Complete Graph Minors and the Graph Minor Structure Theorem

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Motivation

Let H be a fixed graph.

Graphs without H-minors can be built using

- graphs on surfaces
- vortices
- apex vertices
- the clique-sum operation

in a "bounded way" [Robertson and Seymour].

Surfaces

Surface := connected compact 2-manifold

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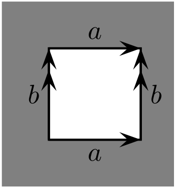
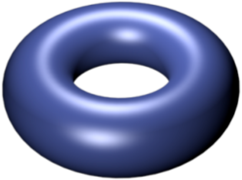
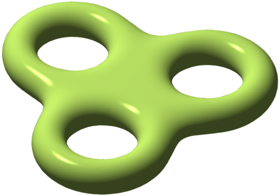
Every surface is either

- $\mathbb{S}_h$:= sphere with h handles (orientable surfaces), or







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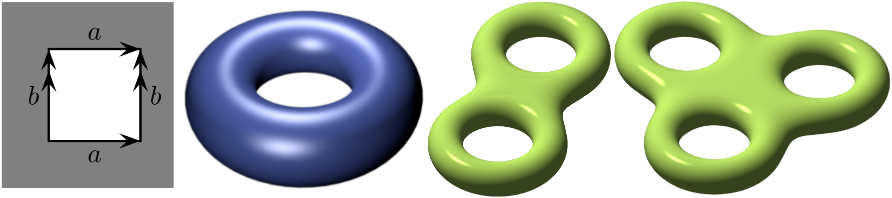
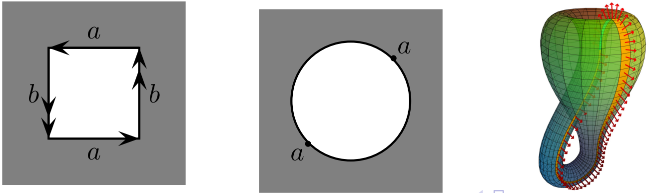


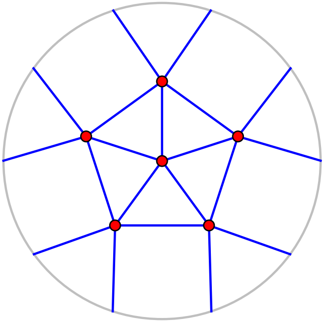
- $\mathbb{N}_c$:= sphere with c cross-caps (non-orientable surfaces)

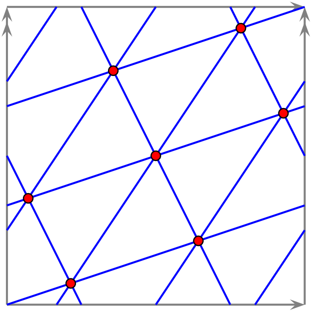


Examples of graphs embedded in a surface

K_6 embedded in projective plane $\mathbb{N}_1$ (antipodal points are identified)

K_7 embedded in torus $\mathbb{S}_1$ (opposite sides are identified)





Euler genus

$$\mathrm{eg}(\mathbb{S}_h) := 2h \qquad \text{Rmk: usual (orientable) genus is } h$$

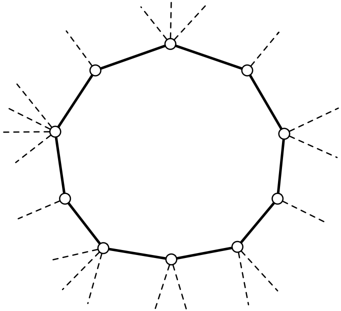
$$\mathrm{eg}(\mathbb{N}_c) := c$$

$$\mathrm{eg}(G) := \min\{\mathrm{eg}(\Sigma) : G \text{ embeds in } \Sigma\}$$

Theorem (Euler-Poincaré Formula)

$$|E(G)| \;\leq\; 3\big(\,|V(G)| + \mathrm{eg}(G) - 2\,\big)$$

Vortices



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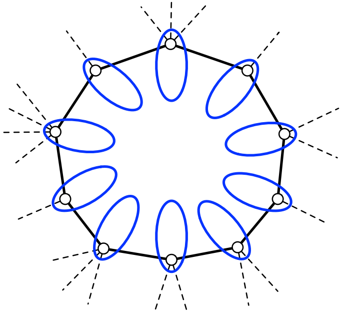

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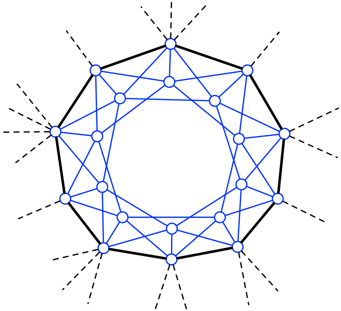

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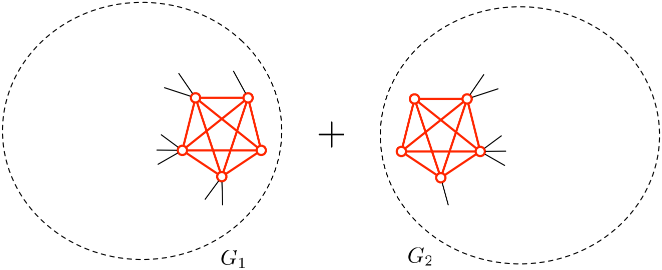

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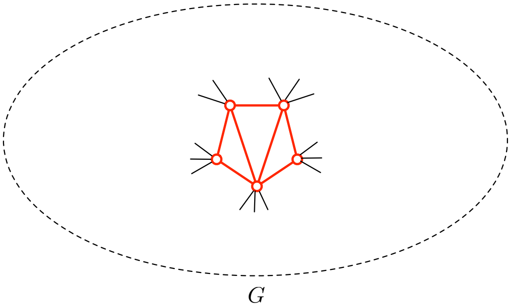

Graph Minor Structure Theorem

Theorem (Robertson and Seymour, GM XVI)

If G has no H-minor, then G can be built using

- graphs on surfaces of Euler genus at most g
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$\rightarrow \mathcal{G}(g, p, k, a)^+$

Clique Minors in Almost Embeddable Graphs

In this context, $H = K_\ell$ usually assumed.

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Our result:

$$\boxed{f(g, p, k, a) = \Theta(a + k\sqrt{p + g})}$$

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$\rightarrow \mathcal{G}(g, p, k)$

Want to show: $f(g, p, k) = O(k\sqrt{p+g})$

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Modification of G in 6 steps:

1. make vortices close to each other
2. split every vortex-boundary
3. simulate vortices using a lexicographic product with K_k
4. reduce chords of vortex-boundaries that are "contractible"
5. shorten ladders defined by (pairs of) vortices
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How $\eta(G)$ changes is controlled at each step

Thank you!