Real-Time Higher-Order Recursion Schemes

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Higher-Order Recursion Schemes

Higher-Order Recursion Scheme (HORS): higher-order grammar that generates trees:

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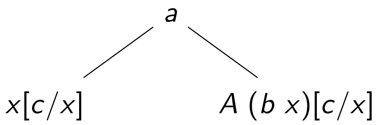

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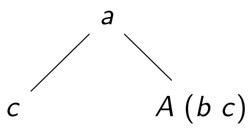

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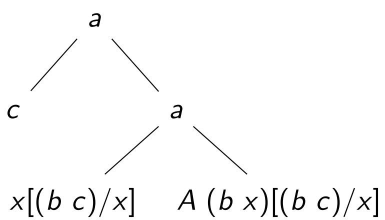

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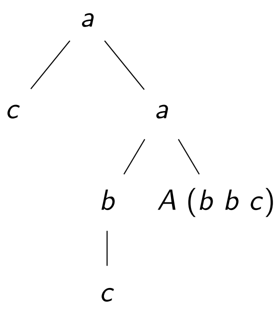

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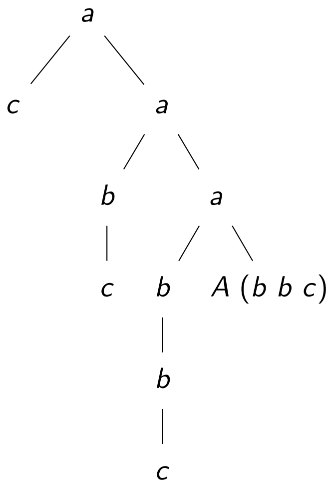

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parity tree automata (APT):

- acceptance explained as two-player game between verifier and spoiler
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Prop.: (Ong'06) For order-k HORS, the HORS model-checking problem is k-EXPTIME-complete.

order of a HORS: order of highest type of argument, e.g.,

- order 1: set of trees,
- order 2: set of trees $\rightarrow$ set of trees,
- ...

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timed automata are well-established, rather simple, and regular in character $\rightsquigarrow$ good fit

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timed automaton introduced as model of real-time systems

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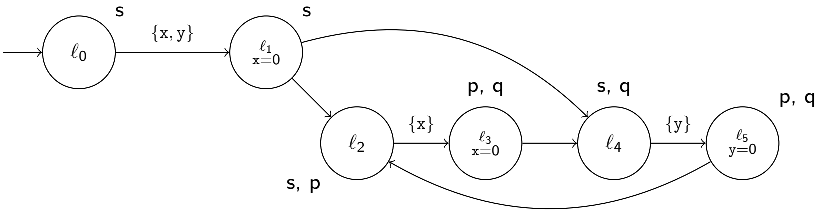

semantics explained via clock evaluations (mappings from clocks $\mathtt{x}, \mathtt{y}, \ldots$ to $\mathbb{R}^{\geq 0}$)

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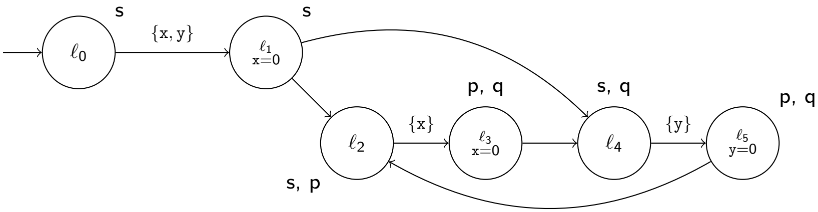

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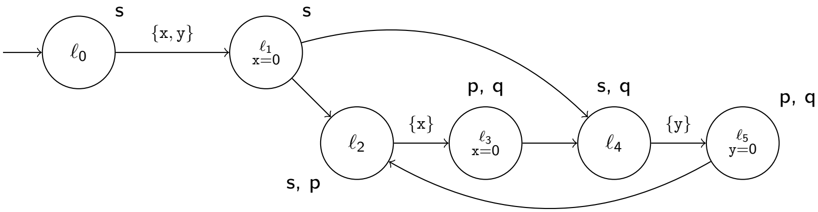

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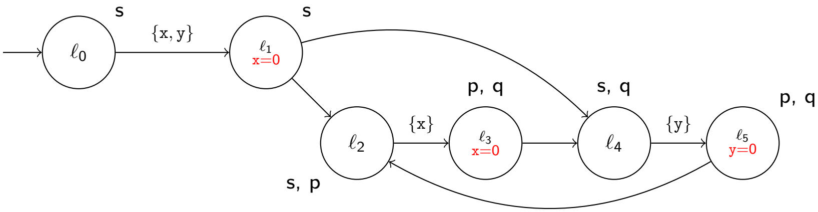

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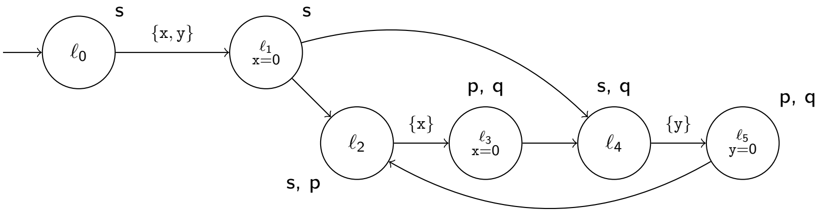

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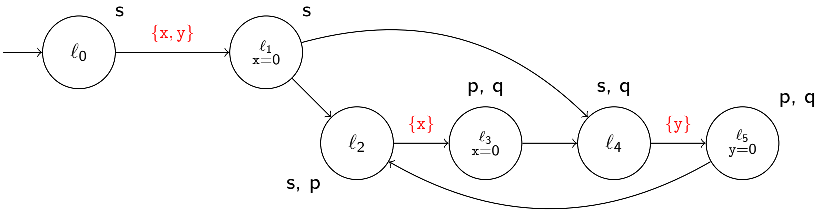

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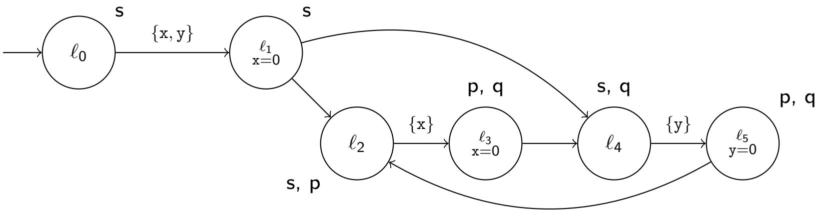

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main challenges:

- timed automata invented as models of systems, not as verification device
- need meaningful interaction between timed automaton and APT for useful specifications

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general flow of acceptance game:

- verifier lets time flow according to current tree label
- then APT transitions

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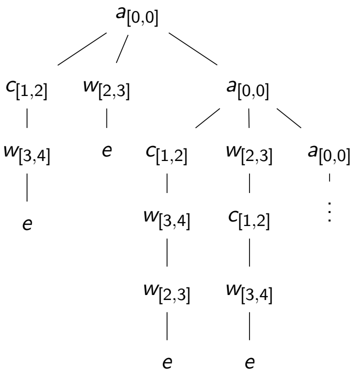

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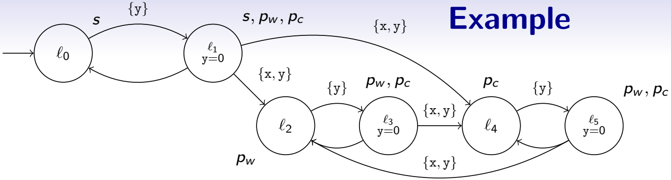

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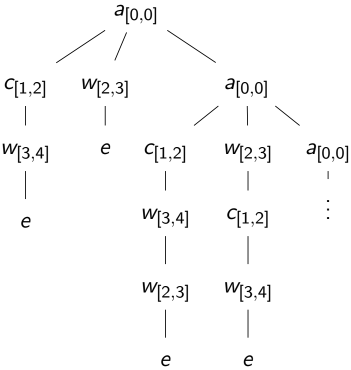

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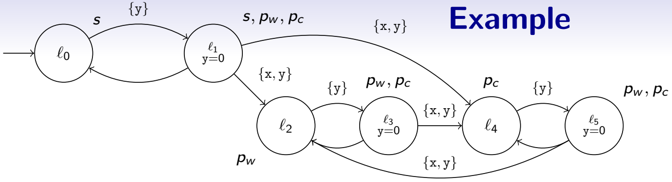

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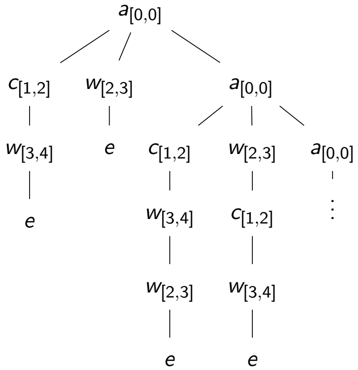

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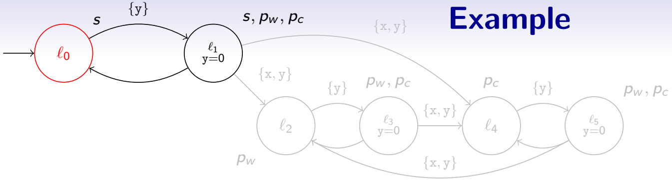

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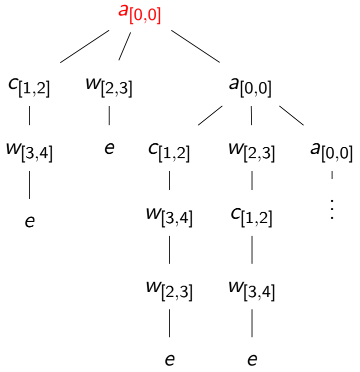

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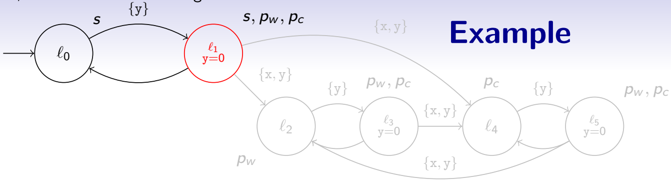

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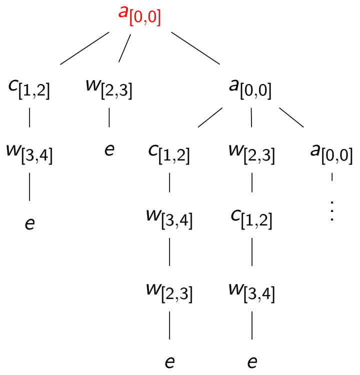

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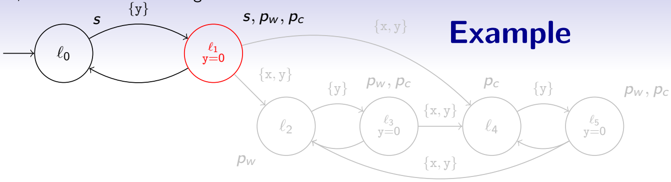

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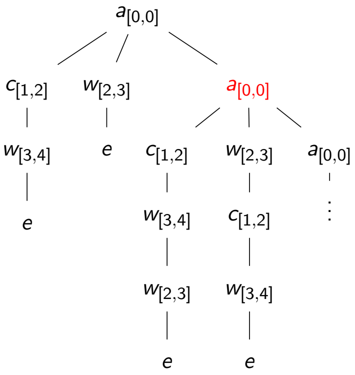

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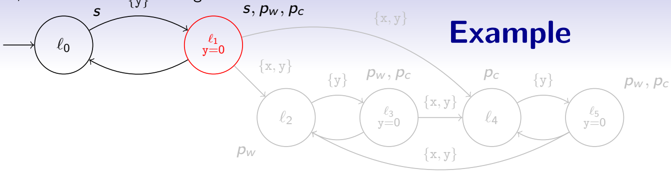

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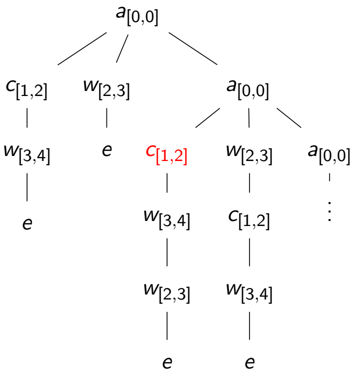

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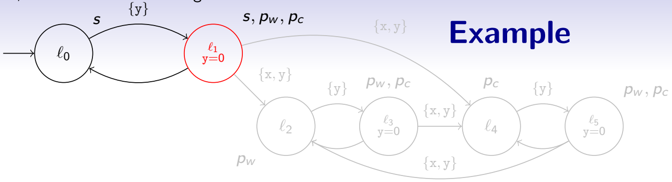

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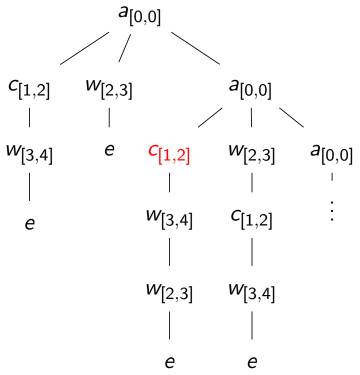

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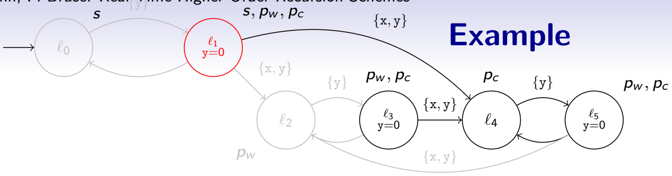

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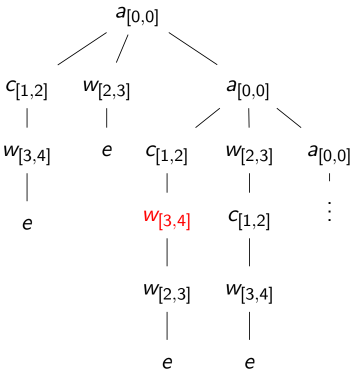

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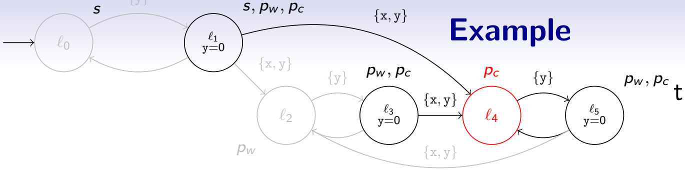

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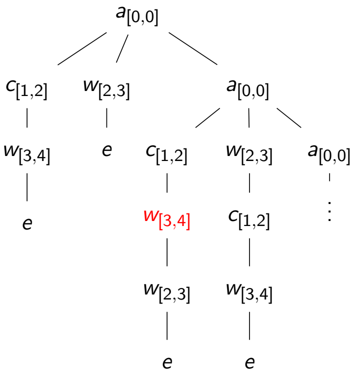

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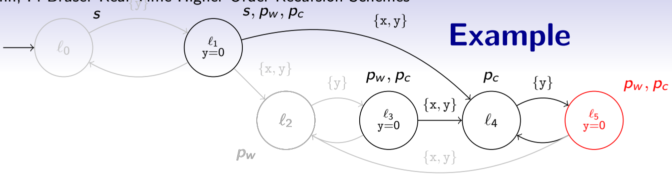

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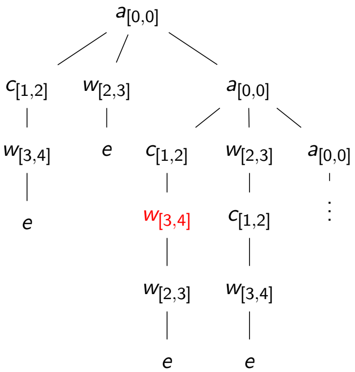

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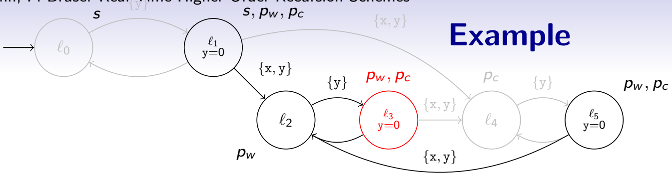

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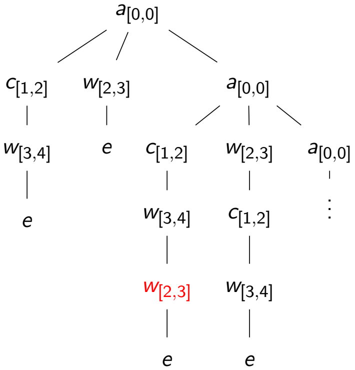

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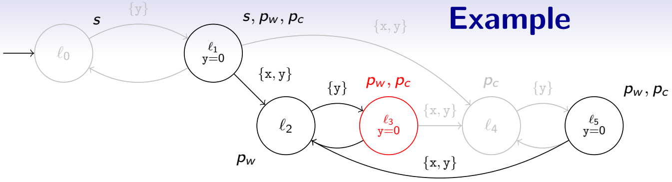

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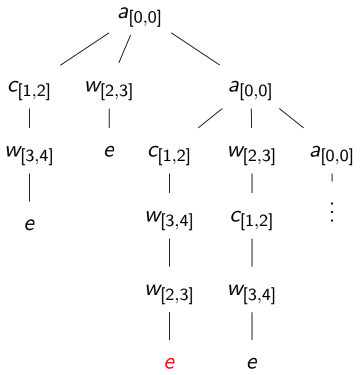

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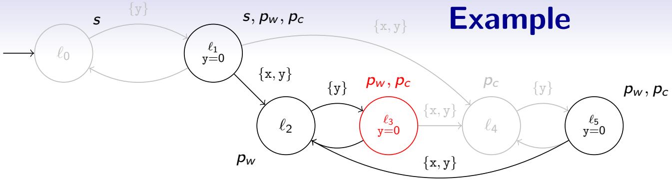

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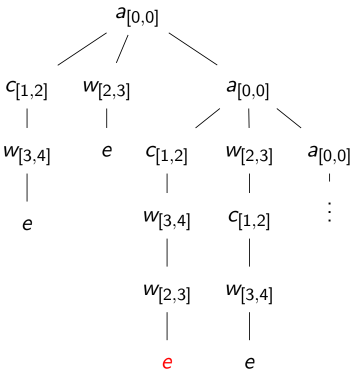

Upper Bounds for Timed-HORS Model-Checking

Theorem 1

Model-checking order-k timed HORS is in $(k+1)$-EXPTIME

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- needs to be emulated on grammar side via gadgets

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#	q_0, i	n	p	u	t	$\square$	$\ldots$	$\square$	#
#	a	n, q_1	p	u	t	$\square$	$\ldots$	$\square$	#
#	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	#
#									#
#	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	#
#	$q_f, \square$	$\square$	$\square$	$\square$	$\square$	$\square$	$\square$	$\square$	#

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$\#$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\#$
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$\rightsquigarrow$ existence of a table can be checked recursively checking for local consistency!

Counting with HORS, APT and TA

- working with an acc. witnessing table requires encoding numbers in $[0, 2^{2^{|w|}} - 1]$
- re-use idea from TIME'22 [B./Lange]: integral clock values in $[0, 2^{|w|} - 1]$ as bits

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Proof of Theorem 2: verify structure of tree from Lemma 4 using an extension if the APT from Lemma 1, and the TA from Lemma 2/3.

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