Ultimate greedy approximation of independent sets in subcubic graphs

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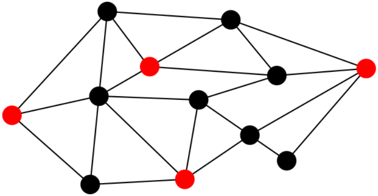
Greedys

Follow the way of best **local** choices in order to reach a good **global** solution.

- simple and natural
- Low processing time
- Efficient

Context : `Maximum Independent Set` **(**`MIS`**)**



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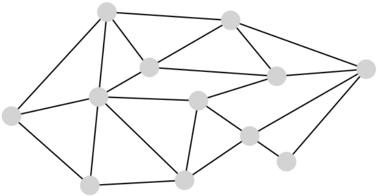

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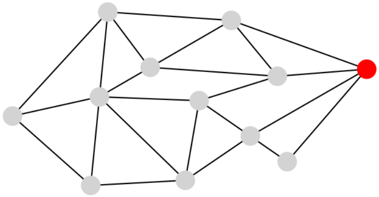

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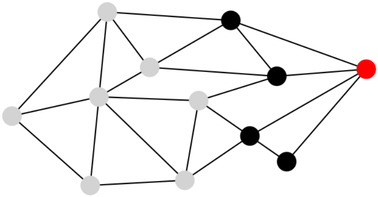

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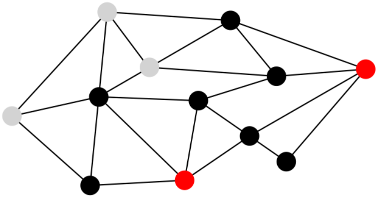

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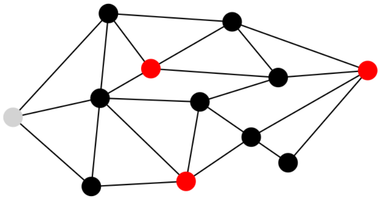

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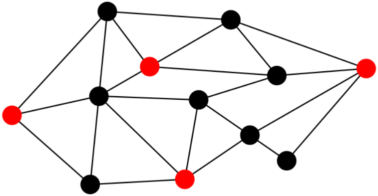

Greedy(G)

While $G \neq \emptyset$:

- Pick $v \in G$ with **minimum degree**.
- Remove v and its neighbours from G.

Guarantees

(Hastad) : `MIS` is hard to approximate within $n^{1-\epsilon}$.

Theorem (Halldórsson and Radhakrishnan, 1994)

Greedy is a (tight) $\dfrac{\Delta+2}{3}$-approximation algorithm for graphs with maximum degree at most Δ.

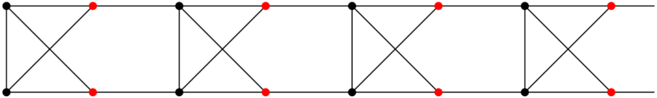

Figure 1: *Tight example with* $\Delta = 4$

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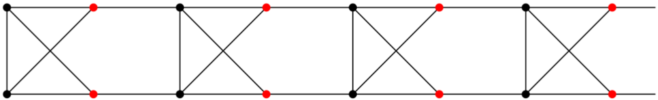


Figure 1: *Tight example with* $\Delta = 4$

$\longrightarrow$ Can we find any better **advised greedy algorithm** ?
(using tie-breaking advices)

Our result for $\Delta \leq 3$

Theorem **[Halldórsson and Radhakrishnan, 1994]**

The (basic) greedy algorithm that has an approximation ratio $5/3 = 1.6666\ldots$

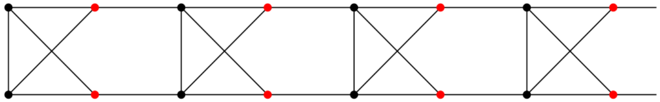
Theorem **[Halldórsson and Yoshihara, 1995]**

There exists an **advised** greedy algorithm, called **MoreEdges** that has an approximation ratio $3/2 = 1.5$.

Our main result

There exists an **advised** greedy algorithm, called **Greedy*** that the **best possible** approximation ratio $5/4 = 1.25$ and runs in time $O(n^2)$.

Too much effort for high degree graphs



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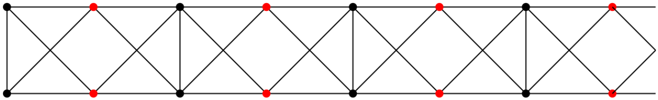


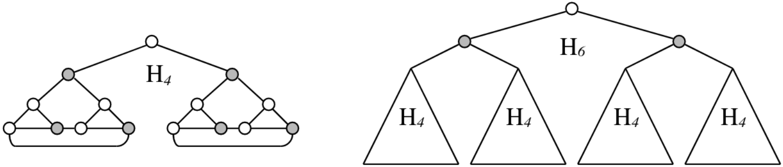
Theorem

Any Greedy algorithm has an approximation ratio at least
$$\frac{\Delta+1}{3} - O(1/\Delta)$$

$\frac{\Delta+1}{3} - O(1/\Delta) \approx \frac{\Delta+2}{3}$ for high degree graphs

(Exercise: Greedy is optimal for graphs with maximum degree at most 2.)

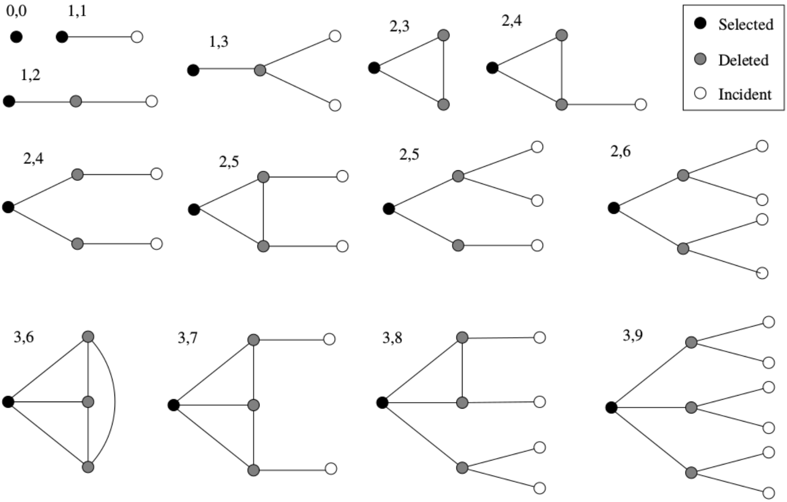
$5/4$ **is tight.** [Halldórsson and Yoshihara, 1995]



$$\frac{\textbf{OPT}(H_i)}{\textbf{Greedy}(H_i)} \longrightarrow \frac{5}{4}$$

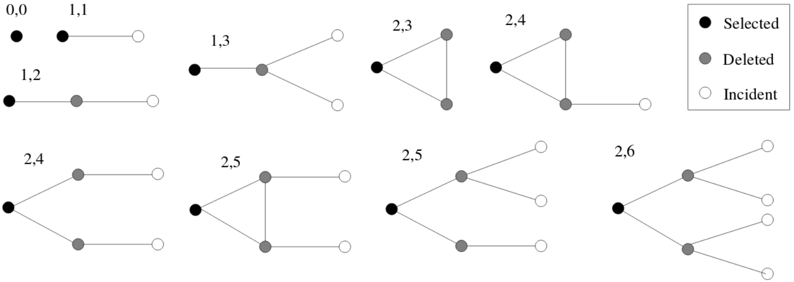
Part I: How to find good advices and analyse it ?

Reductions



0,0
1,1
1,2
1,3
2,3
2,4
Selected
Deleted
Incident
2,4
2,5
2,5
2,6
3,6
3,7
3,8
3,9

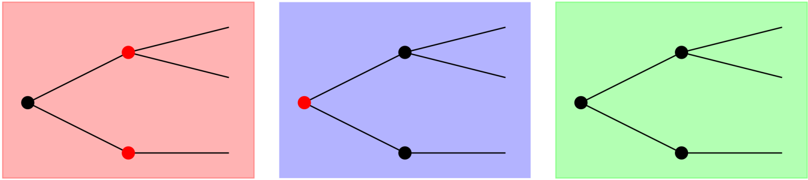
Reductions



$\longrightarrow$ We pick a degree 3 vertex at most once

Good/bad reductions

Let *OPT* denote a fixed maximum independent set.



Bad: $OPT = 2$
$(\text{Greedy} = 1)$

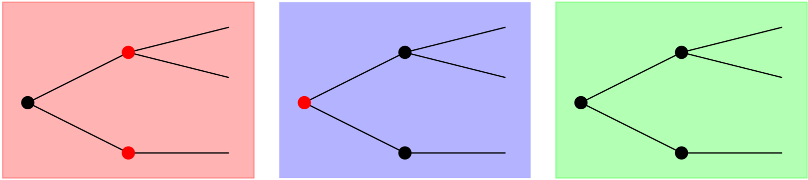
Good: $OPT = 1$
$(\text{Greedy} = 1)$

"Very" good: $OPT = 0$
$(\text{Greedy} = 1)$

For an $5/4$-approximation,

- 3 good reductions pay for 1 bad reduction.
- 1 good $+$ 1 very good pay for 2 bad.

A first potential Ψ



Bad: $OPT = 2$ (Greedy $= 1$) — $\Psi(\text{bad}) = -3$

Good: $OPT = 1$ (Greedy $= 1$) — $\Psi(\text{good}) = 1$

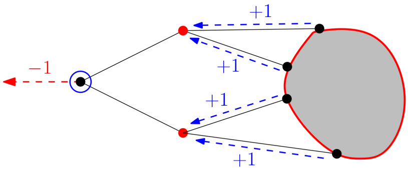
"Very" good: $OPT = 0$ (Greedy $= 1$) — $\Psi(\text{very good}) = 5$

Definition: $\Psi(\text{reduction}) = 5 - 4 \cdot |OPT \cap \text{reduction}|$

claim: $5/4$-approximation $\iff \sum \Psi_i \geq 0$.

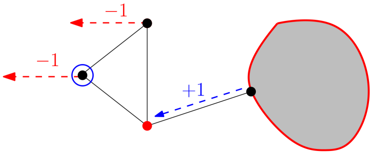
Our potential Φ

Bad reduction



$\Phi = -3 + 4 - 1 = 0$

Good reduction



$\Phi = 1 + 1 - 2 = 0$

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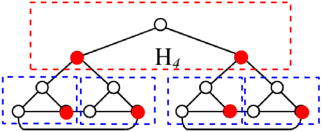
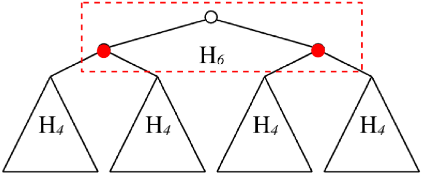
a vertex in *OPT* makes a loan from its neighbors ($\notin$ *OPT*).

a vertex not in *OPT* pays its debt $\leq \Delta - d(v)$.

Definition: $\Phi(\text{reduction}) := \Psi(\text{reduction}) + \textit{loan} - \textit{debt}$

One example





$\longrightarrow$ The potential Φ of each reduction is 0.

$$\Phi^{a,b}(\text{reduction}) := a - b \cdot |OPT \cap \text{reduction}| + loan - debt$$

Proposition

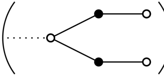
If $\Phi^{a,b} \geq 0$ then **Greedy** is an a/b-approximation algorithm.

Claim: $\Phi^{5,3} \geq 0$

$\longrightarrow$ can be generalized to any maximum degree Δ

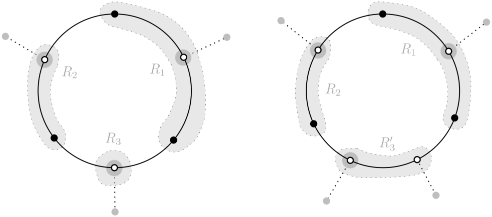
First advice: MoreEdges is a $3/2$-approximation

Claim: We have $\Phi^{6,4} \geq 0$ for all reduction except

$\Phi_I^{6,4}\Big($  $\Big) = 6 - 4 \cdot 2 + 2 - 1 = -1$

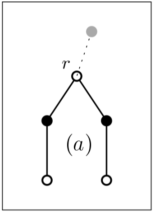
advice (More Edges): Always prefer a degree two vertex <u>with two (or one) neighbors of degree three.</u>

Analysis: If we execute this reduction then the graph is a cycle.

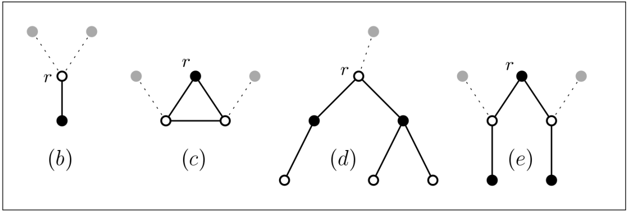


Final goal: an 5/4-approximation

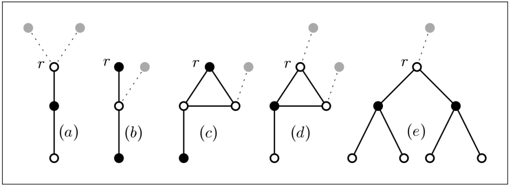
●: in the ind. set; ○: not in the ind. set; ●: was here before;



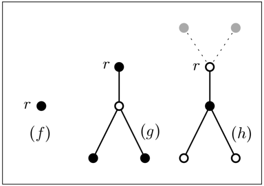
$\Phi = -2$



$\Phi = -1$

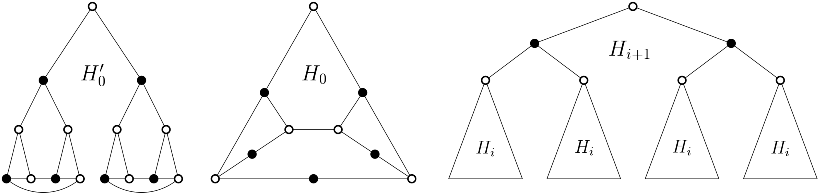


$\Phi = 0$



$\Phi = 1$

An order over reductions is not enough



$\Phi(H_0) = -1$, if the top vertex is picked

$\Phi(H_{i+1}) = 0 + 4 \cdot \Phi(H_i) \longrightarrow -\infty$

advice (DeepestVertex) : Build a Breadth-first search tree, and pick a deepest degree two vertex

(if there remain ties after MoreEdges advice)

Ultimate Greedy*

Greedy*(G)

While $G \neq \emptyset$:

- Pick $v \in G$ with **minimum degree**.

  Break (potential) ties with MoreEdges

  Break ties with DeepestVertex

- Remove v and its neighbours from G.

Theorem: $\frac{5}{4}$-approximation

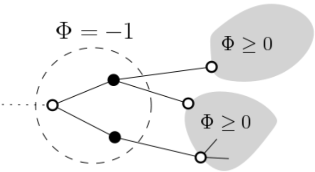
Proof 1/1

(informal) Definition: A graph is problematic if there exists a "special" degree 2 vertex in OPT.

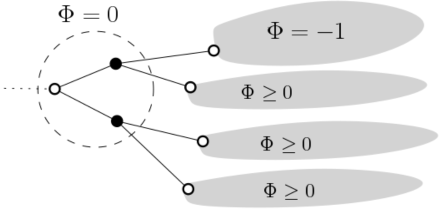
(informal) Lemma

For any sequence of reductions $R_1, \ldots, R_k$ in a connected graph:

1. $\sum \Phi(R_i) \geq -1$
2. if $\sum \Phi(R_i) = -1$, then the graph is problematic.



MoreEdges



DeepestVertex

Part II: Finding good advices is hard

Definition: Given a graph, a greedy set is an (maximal) independent set that can be output by the basic greedy algorithm.

Name: `MaxGreedy`

input: A graph G

output: a greedy set with maximum size

Hardness results (here $P \neq NP$)

planar cubic graphs: NP-hard

Graphs with degree $\leq \Delta$: No $O(\Delta^{1-\epsilon})$-approximation

general graphs : No $O(n^{1-\epsilon})$-approximation

bipartite graphs : No $O(n^{1/2-\epsilon})$-approximation

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Thanks! And happy new year!