Approximating maximum integral multiflows on bounded genus graphs

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Problem 3

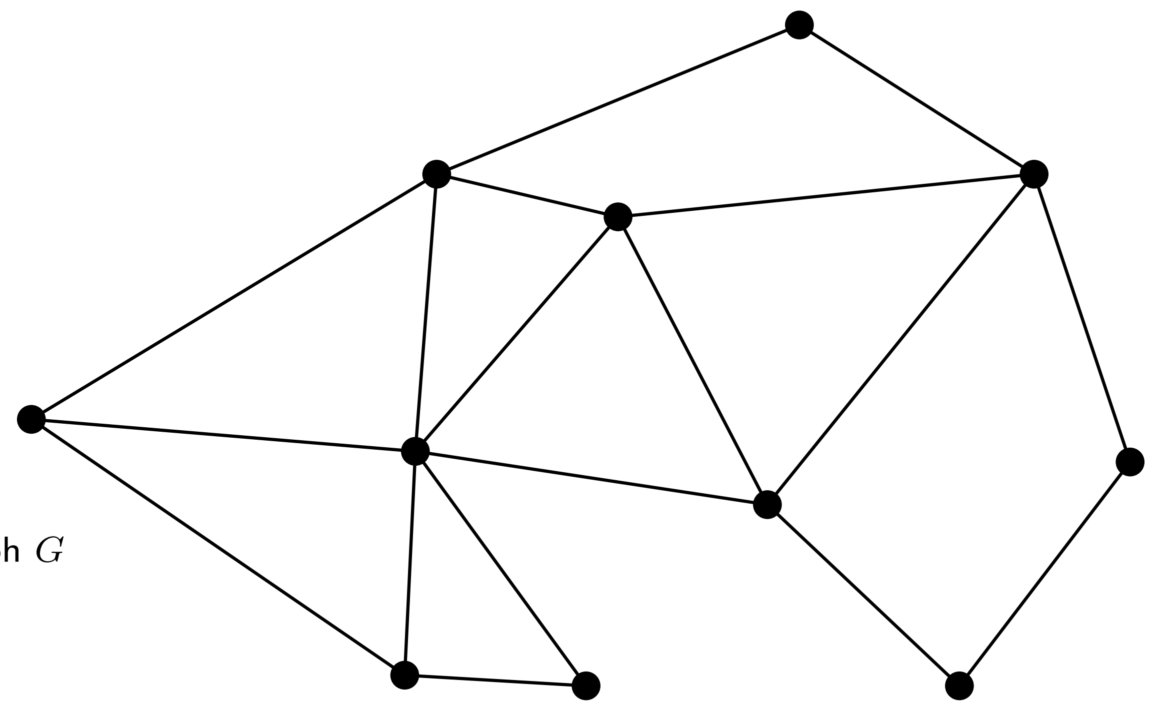
The Edge Disjoint Paths Problem

Input: A *supply* graph G, and a family H of *demand* pairs of vertices $(s_1, t_1), \ldots, (s_k, t_k)$.

Output: A family of edge-disjoint paths in G.

Goal: Maximize the number of pairs (s_i, t_i) connected by a path.

example.



the supply graph G

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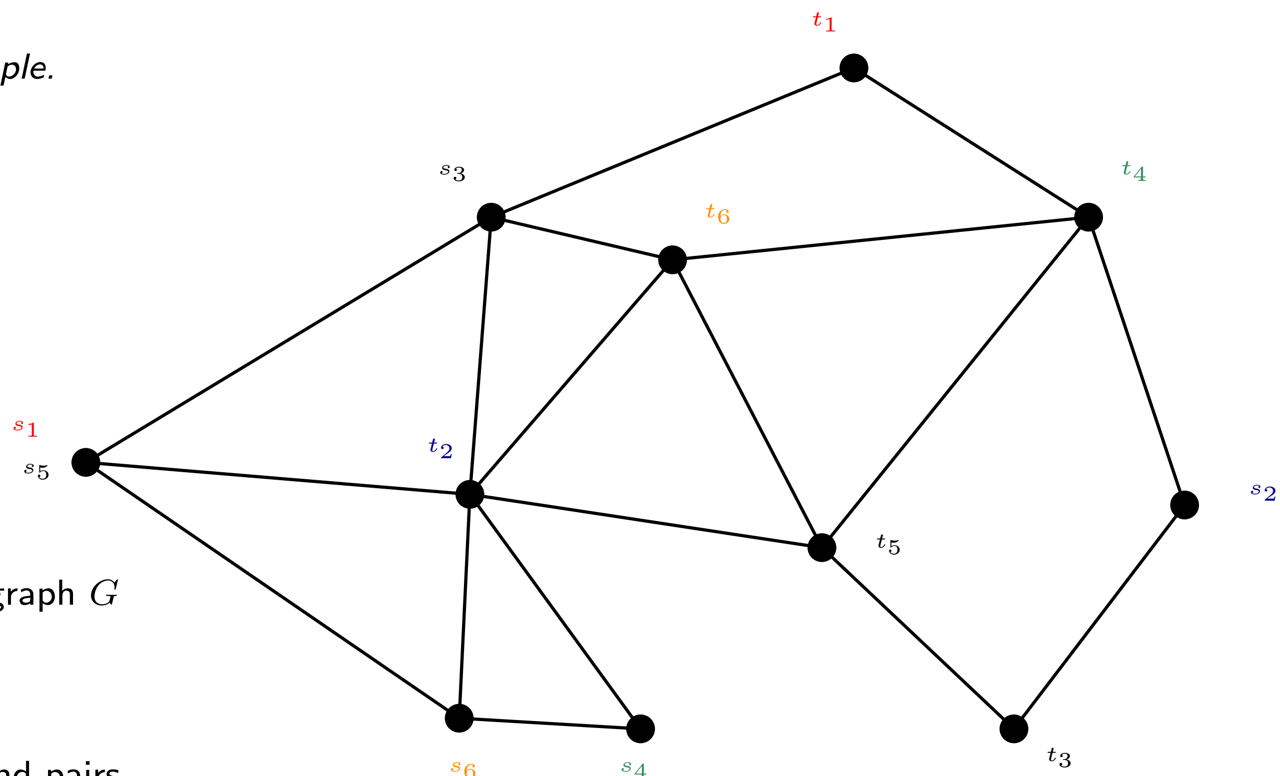

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$H = 6$ demand pairs

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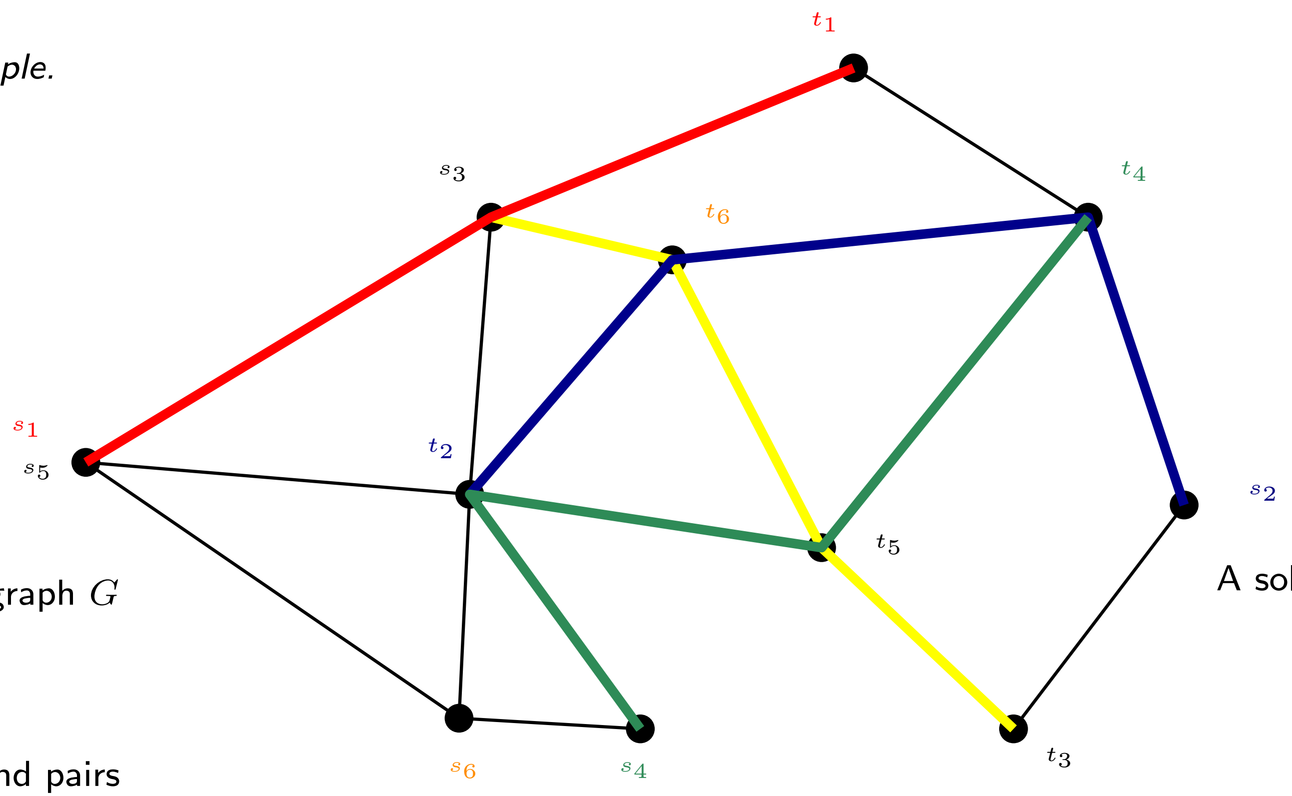

the supply graph G

A solution with 4 paths

$H = 6$ demand pairs

Previous work.

Hardness.

Even when G is planar sub-cubic

No $2^{o(\sqrt{\log n})}$-approx

(assuming NP$\neq$DTIME$(n^{O(\log n)})$)

[Chutzoy, Kim, Nimavat, 2017]

Best algorithms.

$O(\sqrt{n})$-approx

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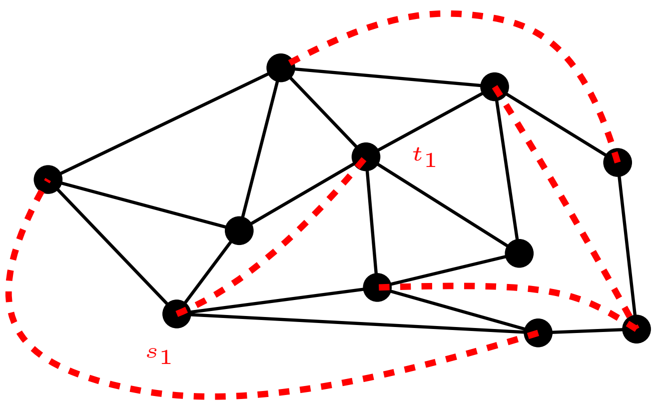

$$G + H$$

- When the **supply graph** + **demand edges** is planar.

Theorem. [Seymour, 1981] When $G + H$ is planar and Eulerian, the cut condition is sufficient.

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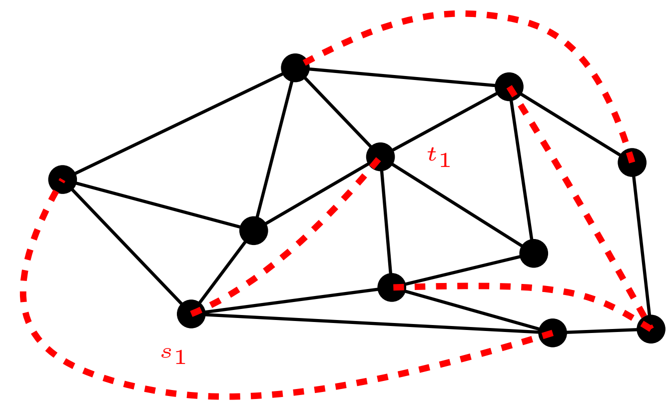

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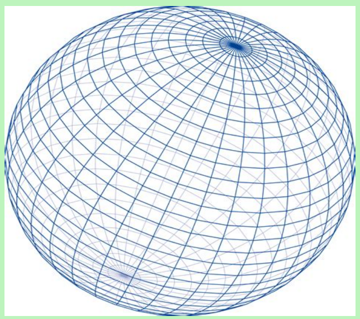
Theorem. When $G + H$ is planar, there is a $O(1)$-approximation algorithm for the Maximum integral Multiflow problem .

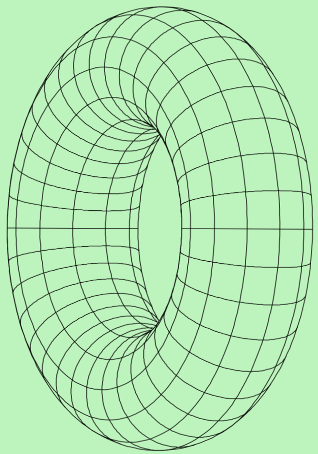
[Huang, M., Mathieu, Schewior, Vygen 2020] [Garg, Kumar, Sebo, 2020]

Surfaces.

Definition: a *surface* is a space that is locally homeomorphic to a disk.

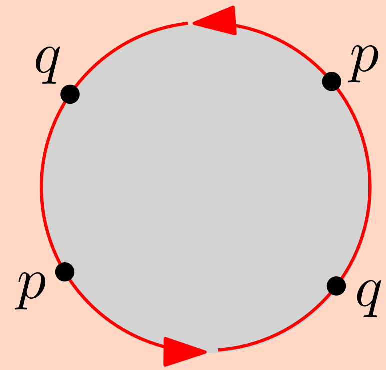


the sphere

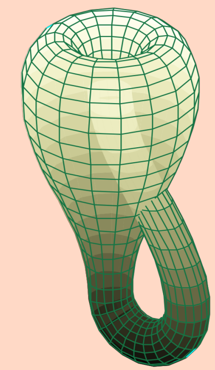


the torus

Orientable



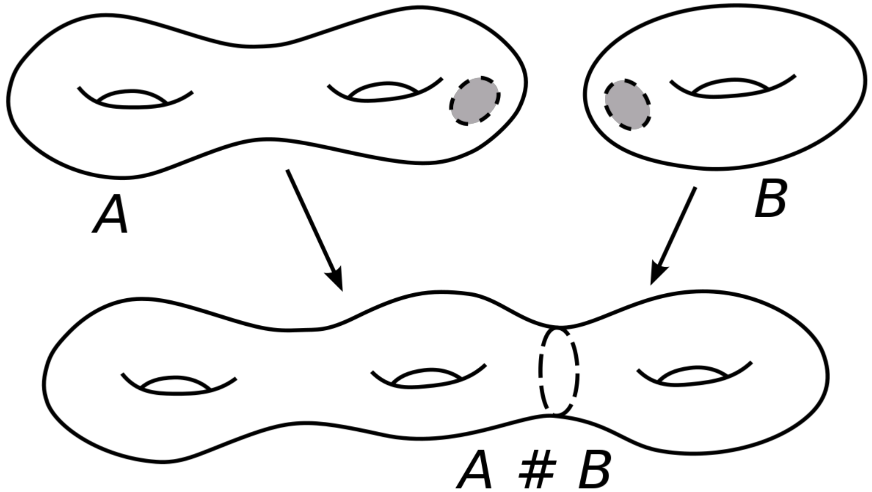
the (real) projective plane



the Klein bottle

Non-orientable

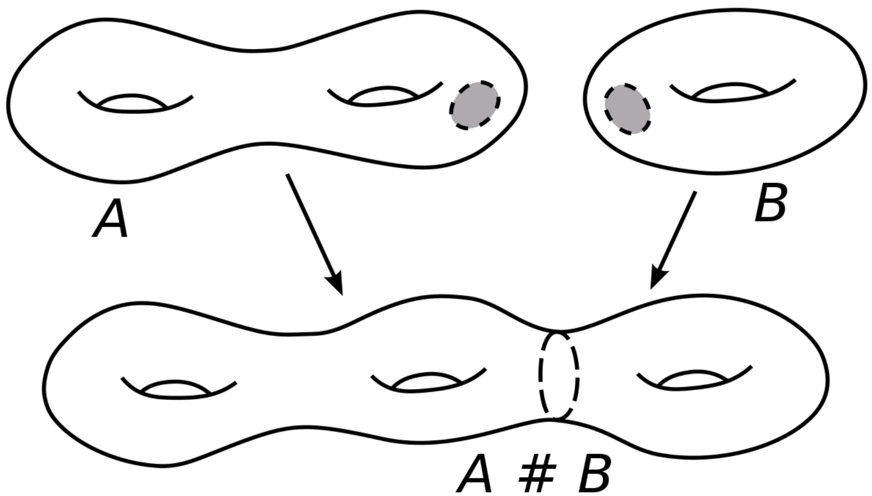
connected sum of two surfaces



Theorem: Surfaces can be classified (up to homeomorphism) into two groups:

- orientable: sphere, torus and connected sum of $g \geq 2$ tori
- non-orientable: projective plane and and connected sum of $g \geq 2$ projective planes.

connected sum of two surfaces



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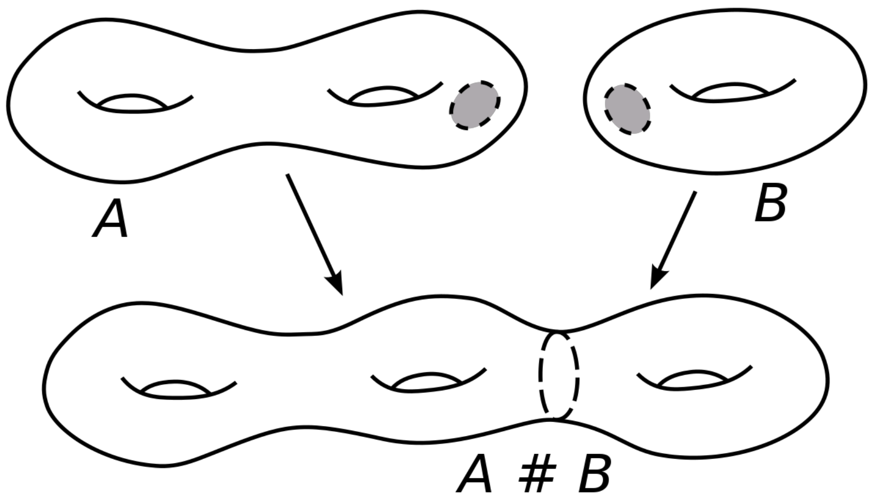

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def: A graph has *genus* g if it can be embedded on an orientable surface of genus g.

$\longrightarrow$ planar graphs = genus-0 graphs

Our contribution.

Theorem.
There is a $O(g^2 \log g)$-approximation algorithm for Edge-disjoint Paths when $G+H$ is a genus-g graph.

→ the ratio between the *optimal fractional solution* and the solution is at most $O(g^2 \log g)$.

→ the algorithm generalizes to arbitrary capacities (maximum integral multiflows)

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Corollary. (Approximate Max-Cut-Min-Flow theorem)

When $G + H$ has bounded genus, the cardinality of a minimum **multicut** is at most a constant times the maximum value of an integer **multiflow**.

- this ratio can be as large as $\Theta(|H|)$ even when G is planar.

Preliminaries

Let $G = (V, E)$ and $H = (V, D)$ such that $G + H$ has genus at most g.

→ w.l.o.g we assume that G+H is a connected graph.

Let $\mathcal{C}$ denote the set of *D*-cycles in $G + H$ that contain exactly one edge in D.

EDP ILP

$$\max \sum_{C \in \mathcal{C}} f_C$$

$$\sum_{C \in \mathcal{C}, e \in C} f_C \leq 1 \quad \text{for all } e \in D \cup E$$

$$f_C \in \{0, 1\} \quad \text{for all } C \in \mathcal{C}$$

LP-relaxation

$$\max \sum_{C \in \mathcal{C}} f_C$$

$$\sum_{C \in \mathcal{C}, e \in C} f_C \leq 1 \quad \text{for all } e \in D \cup E \tag{1}$$

$$f_C \geq 0 \quad \text{for all } C \in \mathcal{C}$$

def: Given a feasible solution f to (1), we define its *support* as $\mathcal{C}(f) := \{C \in \mathcal{C}, f_C > 0\}$.

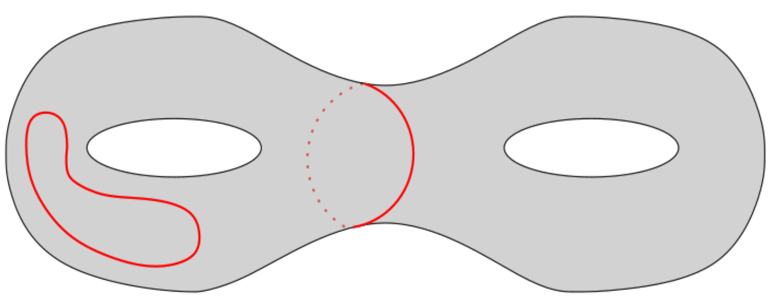
The algorithm.

Step 1. solve the linear program to obain an optimal fractional solution.

Step 2. Uncross the solution to obain another fractional solution $\bar{f}$ such that any two cycles in the support of $\bar{f}$ cross at most once.

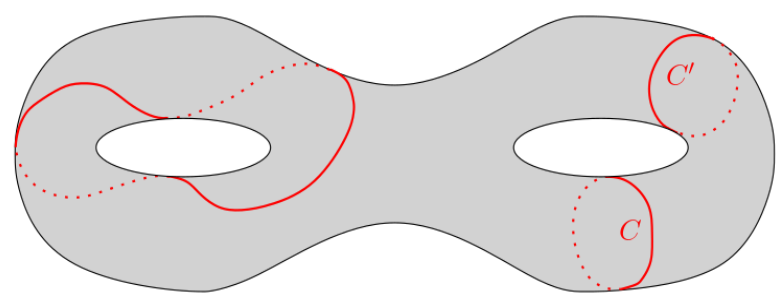
Step 3. If at most half of the value is contributed by separating cycles, construct an integral solution of size at least $\bar{f}/O(\sqrt{g})$.

(similar to the planar case)



separating cycles

Step 3'. If at most half of the value is contributed by non-separating cycles, construct an integral solution of size at least $\bar{f}/O(g^2 \log g)$.



non-separating cycles

Step 1: compute a maximum fractional solution f^*

LP-relaxation

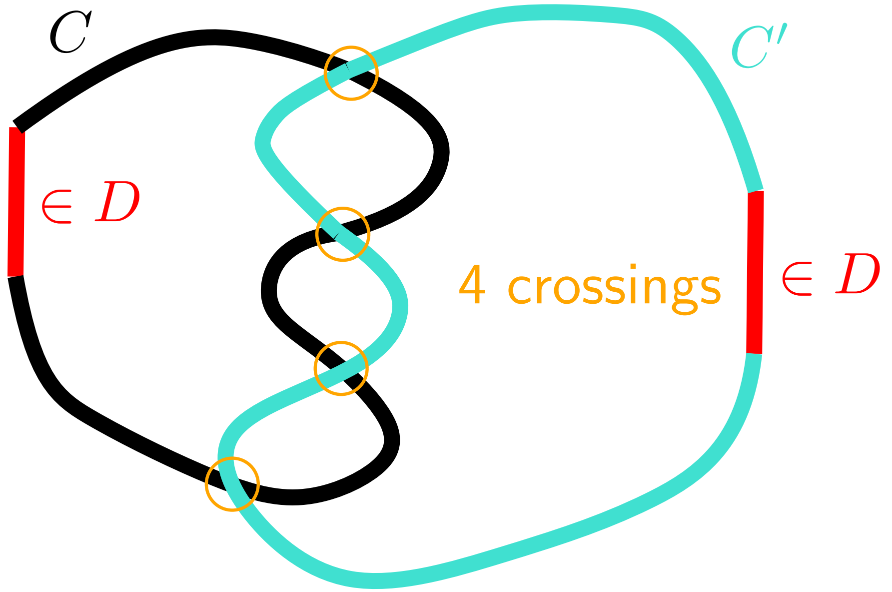
$$\begin{aligned} &\max \textstyle\sum_{C \in \mathcal{C}} f_C & & \\ &\textstyle\sum_{C \in \mathcal{C}, e \in C} f_C \leq 1 & & \text{for all } e \in D \cup E \\ &f_C \geq 0 & & \text{for all } C \in \mathcal{C} \end{aligned} \qquad (1)$$

$\approx$

$$\begin{aligned} &\max \textstyle\sum_{d \in D} x_d^d & & \\ &\textstyle\sum_{d \in D} \sum_{e = uv \in E \cup D} x_e^d \leq 1 & & \text{for all } u \in V \\ &x_e^d \geq 0 & & \text{for all } (e, d) \in (E \cup D, D) \end{aligned}$$

Step 2: making the flow f^* minimally crossing.

Goal: another fractional solution $\bar{f}$ such that any two D-cycles in the support of $\bar{f}$ cross at most once.



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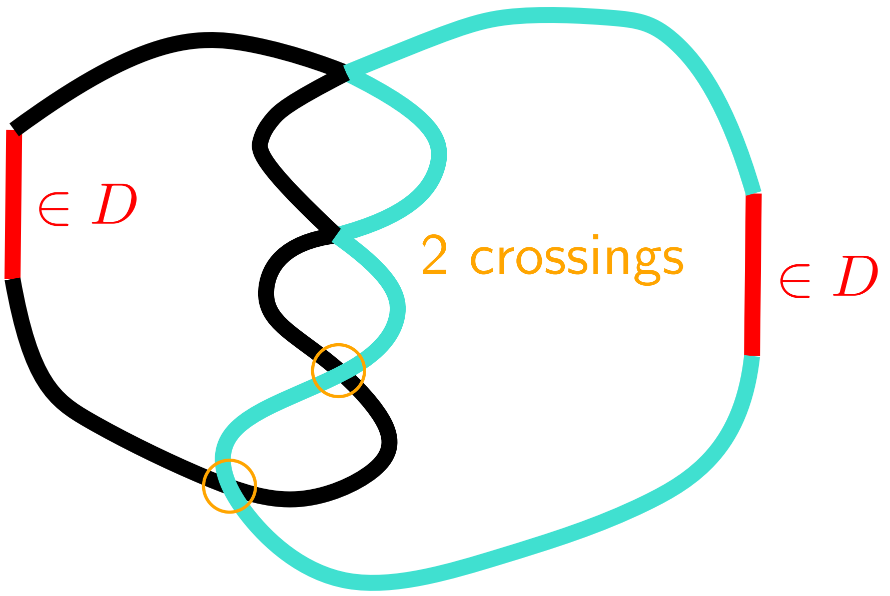

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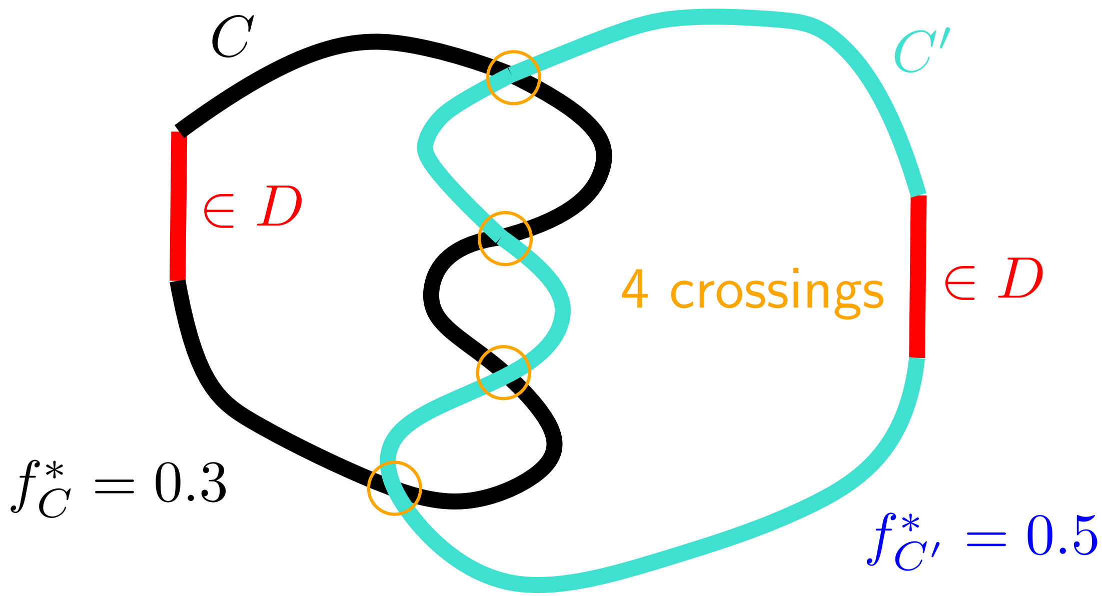

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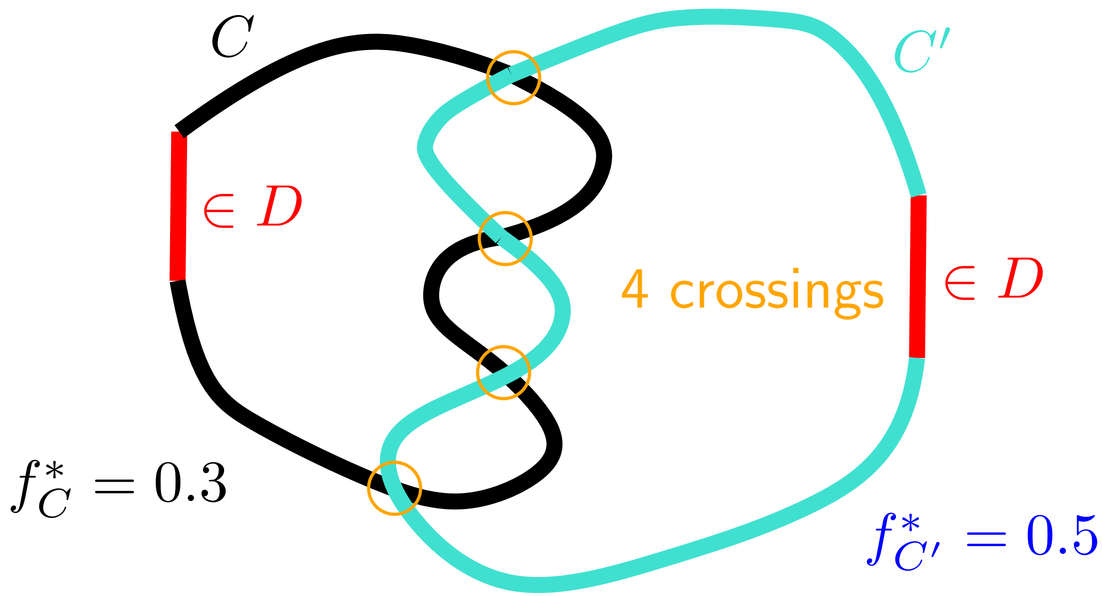

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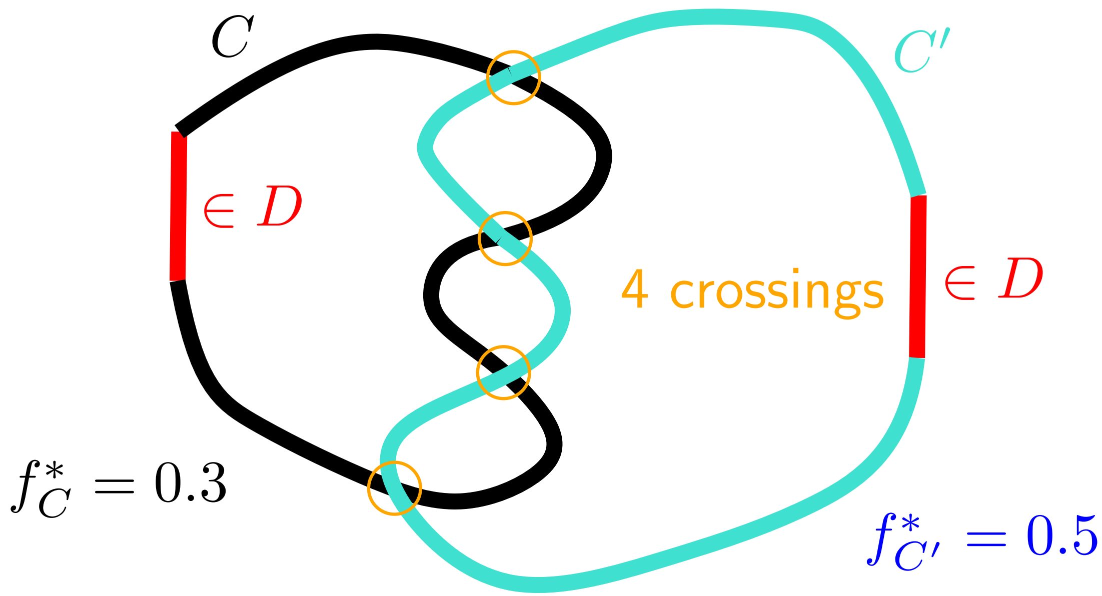

difficulties:

- D-cycles have weights
- uncrossing must must preserve one demand edge per cycles
- the overall numer of crossings might not decrease

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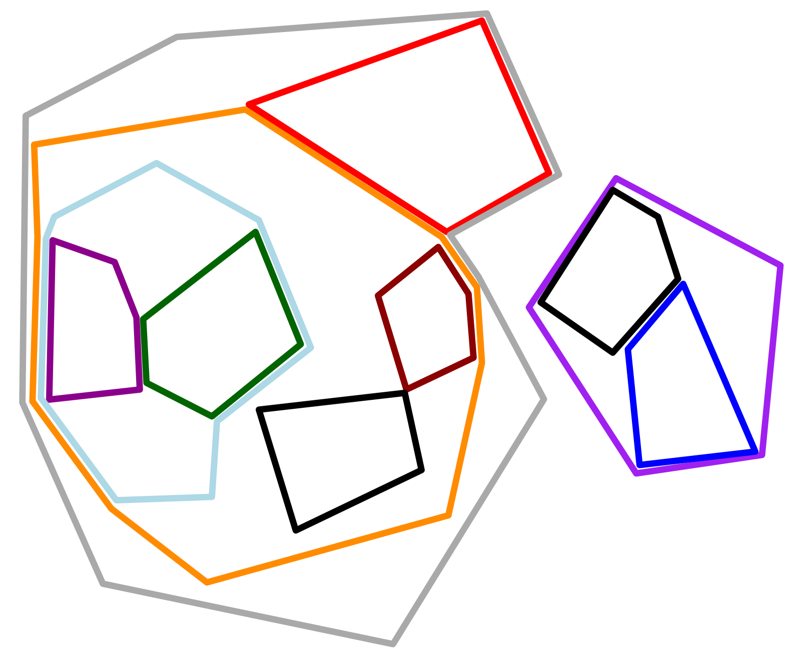

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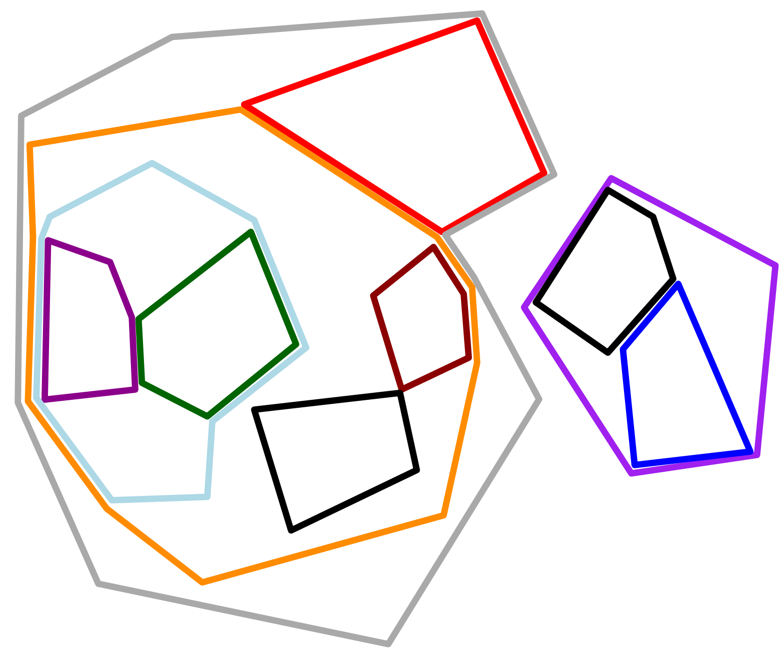

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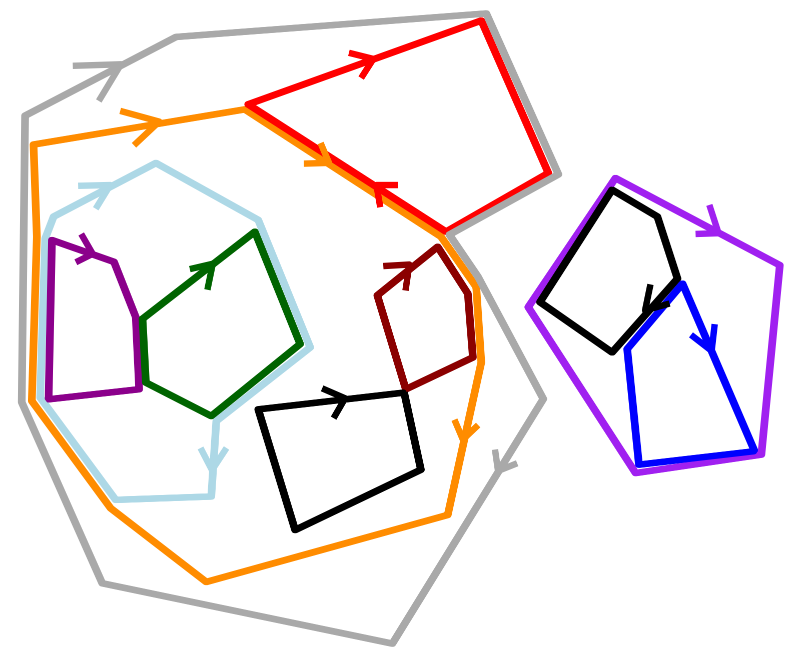

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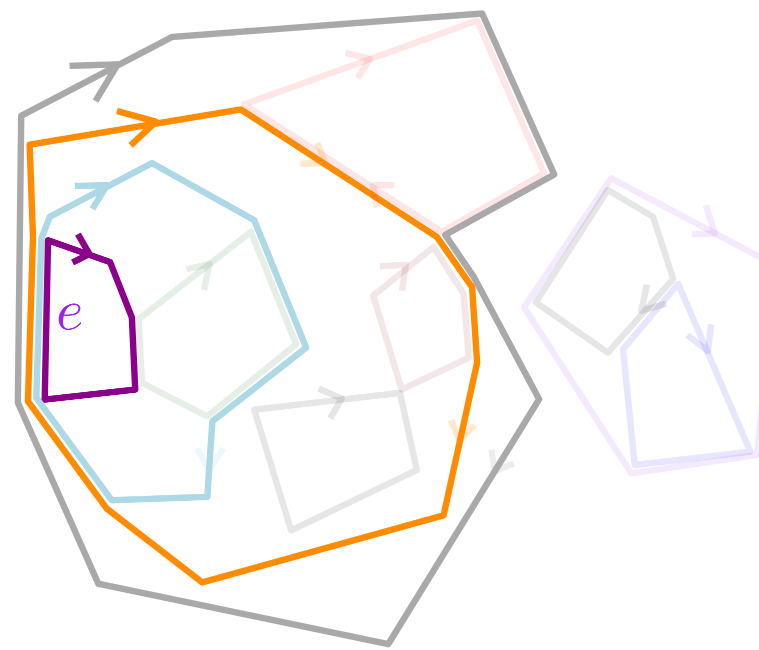

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→ cycles in $\mathcal{C}(f^{half})$ may not be edge-disjoint

- Construct the intersection graph I with vertex set $\mathcal{C}(f^{half})$ such that C and C' are adjacent if there contain common edges.

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Theorem [Heawood, 1890]: A genus-g graph has a vertex of degree at most $\lfloor \frac{7+\sqrt{1+48g}}{2} \rfloor$.

- Using the classic greedy coloring algorithm we can compute an independent set of size at least $|\mathcal{C}(f^{half})|/O(\sqrt{g})$

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Conclusion: We obtained a integral solution of at least $|f^*|/O(\sqrt{g})$ edge-disjoint paths, where $|f^*|$ is the value of the optimal fraction solution.

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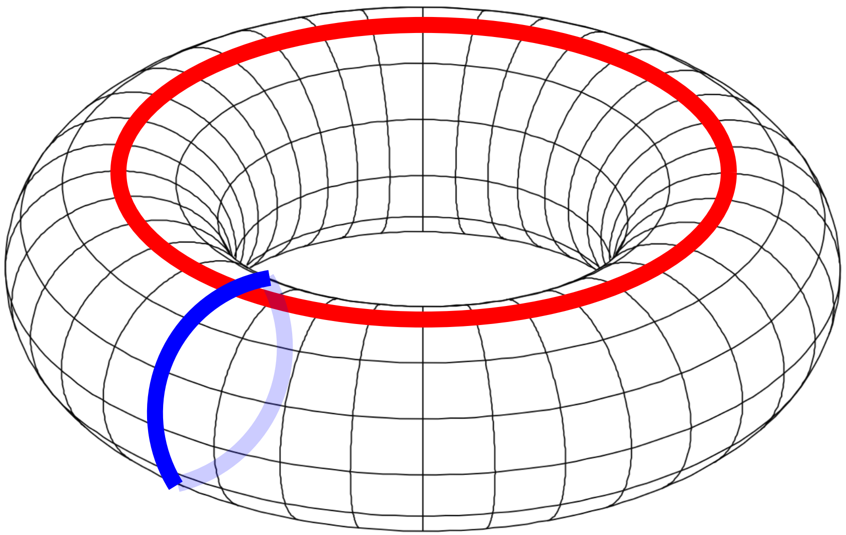

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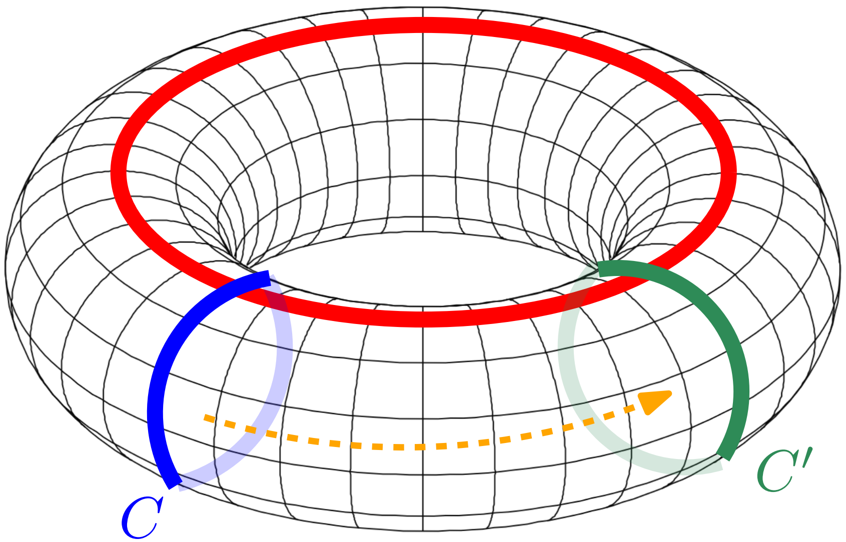

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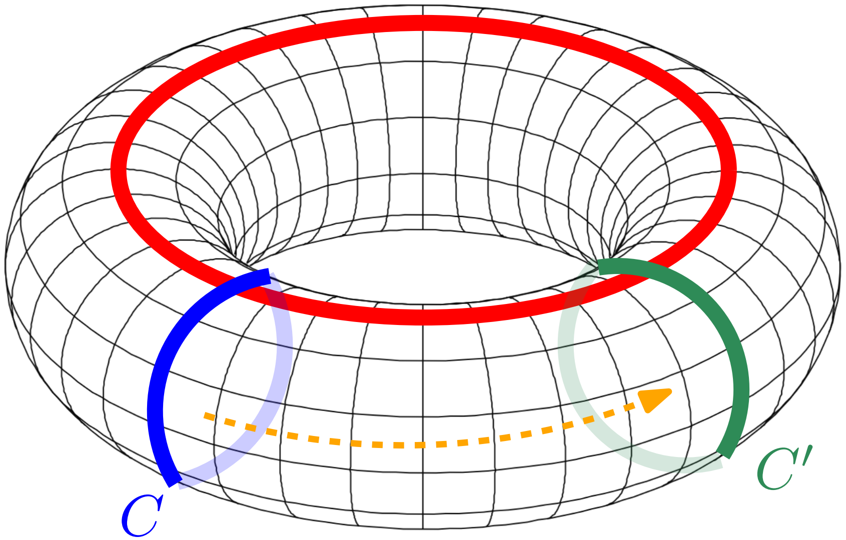

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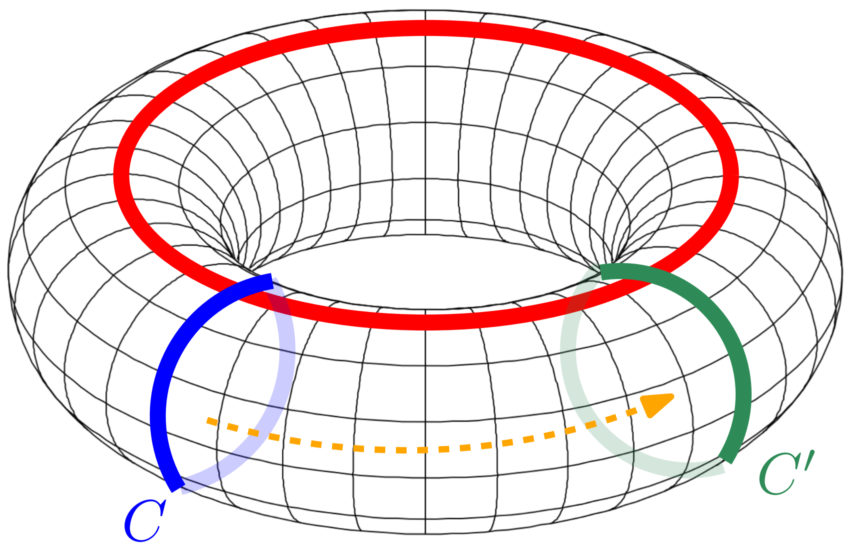

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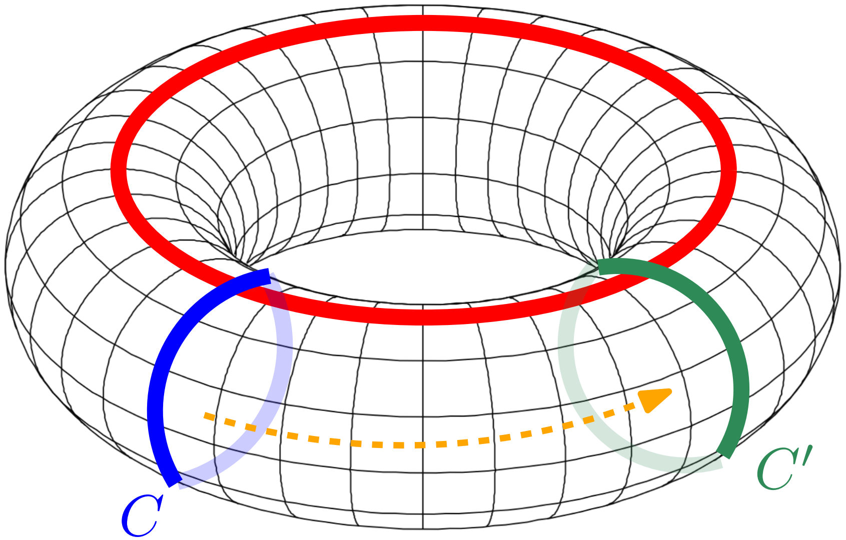

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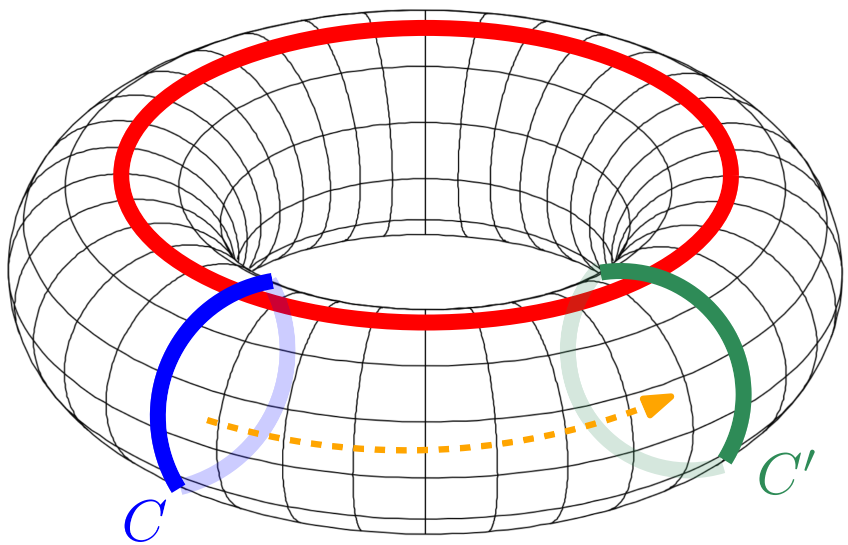

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conclusion: we obtained an integral solution with size at least $f^*/O(g^2 \log g)$

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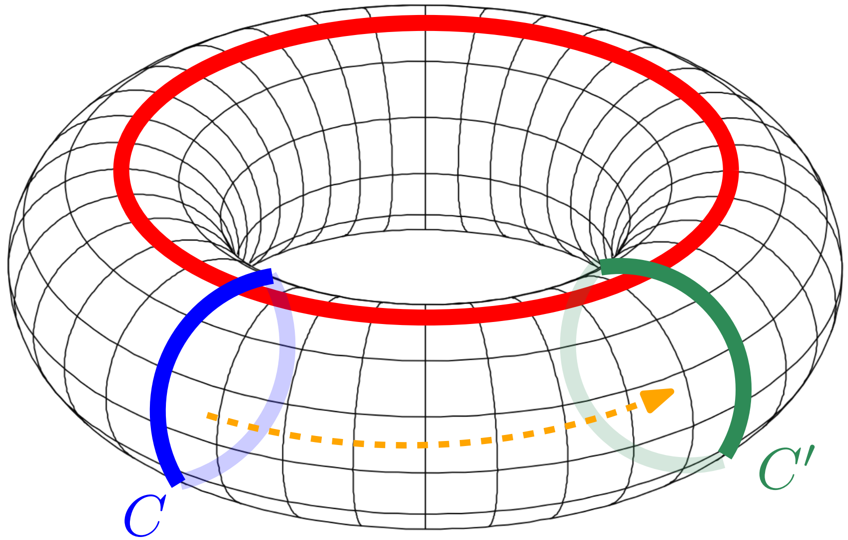

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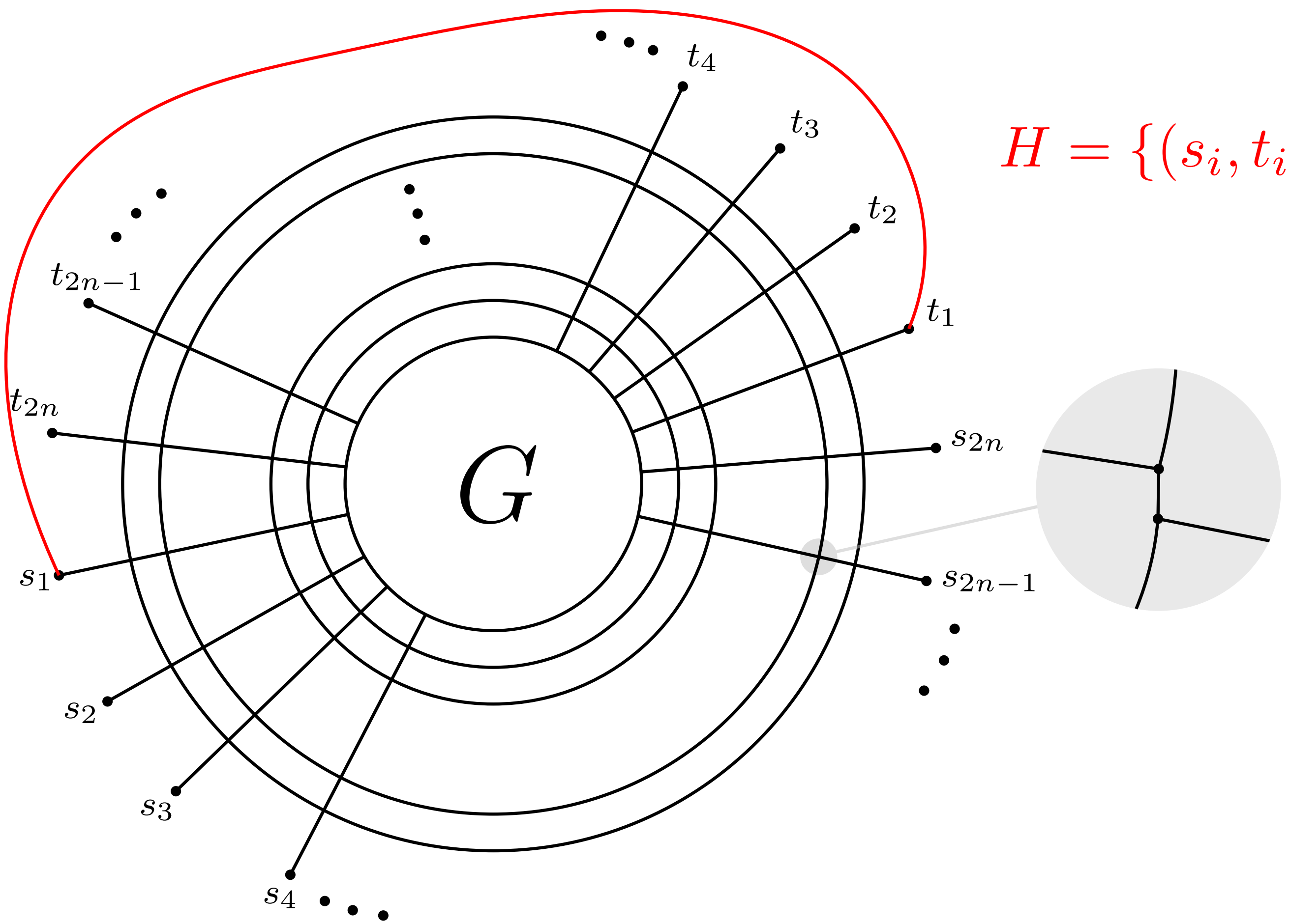
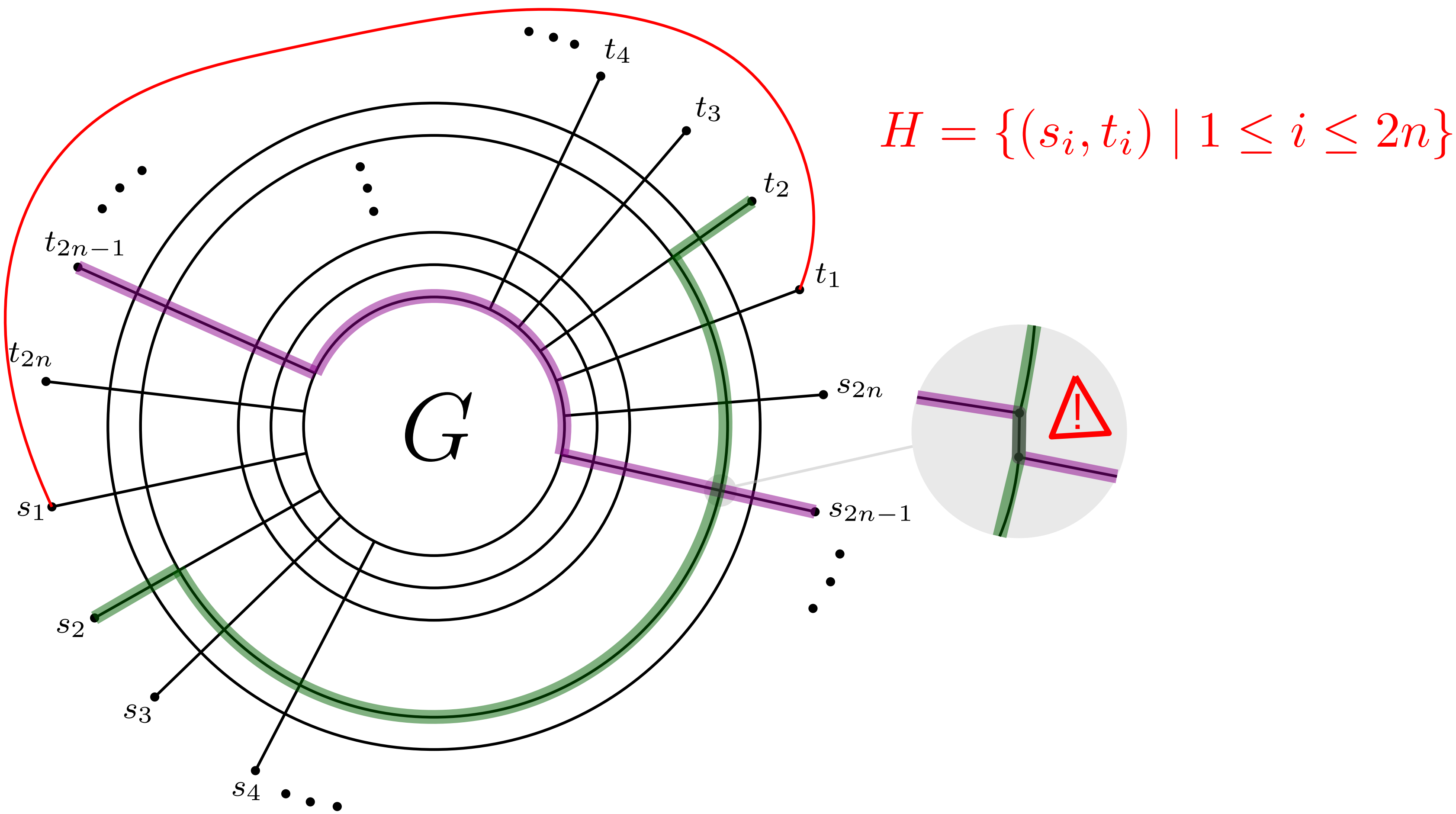
A lower bound on the integrality gap



$H = \{(s_i, t_i) \mid 1 \leq i \leq 2n\}$

A lower bound on the integrality gap



1) The integrality gap of the Edge-disjoint path LP is n

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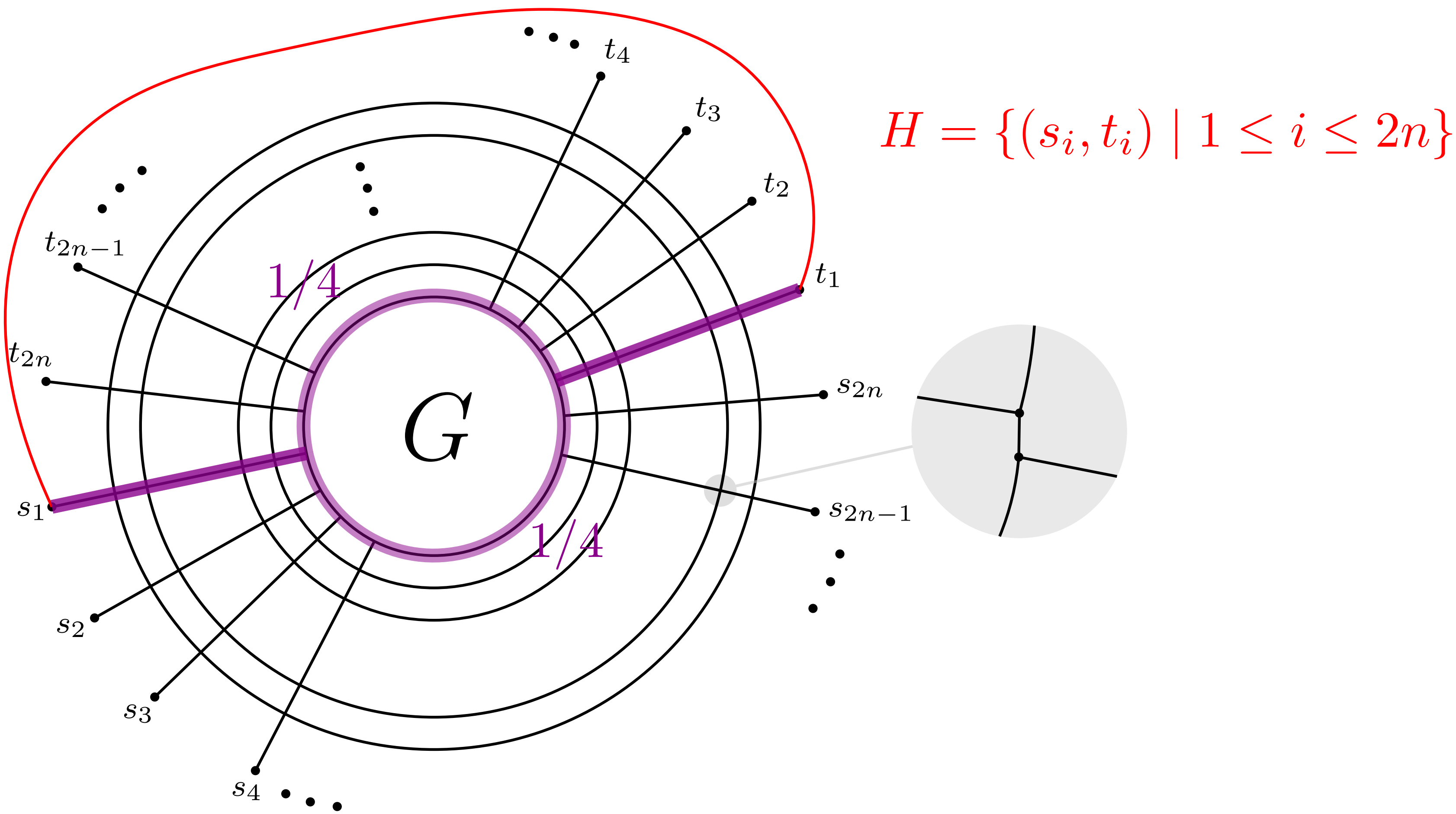

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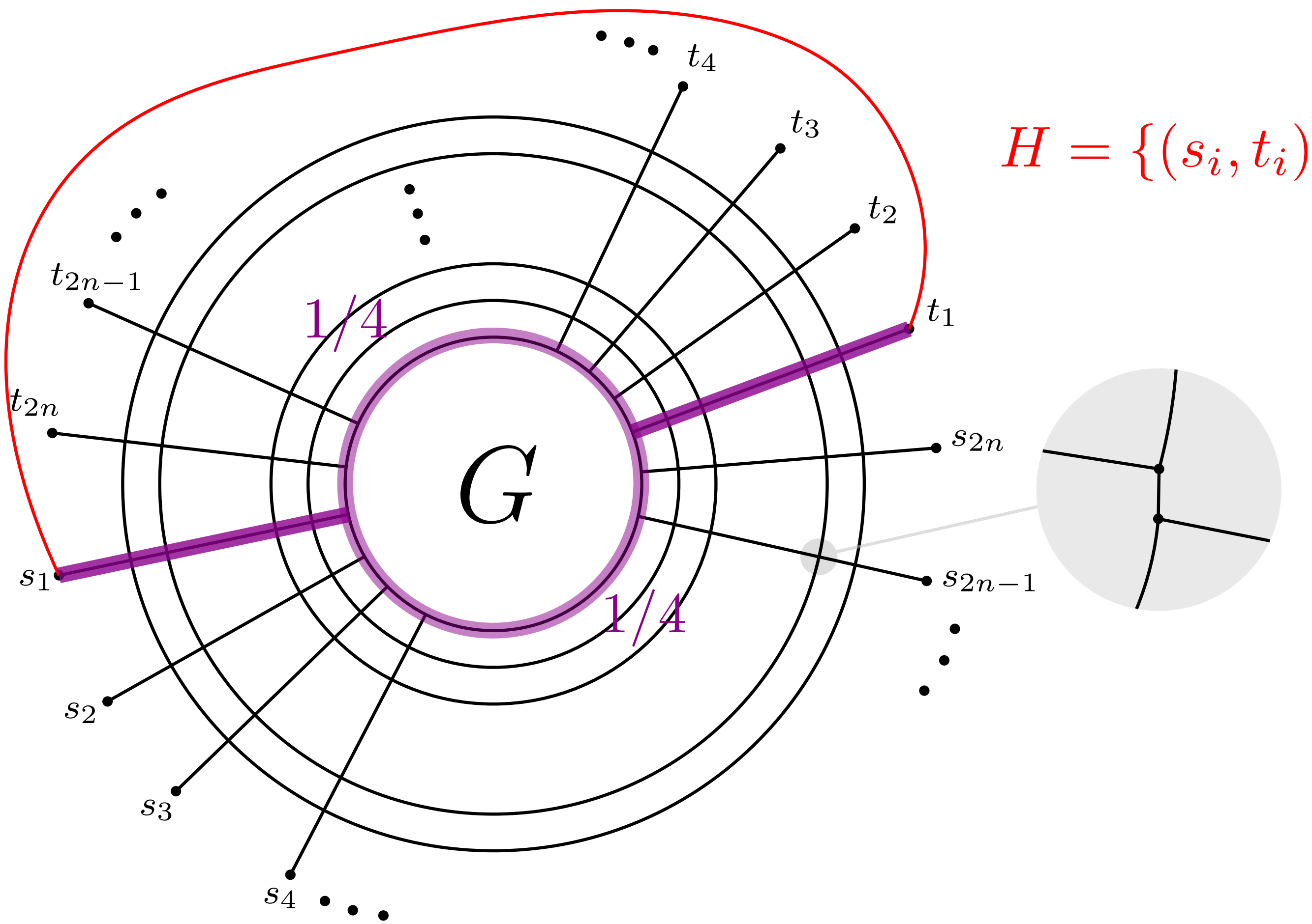

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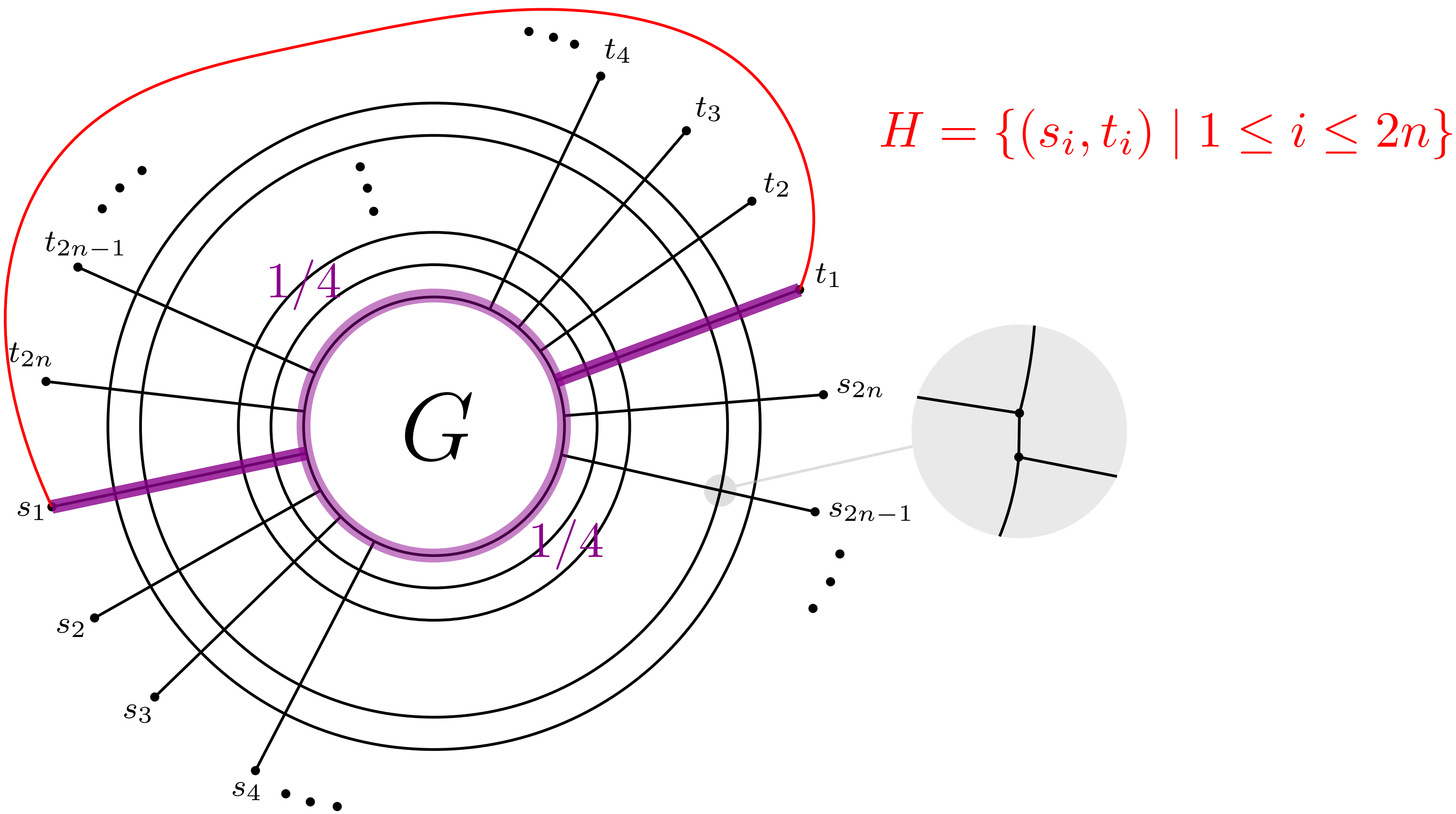

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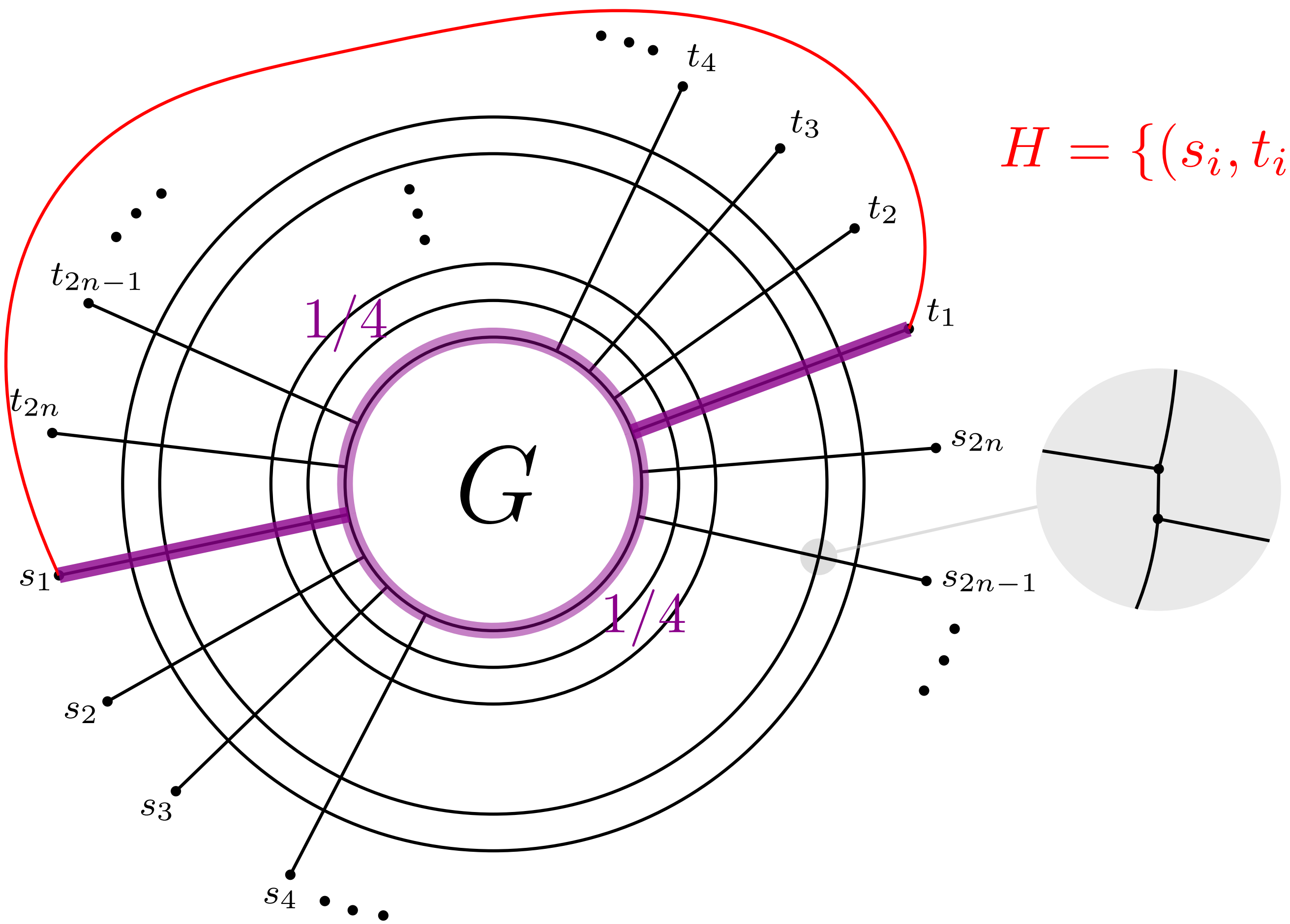

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Open question: A good approximation ratio when $G + H$ can be embedded in the real projective plane ?

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an c-approximation for Edge-disjoint Path.

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