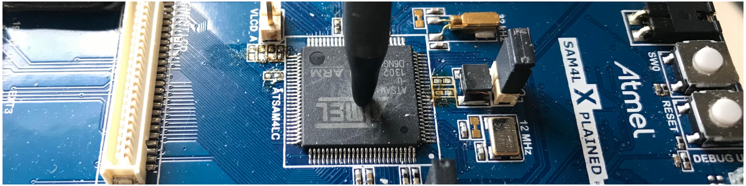


A Decade of Masking Security Proofs

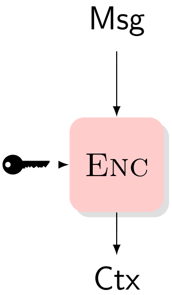
Loïc Masure

PROOFS, 18 September 2025, Kuala Lumpur

Agenda

Content

Context : Side-Channel Analysis (SCA)



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Black-box cryptanalysis: 2^N

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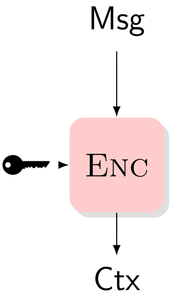

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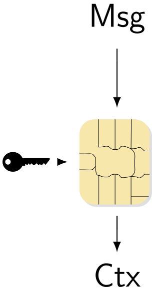

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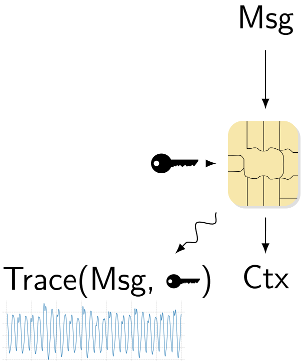

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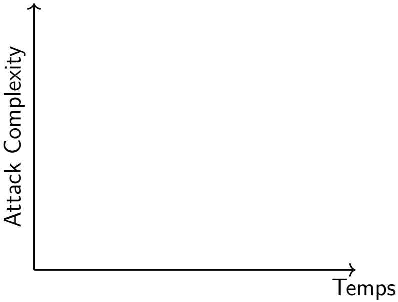
Evaluate Security against Side-Channel Attacks



a

Attack approach (industry):

[a] Shamelessly stolen to O. Bronchain

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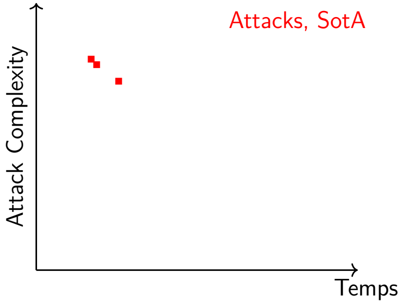

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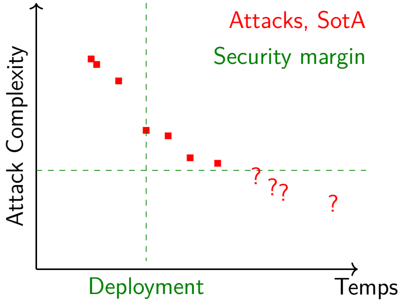

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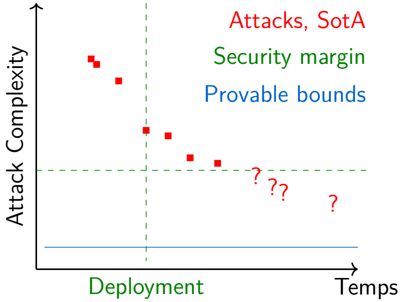

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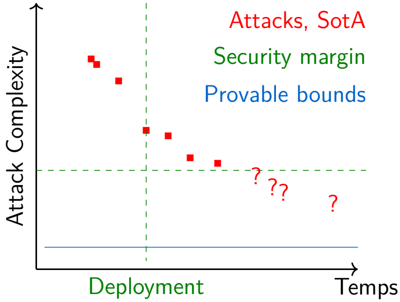

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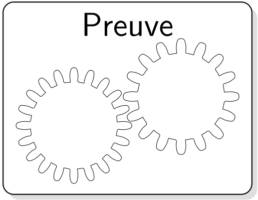
Today's agenda: evaluation by proofs

Security Proof



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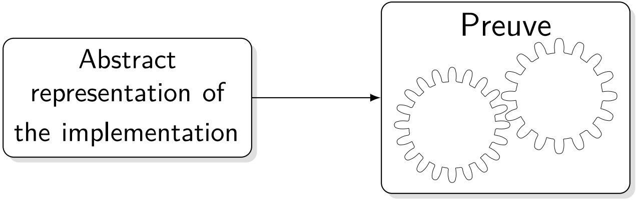

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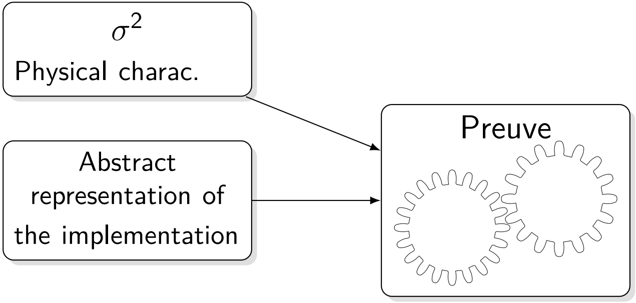

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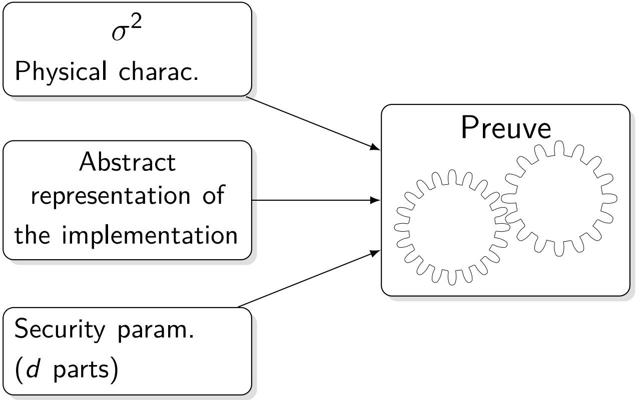

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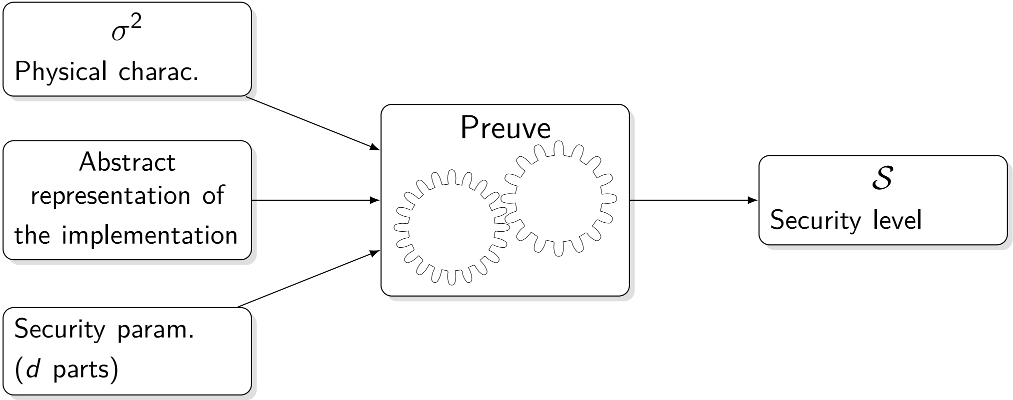

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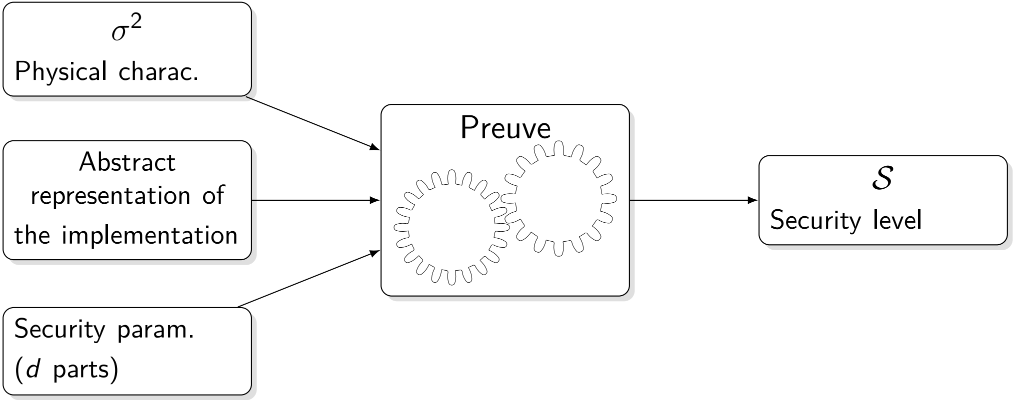

"Any successful attack requires $\mathcal{S}$ observations"

Content

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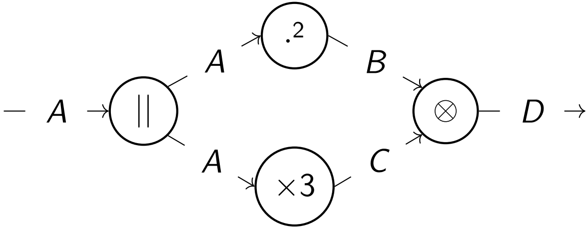
Statement of the Problem



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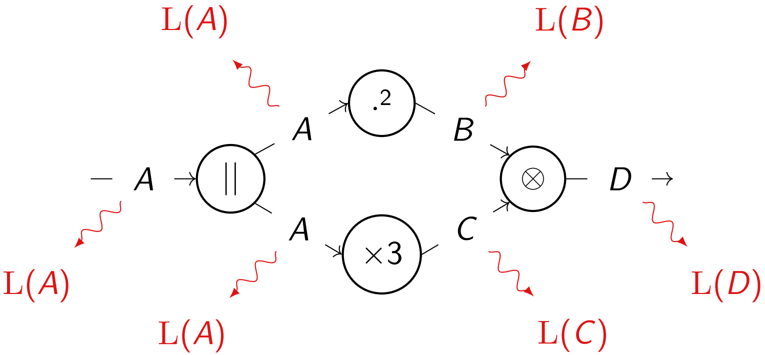

For each wire X, a leakage function $\mathrm{L}(X)$ is revealed to the adversary.

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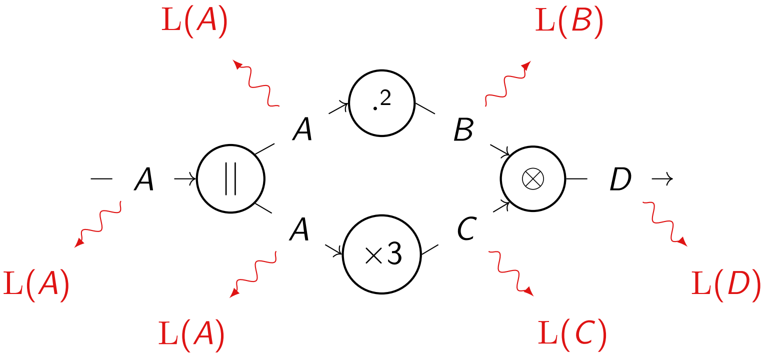
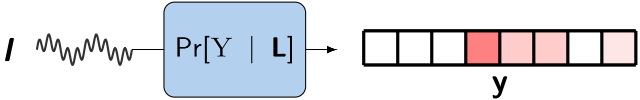


For each wire X, a leakage function $\mathrm{L}(X)$ is revealed to the adversary.

How informative **L** about A?

The Noisy Leakage Model



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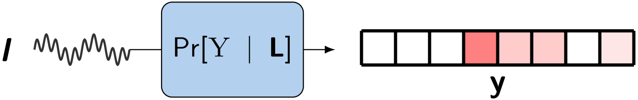
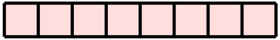


If, the adversary gets:



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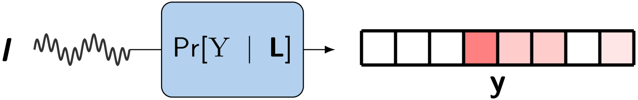

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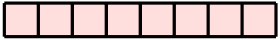

Very noisy leakage

Y indistinguishable from blind guess

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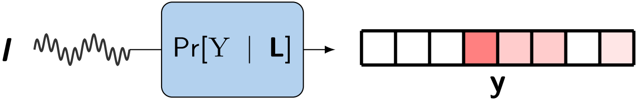

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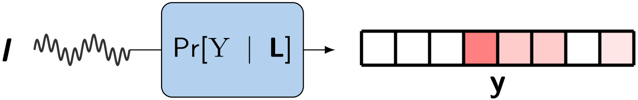

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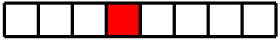
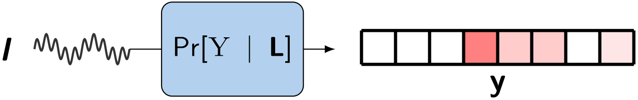


Low-noise leakage
Exact prediction for Y

The Noisy Leakage Model



δ-noisy adversary

Any intermediate computation Y leaks $\mathrm{L}(\mathrm{Y})$ such that:

$$\mathsf{SD}(\mathrm{Y};\mathrm{L}) = \mathop{\mathbb{E}}_{\mathrm{L}}\left[\mathsf{TV}\left(\underbrace{\Pr[\mathrm{Y} \mid \mathrm{L}]}_{\Pr[\mathrm{Y} \mid \mathrm{L}]}, \underbrace{\Pr[\mathrm{Y}]}_{\Pr[\mathrm{Y}]}\right)\right] \leq \delta$$

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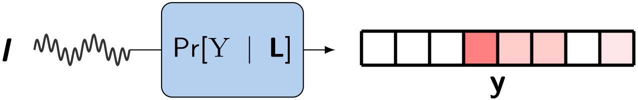

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Main assumption: every observed leakage is δ-noisy

Masking: what is that ?

Masking, a.k.a. *MPC on silicon*:[12] secret sharing over a finite field $(\mathbb{F}, \oplus, \otimes)$

$$\mathrm{Y}(\text{secret})$$

Masking: what is that ?

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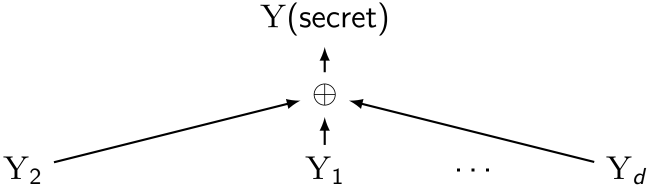

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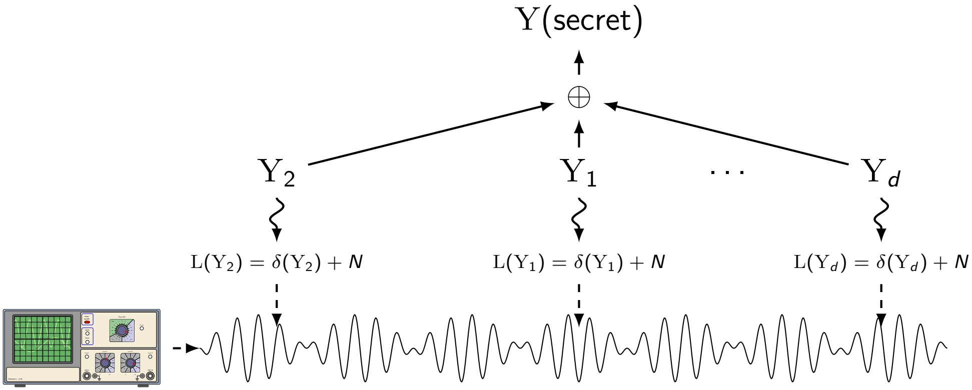
Masking, a.k.a. *MPC on silicon*:[1,2] secret sharing over a finite field $(\mathbb{F}, \oplus, \otimes)$

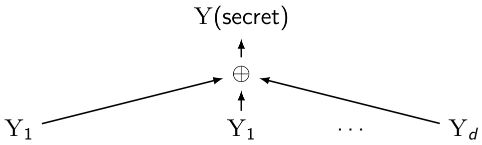


[1] Chari et al., "Towards Sound Approaches to Counteract Power-Analysis Attacks".
[2] Goubin and Patarin, "DES and Differential Power Analysis (The "Duplication" Method)".

The Effect of Masking

Y(secret)

The Effect of Masking



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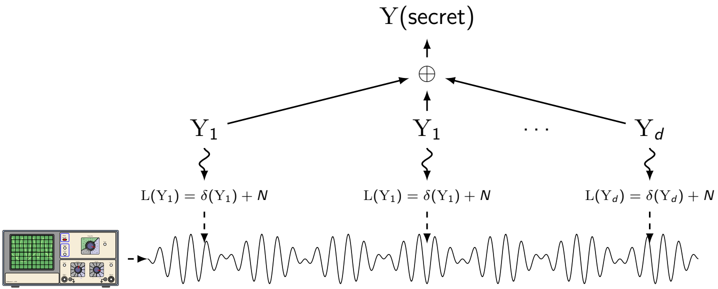

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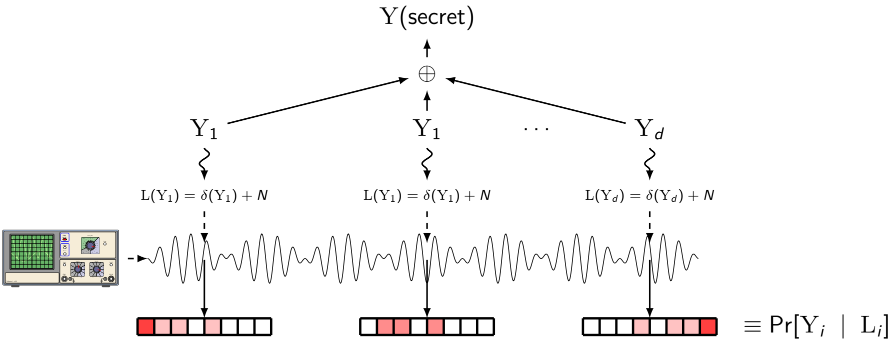

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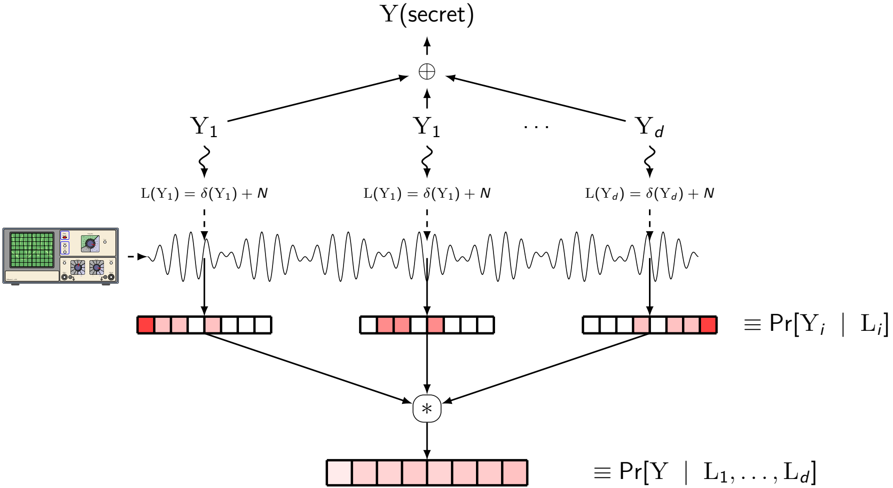

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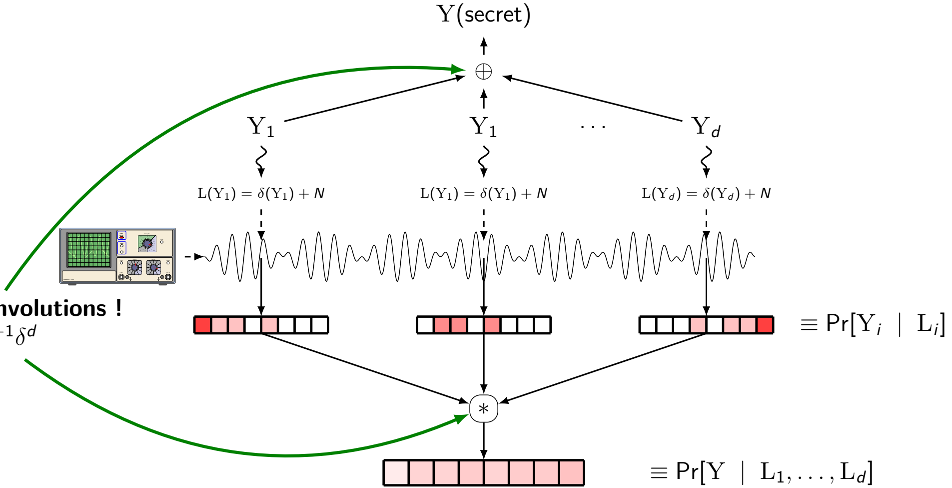

Masking ≡ convolutions !

$\mathrm{SD}(\mathbf{Y}; \mathbf{L}) \leq 2^{d-1}\delta^d$

Content

Computing over Masked Secrets

Idea to make a masked circuit

[4] Ishai, Sahai, and Wagner, "Private Circuits: Securing Hardware against Probing Attacks".
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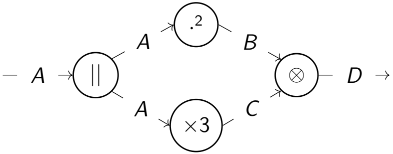

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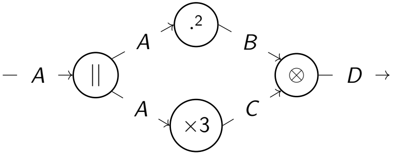

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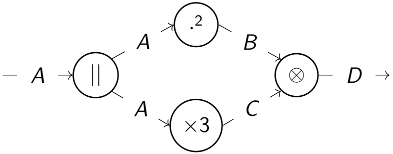

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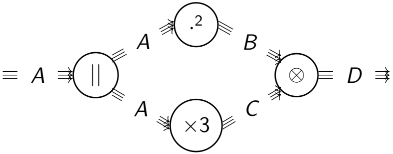

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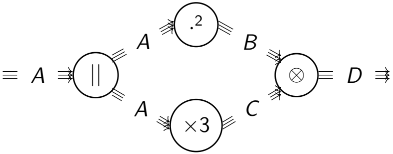

- View your algorithm as a circuit
→ Made of `not`, `and` gates [3]
→ Made of $\oplus$, $\otimes$ gates [4]
- Replace each gate by a masked *gadget*
- Et voilà !**

For now, let's assume the whole circuit to be *probing secure*: every subset of $d-1$ wires is independent from the secret.

[4] Ishai, Sahai, and Wagner, "Private Circuits: Securing Hardware against Probing Attacks".
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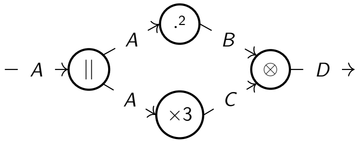
Security Proof for a Gadget

Consider a gadget with ℓ intermediate computations:



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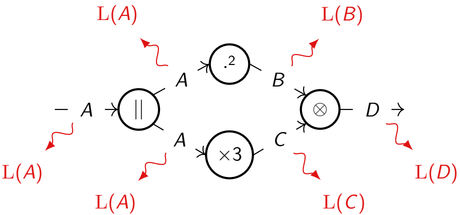

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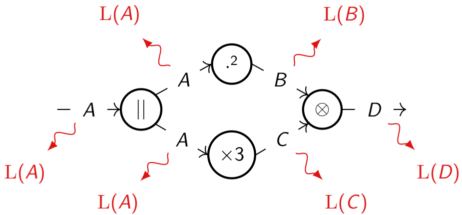

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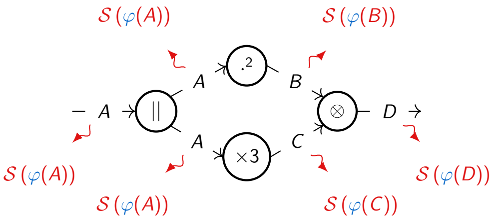

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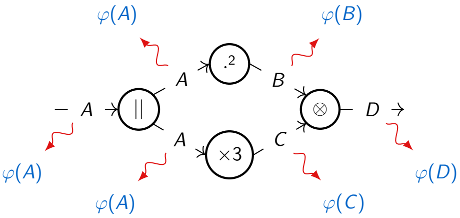

Data-processing Inequality

If for any x the leakage function $\mathrm{L}(x)$ may be expressed as $\mathcal{S}(\varphi(x))$, then:
advantage from $\mathrm{L}(x) \leq$ advantage from $\varphi(x)$

Reduction from Noisy Leakage to Random Probing

Lemma (Simulatability by Random Probing)

The leakage function L *can be simulated from a random probing adversary:* $\varphi(x)$ *reveals* x *with probability* $\epsilon = 1 - \sum_l \min_x \Pr[\mathrm{L}(x) = l] \leq \delta \cdot |\mathbb{F}|$.[5]

Random probing model: easier to analyze for leakage from computations

[5] Duc, Dziembowski, and Faust, "Unifying Leakage Models: From Probing Attacks to Noisy Leakage".

Security against a Random Probing Adversary

To succeed, at least d out of ℓ wires must be revealed to the adversary:

$$\Pr[\text{Adv. learns sth}] \leq \Pr[\text{At least } d \text{ wires revealed}]$$

[6] Boucheron, Lugosi, and Massart, *Concentration Inequalities: A Nonasymptotic Theory of Independence*, P.24, and Ex. 2.11.

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Theorem (Chernoff Concentration Inequality[6])

If ℓ wires, each independently revealed with proba. ϵ:

$$\Pr[\textit{At least d wires revealed}] \leq \left(\frac{e \cdot \ell \cdot \epsilon}{d}\right)^d$$

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Putting all Together

In our context, $\ell \leq \mathcal{O}\left(d^2\right)$ (for $\otimes$ gadget), and $\epsilon \leq \delta \cdot |\mathbb{F}|$:

Theorem (Security Bound)

For a single gadget with $\ell \leq \mathcal{O}\left(d^2\right)$ intermediate computations:

$$\mathsf{SD}\left(k; \mathbf{L}\right) \leq \left(\mathcal{O}\left(d\right) \cdot \delta \cdot |\mathbb{F}|\right)^d$$

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$$\mathsf{SD}(k; \mathbf{L}) \leq (\mathcal{O}(d) \cdot \delta \cdot |\mathbb{F}|)^d$$

For the whole circuit $\mathbb{C}$,

$$\mathsf{SD}(k; \mathbf{L}) \leq (|\mathbb{C}| \cdot \mathcal{O}(d) \cdot \delta \cdot |\mathbb{F}|)^d$$

Reduction from Noisy Leakage to Random Probing (I)

Assume there exists such a simulator $\mathcal{S}$,

Reduction from Noisy Leakage to Random Probing (I)

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$$\begin{aligned} \Pr[\mathcal{S}(x) = l] &= \ldots, \text{ for all } x \\ \Pr[\mathcal{S}(\perp) = l] &= \ldots \end{aligned}$$

Constraints:

Reduction from Noisy Leakage to Random Probing (I)

Assume there exists such a simulator $\mathcal{S}$, we need to construct it for all inputs:

$$\Pr[\mathcal{S}(x) = l] = \ldots, \text{ for all } x$$
$$\Pr[\mathcal{S}(\perp) = l] = \ldots$$

Constraints:

$\rightarrow$ For all input x, $\Pr[\mathcal{S}(x)]$ should be a p.m.f.

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→ For all input x, $\Pr[\mathcal{S}(x)]$ should be a p.m.f.

→ For the input $\perp$, $\Pr[\mathcal{S}(\perp)]$ should be a p.m.f.

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Constraints:

→ For all input x, $\Pr[\mathcal{S}(x)]$ should be a p.m.f.

→ For the input $\perp$, $\Pr[\mathcal{S}(\perp)]$ should be a p.m.f.

→ For any x, l, $\Pr[\mathcal{S}(\varphi(x)) = l] = \Pr[\mathrm{L}(x) = l]$

Reduction from Noisy Leakage to Random Probing (II)

Let us start from the last constraint. For any x and any l:

$$\Pr[\mathrm{L}(x) = l] \quad = \quad \Pr[\mathcal{S}\left(\varphi(x)\right) = l]$$

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Let us start from the last constraint. For any x and any l:

$$\begin{aligned}\Pr[\mathrm{L}(x) = l] &= \Pr[\mathcal{S}(\varphi(x)) = l] \\ &= \Pr[\varphi(x) = x] \cdot \Pr[\mathcal{S}(x) = l] + \Pr[\varphi(x) = \perp] \cdot \Pr[\mathcal{S}(\perp) = l]\end{aligned}$$

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Hence,

$$0 \leq \Pr[\mathcal{S}(\bot) = l] = \frac{\overbrace{\Pr[\mathrm{L}(x) = l] - \epsilon \cdot \Pr[\mathcal{S}(x) = l]}^{\text{Should not depend on } x}}{1 - \epsilon}$$

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Furthermore, summing (6) over l, by definition of probability distributions,

$$\sum_l \pi(l) = \underbrace{\sum_l \Pr[\mathrm{L}(x) = l]}_{=1} - \epsilon \cdot \underbrace{\sum_l \Pr[\mathcal{S}(x) = l]}_{=1}$$

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Hence,

$$\epsilon = 1 - \sum_l \pi(l) \geq 1 - \sum_l \min_x \Pr[\mathrm{L}(x) = l]$$

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Hence, to have the smallest ϵ,

$$\epsilon = 1 - \sum_l \pi(l) = 1 - \sum_l \min_x \Pr[\mathrm{L}(x) = l]$$

Main Challenge

For the whole circuit $\mathbb{C}$,

$$\mathsf{SD}\left(k; \mathbf{L}\right) \leq \left(|\mathsf{C}| \cdot \mathcal{O}\left(d\right) \cdot |\mathbb{F}| \cdot \delta\right)^{d}$$

Main challenge: get rid of the three factors d, $|\mathsf{C}|$, and $|\mathbb{F}|$

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A few numbers:

$$d(2, 3, 4, ..., 16) \ll |\mathsf{C}|(\approx 10^3, 10^5), |\mathbb{F}|(256, 2^{23}, 2^{50})$$

Bad News

Without further assumption on the circuit, the previous bound is *tight*:

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- d: horizontal attacks ✗
- $|\mathcal{C}|$: horizontal attacks ✗
- $|\mathbb{F}|$: non-uniform wires ✗

Why $|\mathbb{F}|$ is Tight

Consider the following leakage model (with $\delta \approx \frac{2}{|\mathbb{F}|} \cdot \alpha$):

$$L(x) = \begin{cases} x, & \text{with probability } \alpha, \text{ if } x = 0, \\ \bot, & \text{otherwise} \end{cases} . \tag{3}$$

Leakage from a uniform encoding: $\epsilon \leq \frac{|\mathbb{F}|}{2} \cdot (2\delta)^d$ ✓

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→ Each output share uniform over $\{0, 1\}$ instead of $\mathbb{F}$

→ Successful recovery of all shares with probability at least $\alpha^d \approx \left(\frac{\delta|\mathbb{F}|}{2}\right)^d$ ✗

Content

Setting

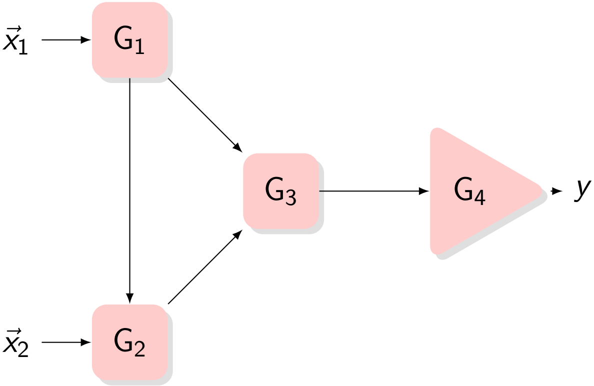
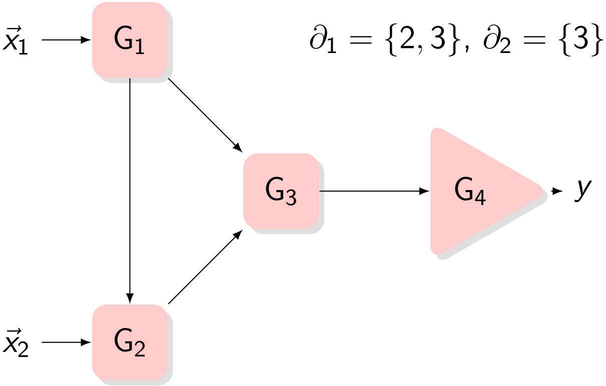


Figure: G_1: SNI copy gadget, G_2, G_3: SNI gadgets, G_4: NIo gadget.

Setting



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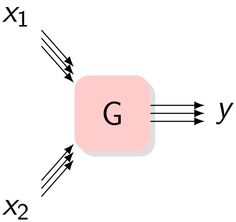
∂_i: set of all subsequent gadgets linked to G_i

Strong Non-Interference[8]

Definition (t-Strong Non-Interference)

A gadget G is t-SNI



[7] Must be connected to different gadgets ✓

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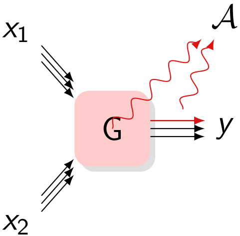
A gadget G is t-SNI if any set W^{G} of internal probes and any set J^{G} of output probes such that $\left|W^{\mathsf{G}}\right| + \left|J^{\mathsf{G}}\right| \leq t$



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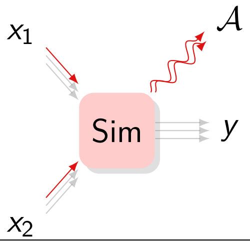

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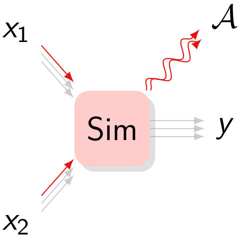

→ Composable : circ. SNI iff all gadgets SNI
→ SNI $\implies$ probing security
→ Extends to multiple outputs[7]

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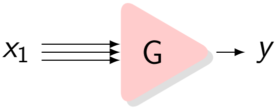
Non-Interference with Public Outputs[10]

Definition (t-NIo)

A gadget is t-NIo



[9] Coron et al., *High-order Polynomial Comparison and Masking Lattice-based Encryption*

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Non-Interference with Public Outputs[10]

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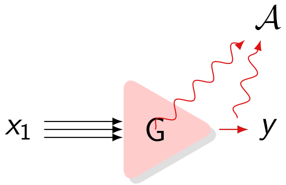

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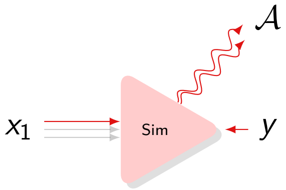
Non-Interference with Public Outputs[10]

Definition (t-NIo)

A gadget is t-NIo if any set of $t_1 \leq t$ internal probes and the output can be jointly simulated from the output and at most t_1 input shares



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Non-Interference with Public Outputs[10]

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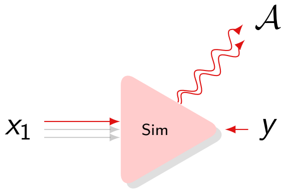

→ Output assumed to be public anyway
→ Built from strong Refreshing [9]

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Composition Theorem

Theorem

Assume: (1) Each output gadget $(d-1)$*-NIo;*

Composition Theorem

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$$\eta \leq \sum_{\substack{i=1 \\ G_i \text{ not output}}}^{|C|} \left(e \cdot \frac{|G_i| + \sum_{j \in \partial_i} |G_j|}{t_i + 1} \cdot \epsilon \right)^{t_i+1} .$$

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Corollary

The d-share ISW compiler is $|C| \cdot \left(\mathcal{O}(d) \cdot |\mathbb{F}| \cdot \delta\right)^d$-noisy leakage secure

Proof Sketch

Apply SNI simulator gadget-wise, in reversed order, until complete or failure

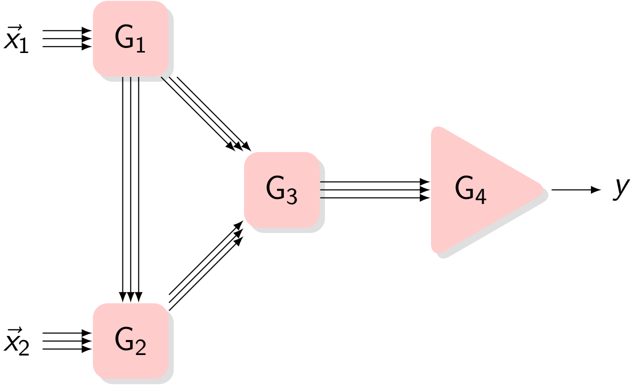

Figure: G_1: SNI copy gadget, G_2, G_3: SNI gadgets, G_4: NIo gadget. $\partial_1 = \{2, 3\}$, $\partial_2 = \{3\}$, $\partial_3 = \{4\}$, $\partial_4 = \emptyset$

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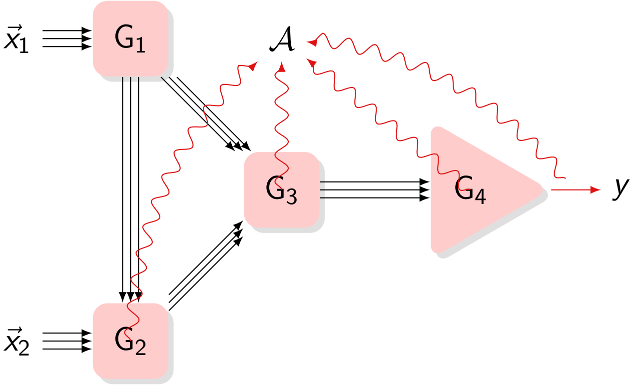

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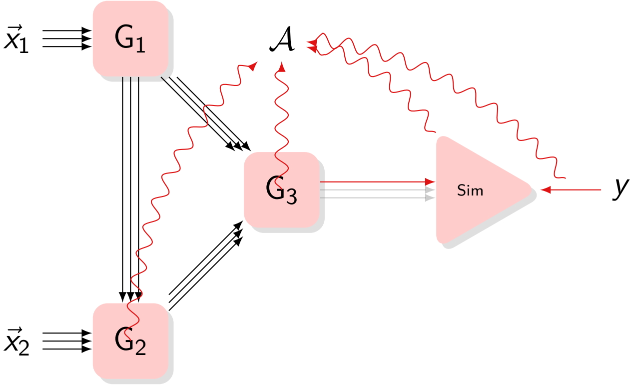

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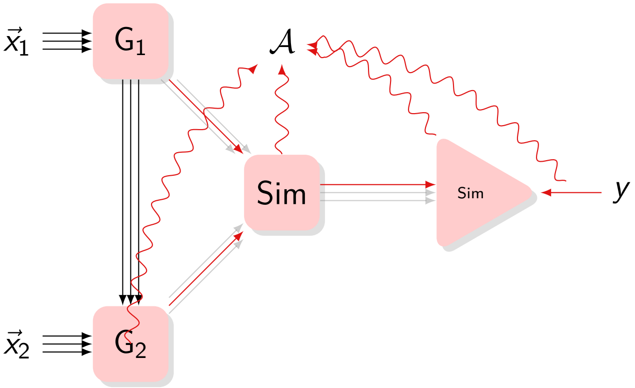

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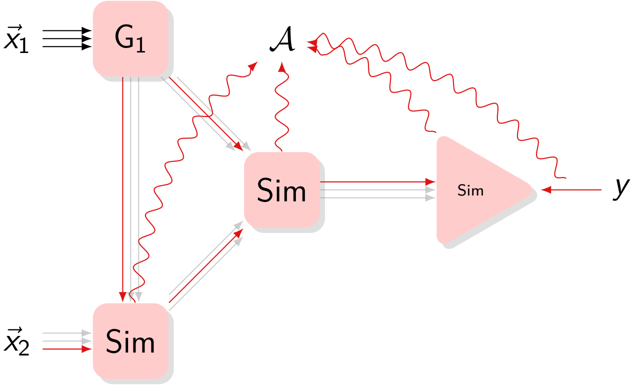

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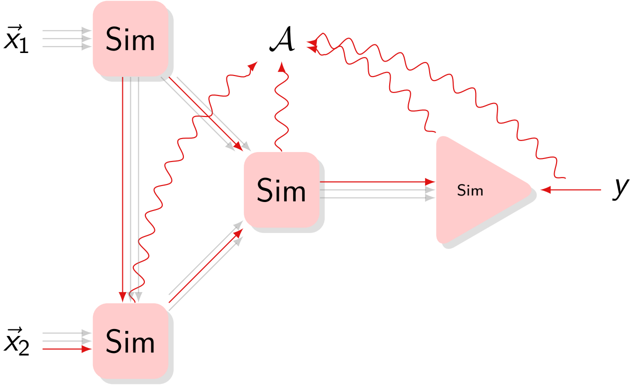

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Proof Sketch

Failure may happen (simulation with abort)

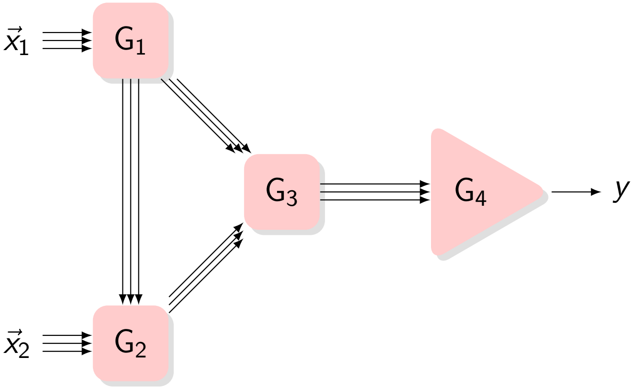

Figure: G_1: SNI copy gadget, G_2, G_3: SNI gadgets, G_4: NIo gadget. $\partial_1 = \{2, 3\}$, $\partial_2 = \{3\}$, $\partial_3 = \{4\}$, $\partial_4 = \emptyset$

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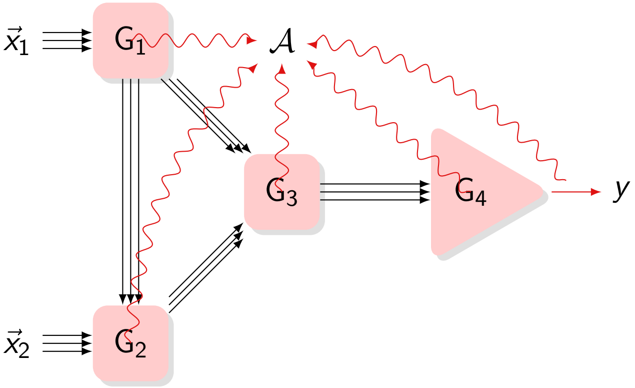

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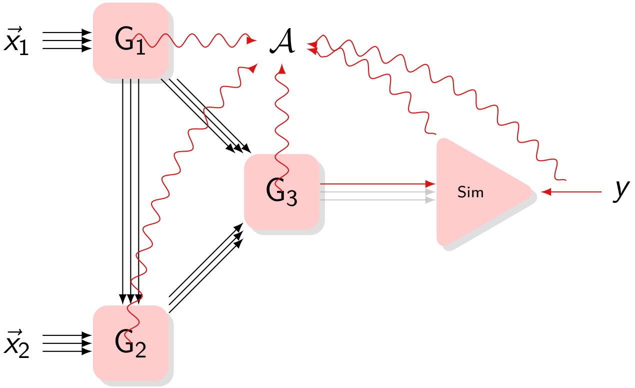

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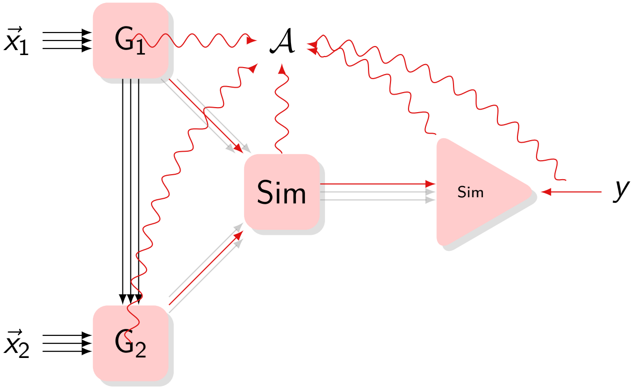

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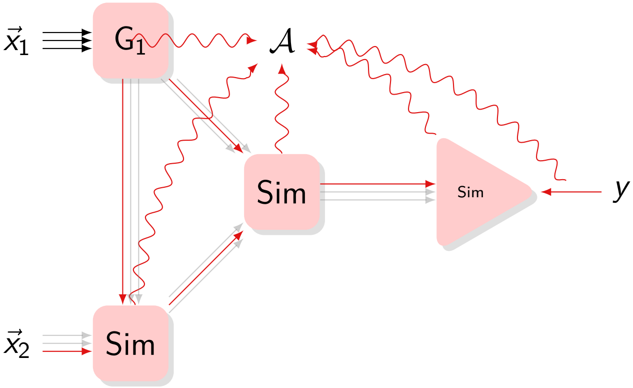

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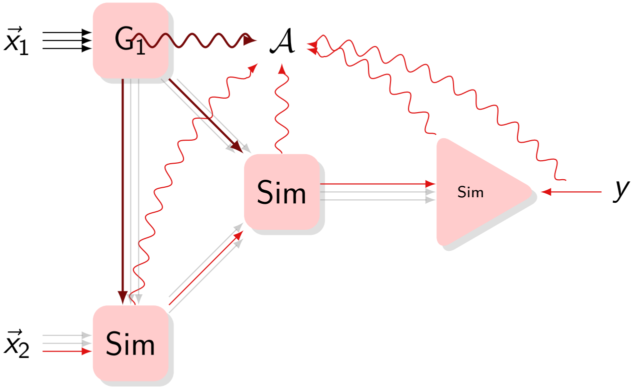

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How Often Does It Fail?

Let bad_i: "simulation failure at step i". This implies:

[11] If G_j is an NIo output gadget, this is also verified.

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$\rightarrow$ $\forall j > i$, t_j-SNI assumption of G_j verified, thereby $\left|J_j^{G_i}\right| = \left|I_i^{G_j}\right| \leq \left|W^{G_j}\right|$[11]

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Hence,

$$\Pr[\mathsf{bad}_i] \leq \Pr\left[\left|W^{G_i}\right| + \sum_{j \in \partial_i} \left|W^{G_j}\right| \geq t_i\right]$$

Using the union bound:

$$\eta = \sum_{\substack{i=1 \\ G_i \text{not output}}}^{|\mathsf{C}|} \Pr[\mathsf{bad}_i]$$

[11] If G_j is an NIo output gadget, this is also verified.

Concluding the Proof

Using Chernoff:

$$\Pr\left[\left|W^{G_i}\right| + \sum_{j\in\partial_i}\left|W^{G_j}\right| > t_i\right] = \Pr\left[\left|W^{G_i} \cup \left(\bigcup_{j\in\partial_i} W^{G_j}\right)\right| \geq t_i + 1\right]$$

$$\leq \left(e \cdot \frac{|G_i| + \sum_{j\in\partial_i}|G_j|}{t_i + 1} \cdot \epsilon\right)^{t_i+1} .$$

Comparison with Previous Works

So far, trade-off was needed:

→ Duc *et al.*:[12] $|C| \cdot \left(\mathcal{O}(d) \cdot |\mathbb{F}| \cdot \delta\right)^{d/2}$

→ Belaïd *et al.*:[13] $|C| \cdot \left(\mathcal{O}(1) \cdot |\mathbb{F}| \cdot \delta\right)^{\approx d/3}$

→ Eurocrypt'25, Asiacrypt'25: tighter composition (yet more complex)

[12] Duc, Dziembowski, and Faust, "Unifying Leakage Models: From Probing Attacks to Noisy Leakage".

[13] Taleb, "Secure and Verified Cryptographic Implementations in the Random Probing Model. (Implémentations cryptographiques sûres et vérifiées dans le modèle random probing)".

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Constraints:

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→ For any x, l, $\Pr[\mathcal{S}(\varphi(x)) = l] = \Pr[\mathrm{L}(x) = l]$

Reduction from Noisy Leakage to Random Probing (II)

Let us start from the last constraint. For any x and any l:

$$\Pr[\mathrm{L}(x) = l] \;=\; \Pr[\mathcal{S}\left(\varphi(x)\right) = l]$$

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&= \epsilon \cdot \Pr[\mathcal{S}\left(x\right) = l] + (1 - \epsilon) \cdot \Pr[\mathcal{S}\left(\bot\right) = l]
\end{aligned}$$

Hence,

$$0 \leq \Pr[\mathcal{S}\left(\bot\right) = l] = \frac{\overbrace{\Pr[\mathrm{L}(x) = l] - \epsilon \cdot \Pr[\mathcal{S}\left(x\right) = l]}^{\text{Should not depend on } x}}{1 - \epsilon}$$

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Hence,

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Is there any ϵ such that $\geq$ and $\geq$ are valid?

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$$0 \leq \pi(l) \leq \Pr[\mathrm{L}(x) = l] \text{ for any } x$$

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Is there any ϵ such that $\geq$ and $\geq$ are valid? From (6), and (7), we get

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In other words,

$$0 \leq \pi(l) \leq \min_{x} \Pr[\mathrm{L}(x) = l]$$

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Furthermore, summing (6) over l, by definition of probability distributions,

$$\sum_l \pi(l) = \underbrace{\sum_l \Pr[\mathrm{L}(x) = l]}_{=1} - \epsilon \cdot \underbrace{\sum_l \Pr[\mathcal{S}(x) = l]}_{=1}$$

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Hence,

$$\epsilon = 1 - \sum_l \pi(l) \geq 1 - \sum_l \min_x \Pr[\mathrm{L}(x) = l]$$

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Hence, to have the smallest ϵ,

$$\epsilon = 1 - \sum_l \pi(l) = 1 - \sum_l \min_x \Pr[\mathrm{L}(x) = l]$$

Reduction to Average Random Probing (I)

For any x and any l:

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\end{aligned}
$$

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&= \epsilon_x \cdot \Pr[\mathcal{S}(x) = l] + (1 - \epsilon_x) \cdot \Pr[\mathcal{S}(\bot) = l]
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Hence, provided that $\epsilon_x < 1$,

$$0 \leq \Pr[\mathcal{S}\left(\perp\right) = l] = \frac{\overbrace{\Pr[\mathrm{L}(x) = l] - \epsilon_x \cdot \Pr[\mathcal{S}\left(x\right) = l]}^{\text{Should not depend on } x}}{1 - \epsilon_x}$$

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$$0 \leq \pi(l, x) \leq \Pr[\mathrm{L}(x) = l] \text{ for any } x$$

So (3) gives

$$\Pr[\mathcal{S}(\perp) = l] \leq \frac{\Pr[\mathrm{L}(x) = l]}{1 - \epsilon_x} \text{ for any } x \text{ s.t. } \epsilon_x < 1$$

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And (3) also gives

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Characterization of ARP-simulable Leakages

Furthermore, summing (6) over l, by definition of probability distributions,

$$\sum_l \pi(l, x) = \underbrace{\sum_l \Pr[\mathrm{L}(x) = l]}_{=1} - \epsilon_x \cdot \underbrace{\sum_l \Pr[\mathcal{S}(x) = l]}_{=1}$$

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Hence, the following result

Theorem (ARP-Simulability)

L *is simulable in the* $\{\epsilon_x\}_x$ *average random probing model iff*[14]

$$1 \leq \sum_{l} \min_{x' : \epsilon_{x'} < 1} \left\{ \frac{\Pr[\mathrm{L}(x') = l]}{1 - \epsilon_{x'}} \right\}$$

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