Nombre minimum d'ensembles stables dans des arbres ayant un nombre de stabilité fixé.

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Fibonacci index and Stability number

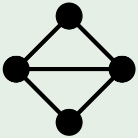
Notations

Simple undirected graph $G = (V, E)$.

Definitions

- **order** $n = |V|$ and **size** $m = |E|$
- **Stability number** $\alpha(G)$:
  number of vertices of a maximum stable set of G
- **Fibonacci index** $F(G)$:
  number of all the stable sets in G (including $\emptyset$)

Example



$\alpha(G) = 2$ and
$F(G) = 1 + 4 + 1 = 6$

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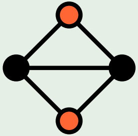

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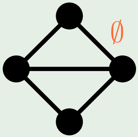

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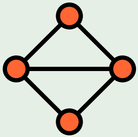

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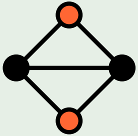

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Fibonacci index and Stability number

Motivation & history

The Fibonacci index $F(G)$ is the number of stable sets in G

- introduced by [Prodinger and Tichy, 1982] in mathematical community
- introduced (independently) by [Merrifield and Simmons, 1989] in chemical community
- $\exists$ correlations between $F(G)$ and the boiling point of a molecular graph G
- widely studied especially recently (both communities)
- rmq : F of paths correspond to the Fibonacci sequence

Known results

Several extremal results for $F(G)$, for example

- Lower and upper bounds (in terms of n) for general, connected graphs and trees [Prodinger and Tichy, 1982]
- Upper bounds for trees inside the class $\mathcal{T}(n, k)$ of trees with order n and a fixed parameter k, e.g., k is
  - the **diameter** [Li et al., 2005]
  - the **maximum degree** [Heuberger and Wagner, 2008]
  - the **number of pending vertices** [Wang et al., 2008]

Known results

However, a few results linking $F(G)$ and the stability number (despite the fact that $F(G)$ and $\alpha(G)$ are both related to stable sets).

Bounds for F in terms of n and α :

- Pederden, Vertergaard (2006) : lower bounds for general and connected graphs

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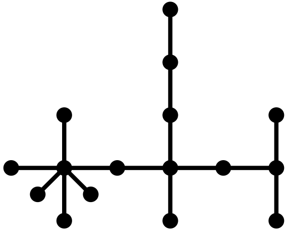
- Bruyère, Mélot (2009) : upper bounds for general, connected graphs and trees
- this work : lower bound for trees

Lower bound for trees

Main results : three theorems

- characterization of the extremal trees
  1. Extremal graphs are Trees of Stars or paths (proved)



Lower bound for trees

Main results : three theorems

- characterization of the extremal trees
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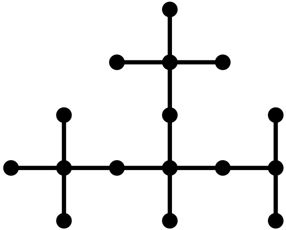
  2. Extremal graphs are Balanced Trees of Stars or paths (proved)



Lower bound for trees

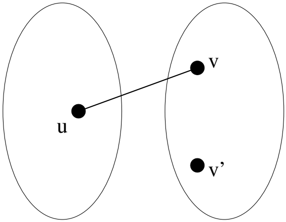
Main results : three theorems

- characterization of the extremal trees
  1. Extremal graphs are Trees of Stars or paths (proved)
  2. Extremal graphs are Balanced Trees of Stars or paths (proved)
  3. How are these stars connected together ? (proved in particular cases but open problem in general)

Idea of proofs

Th. 1 : extremal graphs are trees of stars (or paths)

A simple operation on trees : rotating an edge.



Rotate uv around v :

- remove edge uv
- let $v' \in$ the component of $T - uv$ containing v
- add edge uv'
- resulting graph is still a tree

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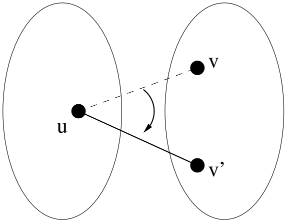


Rotate uv around v :

- remove edge uv
- let $v' \in$ the component of $T - uv$ containing v
- add edge uv'
- resulting graph is still a tree

Idea of proofs

Th. 1 : extremal graphs are trees of stars (or paths)

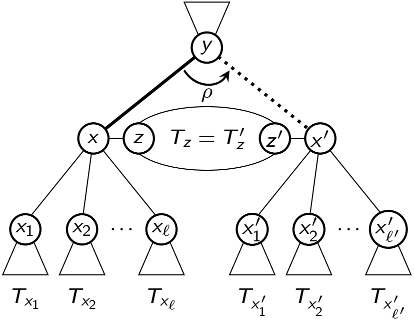
Key idea of the proof : show that if T is not a tree of stars (or a path), then $\exists$ a good rotation that can be applied on T.

A rotation $\rho(T) = T'$ is good if :

- $\alpha(T') = \alpha(T)$
- $F(T') < F(T)$

Idea of proofs

Th. 1 : extremal graphs are trees of stars (or paths)



$$X := \prod_{i=1} F(T_{x_i}), \qquad \overline{X} := \prod_{i=1} F(T_{x_i} - x_i),$$

$$X' := \prod_{i=1}^{\ell'} F(T_{x'_i}), \quad \overline{X}' := \prod_{i=1}^{\ell'} F(T_{x'_i} - x'_i),$$

$$\overline{Z}_z := F(T_z - z), \qquad \overline{Z}_{z'} := F(T_z - z').$$

Lemma

$$F(\rho(T)) < F(T) \iff X\overline{X}'\overline{Z}_{z'} > \overline{X}X'\overline{Z}_z$$

Idea of proofs

Th. 1 : extremal graphs are trees of stars (or paths)

Using the previous lemma implies to study ratio like

$$\frac{F(T)}{F(T-v)},$$

in particular if T is a tree of stars.

Idea of proofs

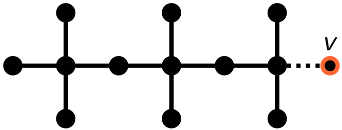
Th. 1 : extremal graphs are trees of stars (or paths)

Interesting property of trees of stars : the golden ratio φ plays an important role

Lemma

Let T be a tree of stars and let v denote a leaf of T. Then

$$F(T)/F(T-v) > \varphi.$$



$$\frac{F(T)}{F(T-v)} \simeq 1.8828 > \varphi \simeq 1.61803$$

Idea of proofs

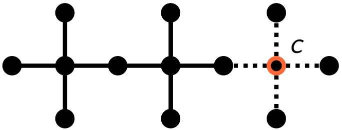
Th. 1 : extremal graphs are trees of stars (or paths)

Interesting property of trees of stars : the golden ratio φ plays an important role

Lemma

Let T be a tree of stars and let c denote a center of T. Then

$$F(T)/F(T-c) < \varphi.$$



$$\frac{F(T)}{F(T-c)} \simeq 1.06638 < \varphi \simeq 1.61803$$

Idea of proofs

Th.1 : extremal graphs are trees of stars (or paths)

- define a set of conditions
- show that these conditions implies that a rotation ρ can be applied s.t. ρ
  - preserves α
  - decreases F
- show that if T is not a tree of stars (or a path), then T statisfies the conditions

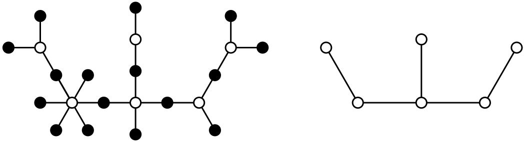
Th.2 : extremal graphs ares balanced tree of stars (or paths)

- also use the rotation lemma
- use technical lemmata on the minimum degree of the centers

Further results

How the balanced stars are linked together in an extremal tree ?

Center-tree of a tree of stars T : vertices = centers of T, edge if two centers at distance 2 in T.



What is the topology of an extremal center-tree ?

Further results

Let T be a balanced tree of stars with order n and stability number α. Let w be a center of T. Then, the degree of w is either $\left\lceil \frac{n-1}{n-\alpha} \right\rceil$ or $\left\lceil \frac{n-1}{n-\alpha} \right\rceil - 1$.

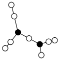
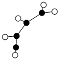
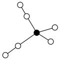
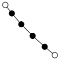
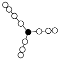
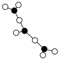
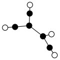
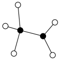
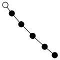
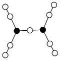
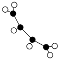
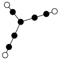
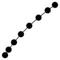
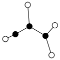
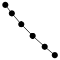
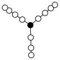
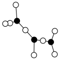
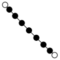
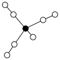
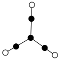
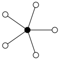
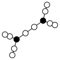
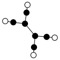
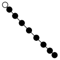
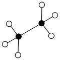
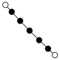
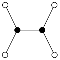
First case : heavy centers
Second case : light centers.

Theorem

Let T be a balanced tree of stars with at most two light centers. If T is extremal, then its center-tree is a path, and each light center of T (if any) is an endpoint of that path.

Further results

In general ? Open problem.

# stars	12	11	10	9	8	7	6
$n = 23$							
$n = 24$							
$n = 25$							
$n = 26$							
$n = 27$							
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