearrowright$ and pointers $\nearrow$?

Scenario	Worst case	Best case
Case $\mathcal{P}$-all	$\Theta(m)$	$\Theta(m)$
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$\mathcal{P}$=Player, $\mathcal{A}$=Adversary

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Very simple algorithm

- Consider any choice of ports $\circlearrowright$ and pointers $\nearrow$
- Run virtually the agent for $\Theta(m \cdot D)$ steps
- The pointers $\nearrow$ are now correctly set again by $\mathcal{P}$

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Useful properties (known results)

Definitions

- Phase: Interval between two traversals of the first edge
- G_i: Graph induced by the edges traversed in Phase i
- Saturated node: Node whose incident edges are all traversed in both directions during the current phase

(Known) Properties

- Each arc is traversed at most once during a phase
- $G_i \subseteq G_{i+1}$
- If $v \in G_i$, then v is saturated in G_{i+1}
- $\Rightarrow G_{i+1} \supseteq \mathrm{neighborhood}(G_i)$

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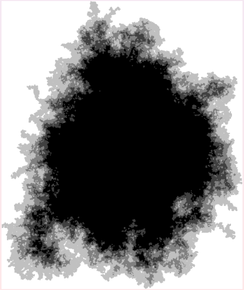

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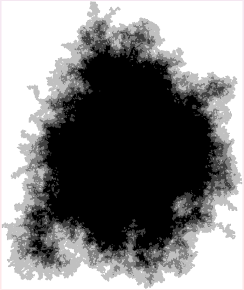

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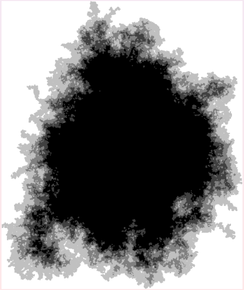

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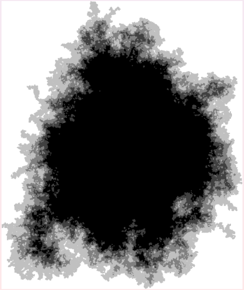

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Case $\mathcal{A}$-*all*	$\Theta(m \cdot D)$	$\Theta(m \cdot D)$

(Sketch of the) Proof

- Let $[u, v]$ be a diameter
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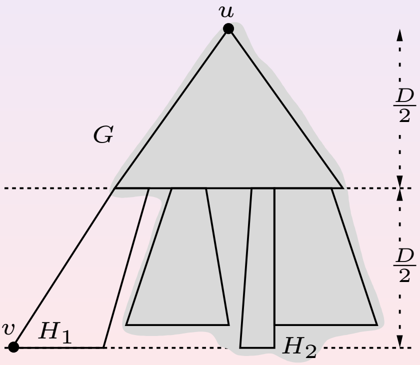

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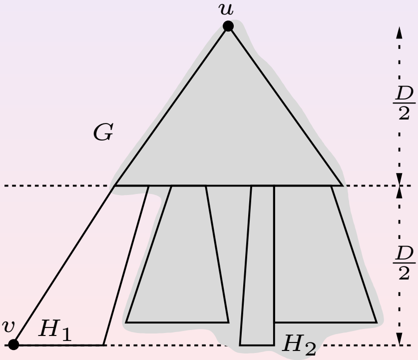

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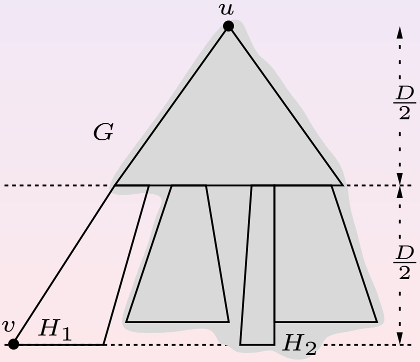

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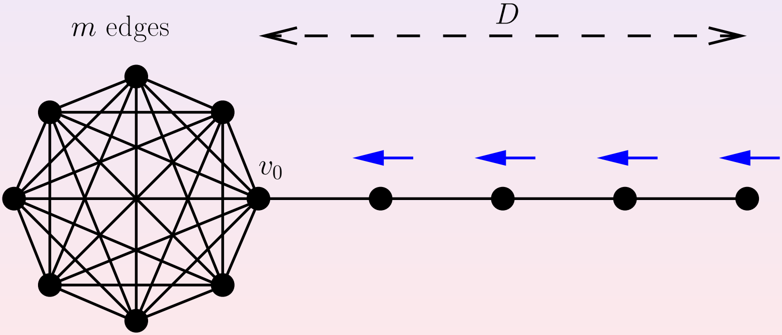

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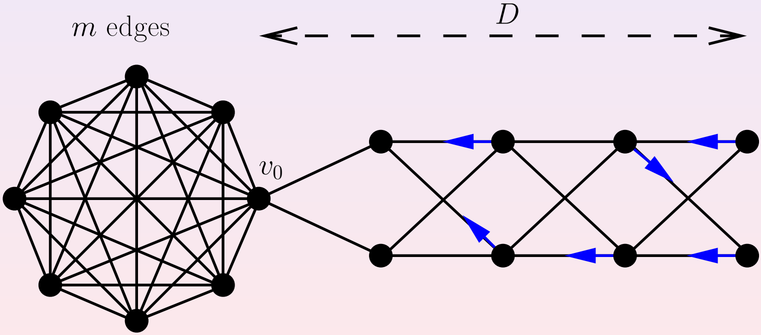

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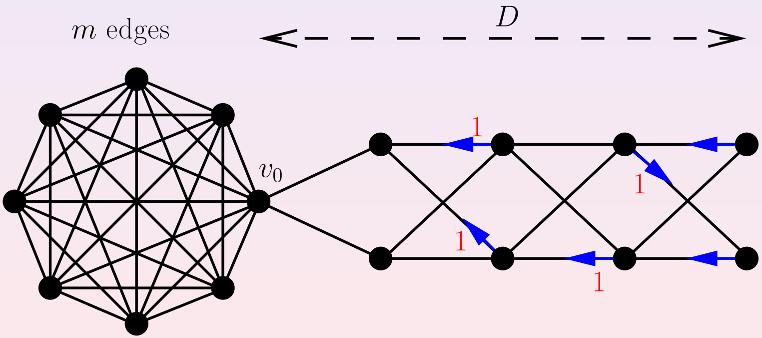

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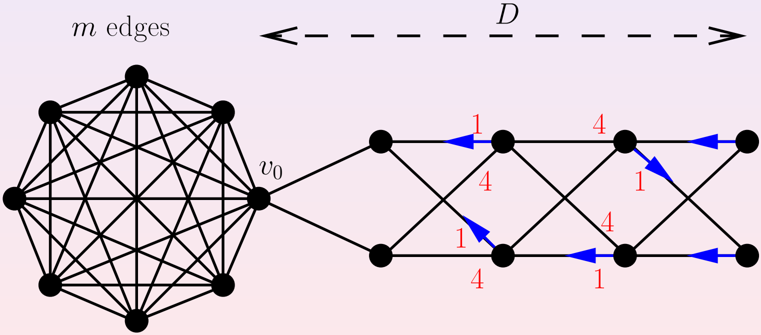

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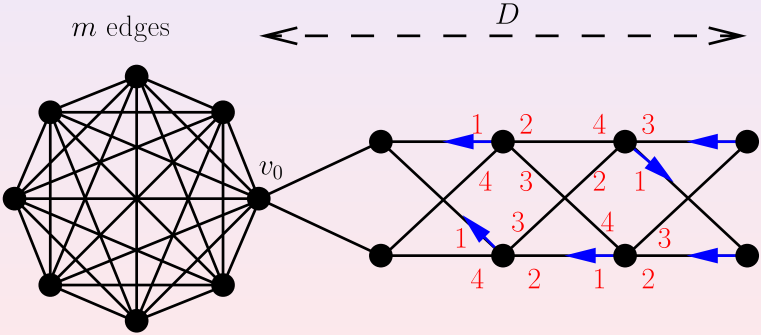

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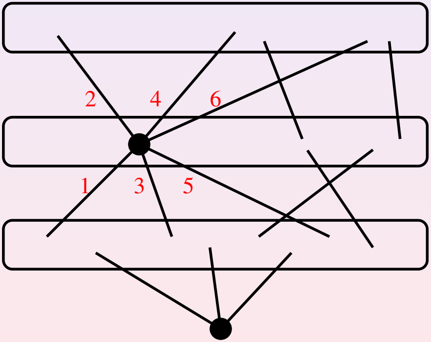

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Open problem

- Does there exist any graph of large diameter with lock-in time $O(m)$ in Case $\mathcal{P}(\circlearrowright)\mathcal{A}(\nearrow)$?
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