# Full Characterization of Extended CTL*

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# Characterization Theorems

- Equivalence results among different formalisms:
  - Modal/Temporal Logics (e.g. LTL, CTL, CTL$^*$ etc)
  - Predicate Logics (e.g. FOL, MSO, etc)
  - Automaton Models (e.g. word/tree automata etc.)
- Useful to study and identify:
  - the precie expressive power of each formalism
  - the decidability (and sometimes complexity) of the associated decision problems.

# Linear Time Landscape

- Complete connections have been established:

| | Finite Words | ω-words | Finite Words | ω-words |
|---|---|---|---|---|
| Temporal Logic | LTL | LTL | ELTL | ELTL |
| Predicate Logic | FOL | FOL | MSO | MSO |
| Automaton Model | Regular<br>Counter Free | ω-Regular<br>Counter Free | Regular | ω-Regular |

# Branching Time Landscape

- Only partial connections have been discovered!
- Difficulties arising with Tree Languages:
  1. No obvious composition operations
  2. Bisimulation invariance / Counting Quantifiers

| Temporal Logic | CTL | CTL* | ECTL* | $\mu$-Calculus |
|---|---|---|---|---|
| Predicate Logic | ? | MPL/bis | MCL/bis | MSO/bis |
| Automaton Model | Symmetric<br>Hesitant<br>Linear Tree A. | ? | ? | Symmetric<br>Alternanting<br>Parity Tree A. |

# Branching Time + Counting Landscape

- Known connections

| Temporal Logic | CTL$^*$ | Counting-CTL$^*$ | ECTL$^*$ | Counting-ECTL$^*$ |
|---|---|---|---|---|
| Predicate Logic | MPL/bis | MPL | MCL/bis | ? |
| Automaton Model | Symmetric Hesitant Counter free Tree A. | Graded Hesitant Counter free Tree A. | Symmetric Hesitant Tree A. | Graded Hesitant Tree A. |

- Focusing on Counting-ECTL$^*$and MCL

| Temporal Logic | Counting-ECTL$^*$ | ? |
|---|---|---|
| Predicate Logic | ? | MCL |
| Automaton Model | Graded Hesitant Tree A. | Hesitant First Order Tree A. |

M. Benerecetti, L. Bozzelli, F. Mogavero et A. Peron - Automata-Theoretic Characterisations of Branching-Time Temporal Logics. - ICALP 2025.

# Monadic Chain Logic MCL

- MCL is a well-known fragment of MSO where second-order quantification is restricted to chains of the given Kripke tree.
- (a chain is a subset of a path of the Kripke tree)

## Syntax

$$\varphi := \mathtt{sing}(X) \mid X \subseteq p \mid X \subseteq Y \mid X \leq Y \mid \neg\varphi \mid \varphi \wedge \varphi \mid \exists^{\mathrm{C}} X.\varphi$$

where $p$ in a set of propositions $AP$ and $X, Y$ in a set of second order variables $\mathtt{Vr}_2$.

- $\mathtt{sing}(X)$ asserts that $X$ is a singleton,
- $X \subseteq p$ means that $p$ holds at each node of $X$,
- $X \leq Y$ means that each node of $Y$ is a descendant of each node of $X$.

# Counting CTL$^*$ (CCTL$^*$)

- CCTL$^*$ extends standard CTL$^*$ with counting operators.

**Syntax**

$$\varphi ::= \top \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathsf{E}\psi \mid \mathsf{D}^{n}\varphi \quad (\textit{state formulae})$$
$$\psi ::= \varphi \mid \neg\psi \mid \psi \wedge \psi \mid \mathsf{X}\,\psi \mid \psi \mathsf{U} \psi \quad (\textit{path formulae})$$

- where $p$ is in a set of propositions $AP$

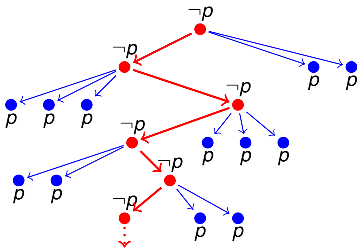
- X and U are the standard “next" and “until" temporal modalities (standard path semantics)
- E is the existential path quantifier
- $\mathsf{D}^{n}\varphi$, with $n \in \mathbb{N} \setminus \{0\}$, is the counting operator:
  there are at least $n$ distinct children of the current node satisfying $\varphi$

# CCTL*-definable property

## Example

$$\mathrm{EG}(\neg p \wedge \mathrm{D}^2 p)$$

there exists a branch where each node does non satisfy $p$ but have at least two children satisfying $p$

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# Counting ECTL$^*$ (CECTL$^*$)

- CCTL$^*$ extends standard CTL$^*$ with counting operators and additional expressiveness of path formulae.

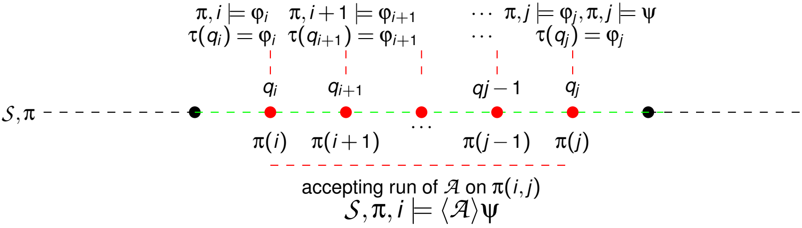
### Syntax

$\varphi ::= \top \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathsf{E}\psi \mid \mathsf{D}^n\varphi$ (*state formulae*)

$\psi ::= \varphi \mid \neg\psi \mid \psi \wedge \psi \mid \langle\mathcal{A}\rangle\psi$ (*path formulae*)

- where $p$ is in a set of propositions $AP$
- $\mathcal{A} = \langle 2^{AP}, Q, \delta, q_I, F, \tau\rangle$ a testing Non-deterministic Word Automaton consisting of:
  1. a Non-deterministic Word Automaton $\langle 2^{AP}, Q, \delta, q_I, F\rangle$
  2. a test function $\tau$ mapping states in $Q$ to CECTL$^*$ formulae.

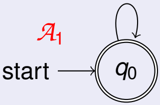
# Path formulae semantics

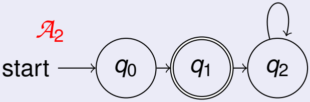



- A test $\tau$ is trivial if it maps each state to $\top$;
- A CECTL$^*$ formula $\varphi$ is counter free if:
  1. the testing automata $\mathcal{A}$ occurring in $\varphi$ is counter free;
  2. either $\mathcal{A}$ is deterministic or $\tau$ is trivial.

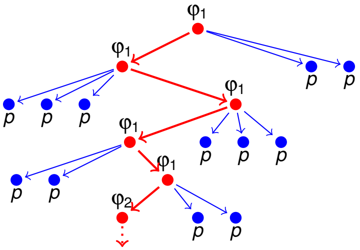
## Example

$$\mathsf{E}\langle\mathcal{A}_1\rangle\langle\mathcal{A}_2\rangle\varphi_2$$




$\tau(q_0) = \varphi_1 \wedge \mathsf{D}^2 p$ $\qquad$ $\tau(q_i) = \top \; i \in \{1,2,3\}$

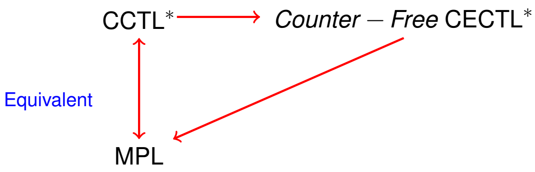


# Expressiveness equivalence of CCTL* and counter-free CECTL*

## Theorem

CCTL* and counter-free CECTL* are equivalent formalisms, i.e., they specify the same class of tree languages.

- Given a CCTL* formula, one can build a equivalent counter-free CECTL* formula.
- by the known equivalence between *Monadic Path Logic* (MPL) and CCTL*, it suffices to show that each counter-free CECTL* formula can be translated into an equivalent MPL sentence.



# Equivalence of MCL and CECTL$^*$

- It has beeen recently shown that each CECTL$^*$ state formula has an equivalent MCL sentence.
- The main result of the paper is the proof of the converse inclusion.
- The proof of this result exploits a compositional argument similar of that given for showing that each MPL sentence has an equivalent CCTL$^*$ state formula in

  F. Moller et A.M. Rabinovich - Counting on CTL$^*$: On the expressive power of Monadic Path Logic, IC, 2003.

## Main result

### Equivalence Theorem

MCL and CECTL$^*$ are equally expressive.

# Equivalence Theorem Proof

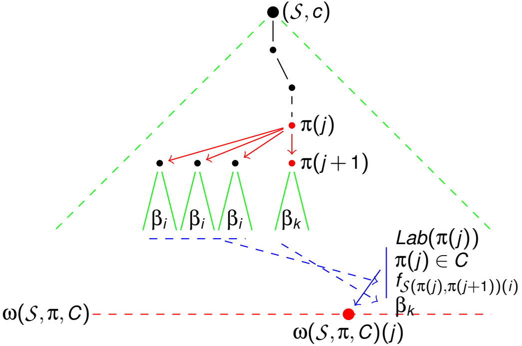
- An $h$-MCL formula is a MCL formula with at most $h$ free (second order) variable $X_1 \dots X_h$;
- An $h$-structure $\mathcal{S}_h$ is a tuple $(\mathcal{S}, C_1, \dots C_h)$ where
  1. $\mathcal{S}$ a Kripke tree over proposintion ($AP$);
  2. $C_1, \dots, C_h$ chains of $\mathcal{S}$.

## Equivalence for $h$-structures

- Two $h$-structures $\mathcal{S}_h$ and $\mathcal{S'}_h$ are $m$-rank equivalent ($\mathcal{S}_h \equiv_m \mathcal{S'}_h$) if no $h$-MCL formula $\varphi(X_1 \dots X_h)$ of rank at most $m$ can distinguish them;
- $\equiv_m$ defines finitely-many equivalence classes over $h$-structures;
- Each equivalence class $\Lambda$ is charaterized by a $h$-MCL formula $\beta$ (called $m$-type) of rank at most $m$ ($\mathcal{S}_h \models \beta$ iff $\mathcal{S}_h \in \Lambda$);
- each $h$-MCL formula of rank at most $m$ is equivalent to a disjunction of $m$-types;
- $\equiv_m$ can be characterized in terms of *Ehrenfeucht-Fraissé games* over $h$-structures.

# Composition Theorem for MCL

- A composition theorem for 1-MCL formulas over 1-structures $(\mathcal{S}, C)$.
- It allows us to express a 1-MCL formula in terms of MSO sentences over infinite words on a suitable alphabet $2^{AP_m}$).



# A Composition Theorem for MCL

## Composition Theorem

For all $m \geq 1$ and 1-MCL formulas $\varphi(X)$ over $AP$ with quantifier rank at most $m$, there is an MCL sentence $\psi$ over $AP_m$ such that for each 1-structure $(\mathcal{S}, C)$ and infinite branch $\pi$ of $\mathcal{S}$ with $C \subseteq \pi$, we have

$$(\mathcal{S}, C) \models \varphi(X) \Leftrightarrow \omega(\mathcal{S}, \pi, C) \models \psi.$$

- Remember that path formulae in CECTL$^*$ can be expressed by Testing Non-deterministic Word Automata;
- Every $1-$MCL formula can be translated via structural induction in an equivalent CECTL$^*$ formula.

## Equivalence Theorem

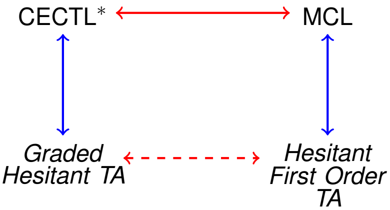
MCL and CECTL$^*$ are equally expressive.

# Corollary

Considering the automata counterparts of CECTL$^*$ and MCL ....

**Equivalence Theorem**

The logics CECTL$^*$ and *MCL* and the classes of automata *HGTA* and *HFTA* are all equivalent formalisms.



# Conclusions

## Summary

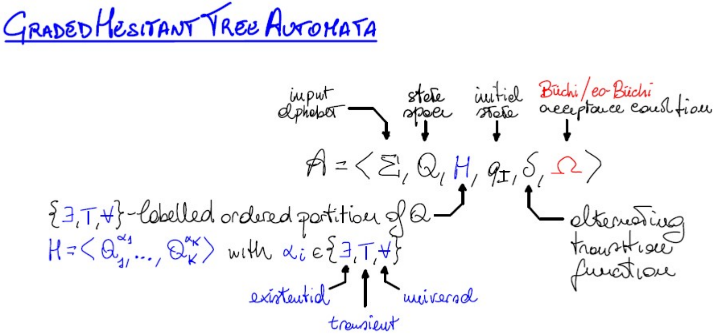
- We have adopted a compositional approach to prove the expressive equivalence of Monadic Chain Logic (MCL) and the counting extension CECTL$^*$ of ECTL$^*$.
- As a corollary we have proved that two extension of Hestitant Tree Automata (HGTA and HFTA) are equivalent.

## Future work

- We are interested in investigating a characterization of Monadic Tree Logic, a fragment of MSO where second order variables range over trees.
  - M. Benerecetti, L. Bozzelli, F. Mogavero et A. Peron - Quantifying over Trees in Monadic Second Order Logic, LICS'23, 2003.
- The goal is to gain some insights into the expressiveness of various extensions of standard temporal logics for strategic reasoning such as Structured Temporal Logic (STL)
  - M. Benerecetti, F. Mogavero et A. Murano - Structure Temporal Logic, LICS'13, 2013.

Thank you!

# Graded Hesitant Tree Automata



# Graded Hesitant Tree Automata