Reasoning About the Sizes of Sets: What we Know, and What we Don't Know Yet

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Abstract

This submission deals with the logical and mathematical properties of determiners and determiner phrases that have something to do with *reasoning about the sizes of sets*. These include determiners such as *most* and determiner phrases such as *there are at least as many* ... *as* ... It studies these on top of a basic syllogistic logic and shows that the resulting logical systems are sound, complete, and (as a result) efficiently decidable. Most of the results in the talk have appeared elsewhere. But putting them would be new, as would a statement of open problems.

The classical syllogistic is the logical system whose sentences are of the form All x are x, Some x are x, and No x are y. The most straightforward semantics evaluates sentences in *models*. These models are familiar from generalized quantifier (GQ) theory, and they just consist of a universe set and sets $[\![x]\!]$ for the variables x. Logical systems related to these go back to Aristotle, of course, and so they are the very root of the western logical tradition. Although sidelined by the Fregean revolution in logic, their study has been revived in recent years. This paper studies what happens when we extend the classical syllogistic by the kinds of determiners which are familiar from GQ theory.

1 Syntax and semantics

For the syntax of our language, we start with a collection of *nouns*. (These are also called *unary atoms* or *variables* in this area, and we shall use these terms interchangeably.) We use upper-case Roman letters like A, B, ..., X, Y, Z for nouns. We are only interested in *sentences* of one of the following forms: All X are Y, Some X are Y, Most X are Y, There are at least as many X are Y, and There are more X are Y. Except in one

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place, the logics have no additional boolean connectives, and certainly no first-order quantifiers. We mentioned sentences No X are Y in the Introduction, but we are ignoring No in what follows; it is open to extend what we do to the larger syllogistic fragment with No.

For the semantics, we use *models* $\mathcal{M}$ consisting of a finite set M together with *interpretations* $[\![X]\!] \subseteq M$ of each noun X. We then interpret our sentences in a model as follows

$\mathcal{M} \models$ All X are Y	iff	$[\![X]\!] \subseteq [\![Y]\!]$
$\mathcal{M} \models$ Some X are Y	iff	$[\![X]\!] \cap [\![Y]\!] \neq \emptyset$
$\mathcal{M} \models$ Most X are Y	iff	$card([\![X]\!] \cap [\![Y]\!]) > \frac{1}{2} card([\![X]\!])$
$\mathcal{M} \models$ There are at least as many X are Y	iff	$card([\![X]\!]) \geq card([\![Y]\!])$
$\mathcal{M} \models$ There are more X are Y	iff	$card([\![X]\!]) > card([\![Y]\!])$

Results The logic with sentences All X are Y, Some X are Y, and Most X are Y was axiomatized in [1]. Logic with Most X are Y and boolean connectives $\wedge$, $\vee$, and $\neg$ was axiomatized in [2]. The logic with sentences All X are Y, Some X are Y, There are at least as many X are Y, and There are more X are Y was axiomatized in [3]. This axiomatization was only for finite universes. In fact, several of the logical axioms are not sound for infinite universes. A more recent (and unpublished) result of [4] obtains the completeness theorem for a different logic that uses the same syntax but interprets things on infinite universes.

Except for the relatively easy paper [3], each of these papers has some combinatorial content that makes for significant complications. The logics themselves are too large to include in a 2-page abstract. Many of the axioms are unusual, and some are so involved that I doubt that anyone in "real life" has ever used them. Still, for theoretical purposes it seems that we should know about them. All of the logics mentioned are decidable in polynomial time (!), except for the one in [2]. (And the logic in [2] is more complex mainly due to the boolean connectives.) By the time of the Quad meeting, all of these logics will have been implemented, and I expect that implementations will be housed on publicly-available websites. The algorithmic side of the story includes automatic generation of counter-models.

Open questions and partial results The overall aim of this talk, and indeed of this area, is to tell the complete story on reasoning about the sizes of sets. My talk will mention open questions which I take to be workable based on what we already know and which also would contribute to the overall goal. Results and questions will be mentioned on: extending the results to bigger fragments (include some ongoing work by one or two undergraduates), work on mass nouns (everything above is just for count nouns), and perhaps some connections to the experimental literature on human syllogistic reasoning.

References

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