Hardness Results and Efficient Algorithms for Graph Powers

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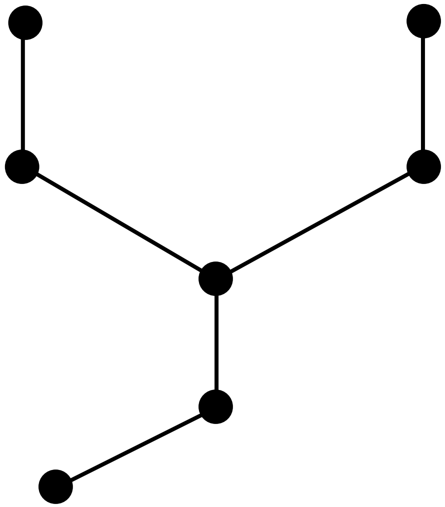
Outline

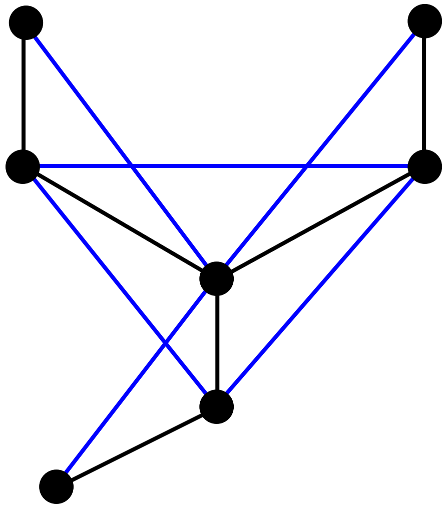
- Introduction
- NP-completeness results for recognizing powers of graphs
- Efficient algorithms for solving SQUARE OF STRONGLY CHORDAL SPLIT GRAPH and CUBE OF GRAPH WITH GIRTH ≥ 10
- Conclusion and open problems

Graph powers

- *k*-th power and *k*-th root of graph.
  - Let $H = (V, E)$ be a graph. Let k be a positive integer. The graph $G = (V, E^k)$ is the *k*-th power of H, and H is called a *k*-th root of G, where $E^k = \{ xy \mid 1 \leq d_H(x,y) \leq k \}$.



The graph H



Square of H

Graph powers

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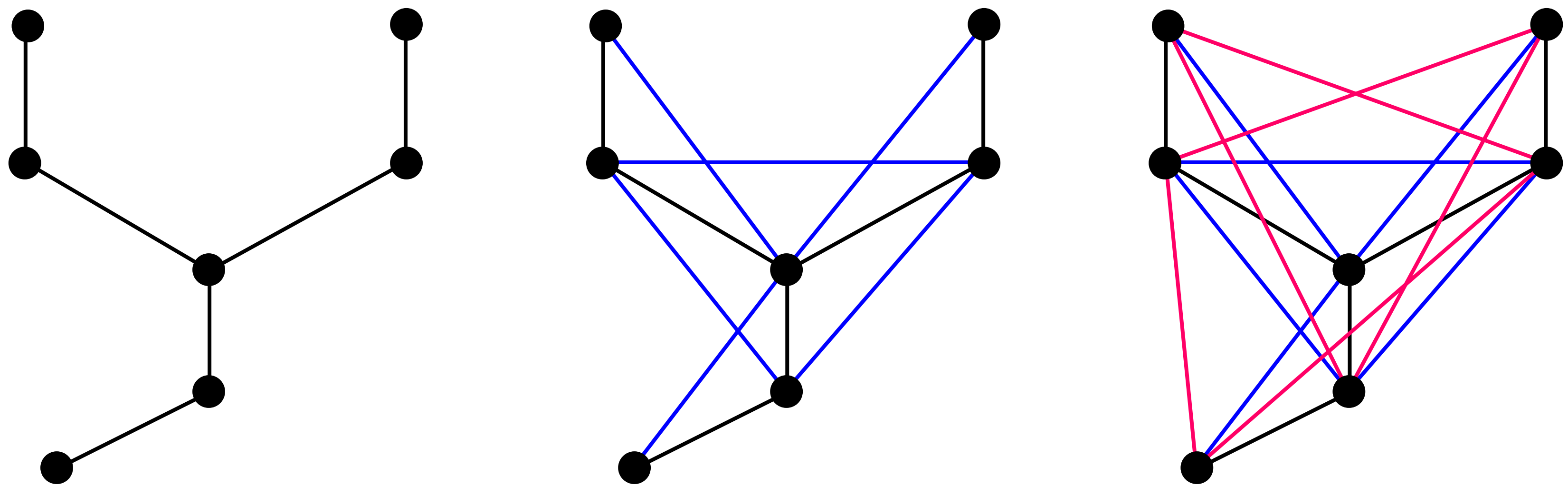


The graph H Square of H Cube of H

Problems on graph powers

- *k*-TH POWER OF GRAPH

  Instance: A graph $G = (V, E)$.

  Question: Is there a graph H such that $G = H^k$?

For a given graph class $\mathcal{C}$:

- *k*-TH POWER OF $\mathcal{C}$ GRAPH

  Instance: A graph $G = (V, E)$.

  Question: Is there a graph H in $\mathcal{C}$ such that $G = H^k$?

- k= 2: SQUARE OF GRAPH and SQUARE OF $\mathcal{C}$ GRAPH
- k= 3: CUBE OF GRAPH and CUBE OF $\mathcal{C}$ GRAPH

Related work

- In 1960 Ross and Harary [*The square of a tree, Bell System Tech. J.39*] first studied the concept of the square of a graph.

- The computational complexity of *k*-TH POWER OF GRAPH was unresolved until 1994 when Motwani and Sudan proved that SQUARE OF GRAPH is NP-complete [Computing roots of graphs is hard*, Discrete Applied Math 54* (1994)].

- In 2006, Lau proved the NP-completeness of CUBE OF GRAPH

  [Bipartite roots of graphs*, ACM Transactions on Algorithm 2* (2006)].

- For $k \geq 4$, the computational complexity of *k*-TH POWER OF GRAPH remains open.
- Conjecture 1 [Lau, 2006]:

  For all fixed $k \geq 4$, *k*-TH POWER OF GRAPH is NP-complete.

Tree powers

- In 1995, Lin and Skiena gave an algorithm that solves SQUARE OF TREE in linear time.
- In 1998, Kearney and Corneil solved *k*-TH POWER OF TREE in cubic time for all fixed *k*.
- In 2006, Chang *et al*. gave $O(n+m)$-*time* algorithms for *k*-TH POWER OF TREE.
- New and simpler linear-time algorithms for SQUARE OF TREE are given by Brandstädt *et al.*(2006)

Powers of bipartite and chordal graphs

- In 2006, Lau showed that SQUARE OF BIPARTITE GRAPH is polynomially solvable, while CUBE OF BIPARTITE GRAPH is NP-C

- Conjecture 2 [Lau, 2006]:
  For all fixed $k \geq 3$, k-TH POWER OF BIPARTITE GRAPH is NP-C

- In 2004, Lau and Corneil proved NP-completeness for SQUARE OF CHORDAL GRAPH and SQUARE OF SPLIT GRAPH, while SQUARE OF PROPER INTERVAL GRAPH can be recognized efficiently.

- For $k \geq 3$, the computational complexity of k-TH POWER OF CHORDAL GRAPH was unknown so far.

Powers versus girth

- The girth of a (connected) graph G, girth(G), is the smallest length of a cycle in G. In case G has no cycles, girth(G)= ∞ .

- Very recently, square roots with girth conditions have been considered by Farzad *et al*.[*Computing graph roots without short cycles,* STACS 2009].

  SQUARE OF GRAPH WITH GIRTH $\leq$ 4 is NP-complete, while
  SQUARE OF GRAPH WITH GIRTH $\geq$ 6 is polynomially solvable.

- Conjecture 3 [Farzad *et al*. STACS 2009]: k-TH POWER OF GRAPH WITH GIRTH $\geq 3k$ -1 is polynomially solvable.

Conjectures of Lau

The following problems are NP-complete.

- *k*-TH POWER OF BIPARTITE GRAPH

  Instance : A graph $G = (V, E)$.

  Question: Is there a *bipartite graph* H such that $G = H^k$?

- *k*-TH POWER OF GRAPH

  Instance : A graph $G = (V, E)$.

  Question: Is there a graph H such that $G = H^k$?

Set splitting

- SET SPLITTING [Garey and Johnson, Problem SP4]
  Instance : Collection D of subsets of a finite set S .
  Question: Is there a partition of S into two disjoint subsets S_1 and S_2 such that each subset in D intersects both S_1 and S_2 ?

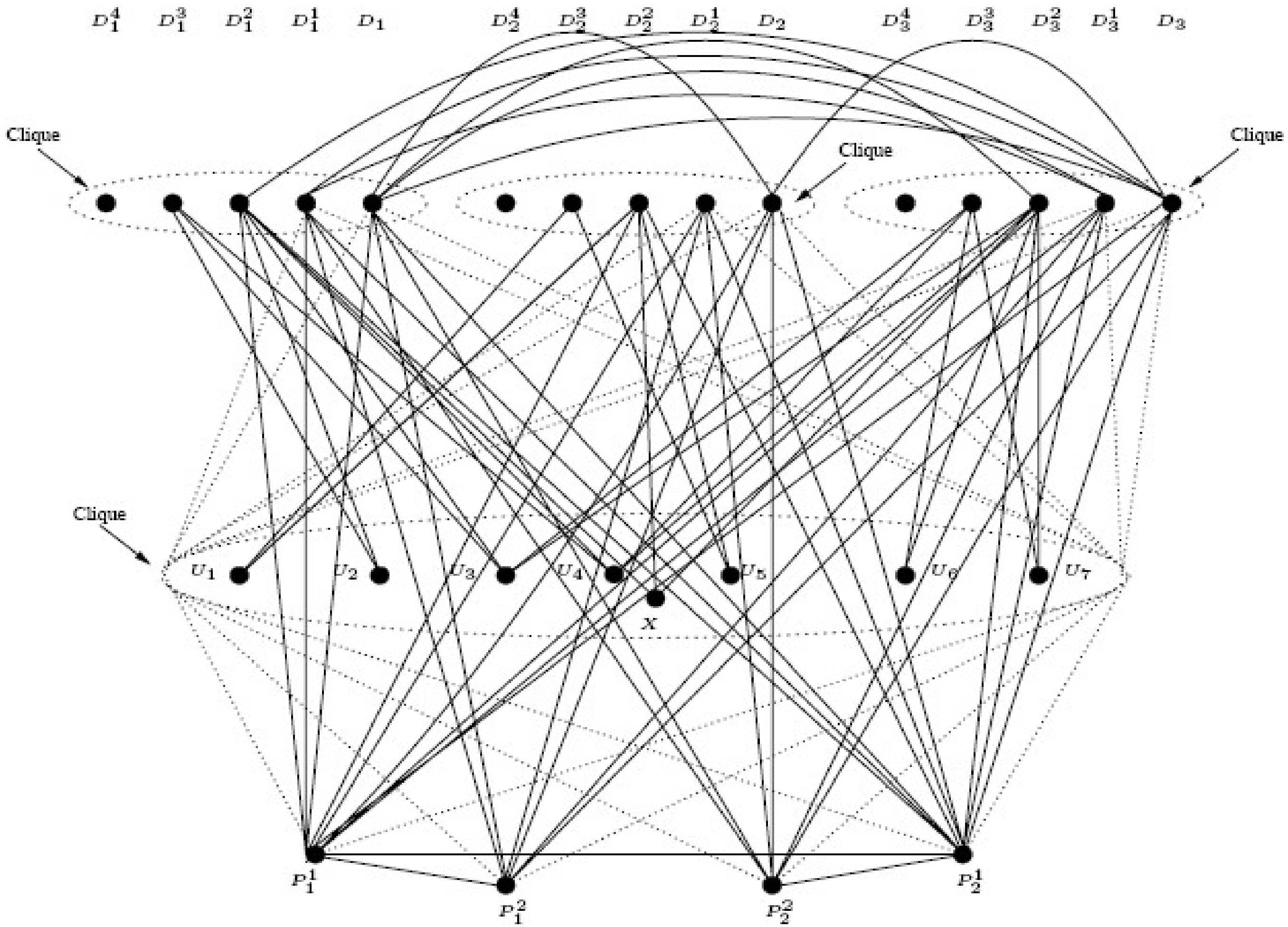
- Given $S = \{u_1 , . . ., u_7\}$ and $D = \{d_1, d_2, d_3\}$ with $d_1 = \{u_2 , u_3 , u_4\}$, $d_2 = \{u_1 , u_5\}$, $d_3 = \{u_3 , u_4 , u_6 , u_7\}$.

- We will reduce SET SPLITTING to k-TH POWER OF BIPARTITE GRAPH

Given $S = \{u_1, \ldots, u_7\}$ and $D = \{d_1, d_2, d_3\}$ with

$d_1 = \{u_2, u_3, u_4\}$, $d_2 = \{u_1, u_5\}$, $d_3 = \{u_3, u_4, u_6, u_7\}$.

An instance $G = G(D,S)$ for 4-TH POWER OF BIPARTITE GRAPH



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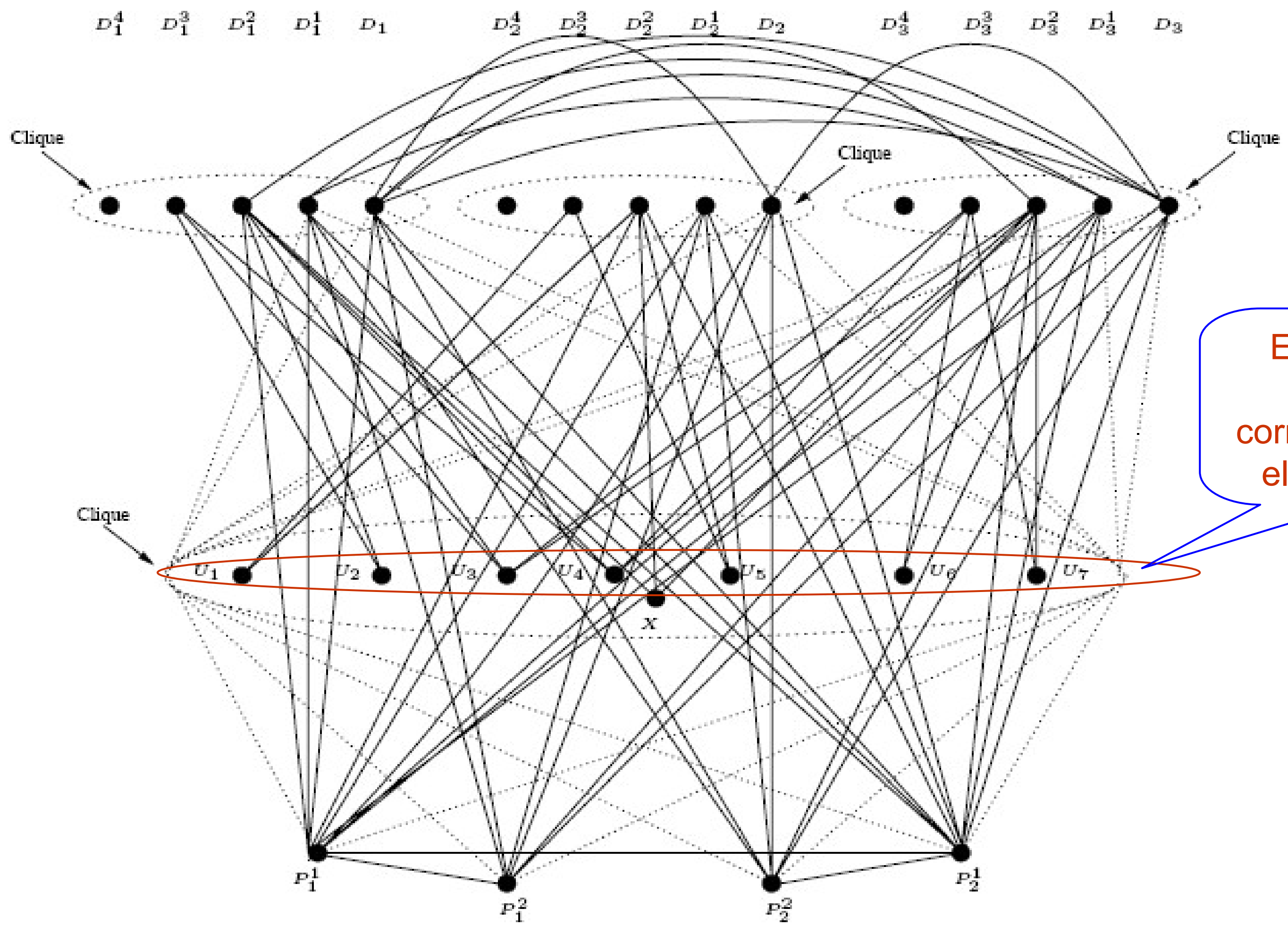

Each 'element vertex' U_i corresponds to the element u_i in S.

Given $S = \{u_1, . . ., u_7\}$ and $D = \{d_1, d_2, d_3\}$ with

$d_1 = \{u_2, u_3, u_4\}$, $d_2 = \{u_1, u_5\}$, $d_3 = \{u_3, u_4, u_6, u_7\}$.

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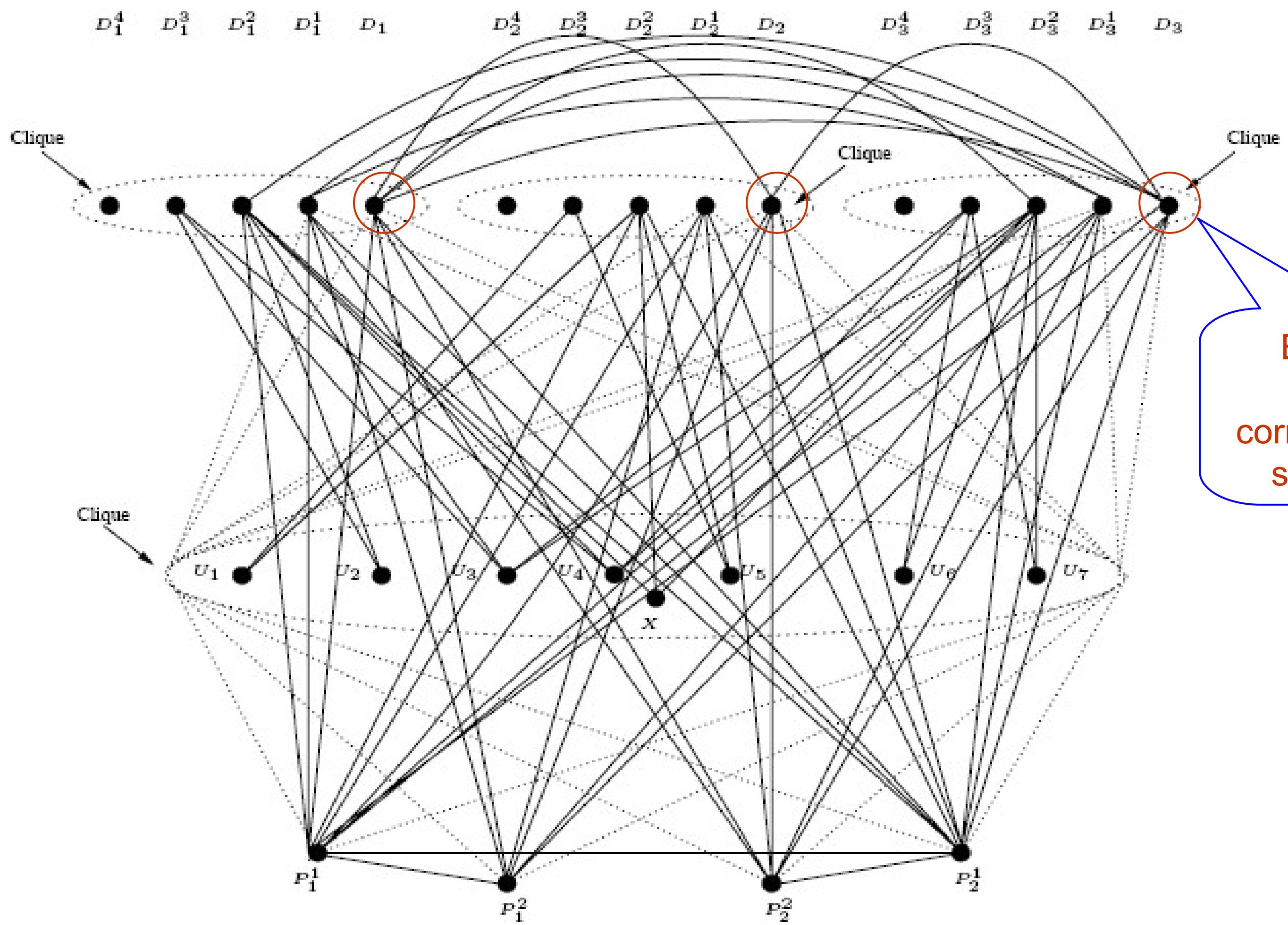
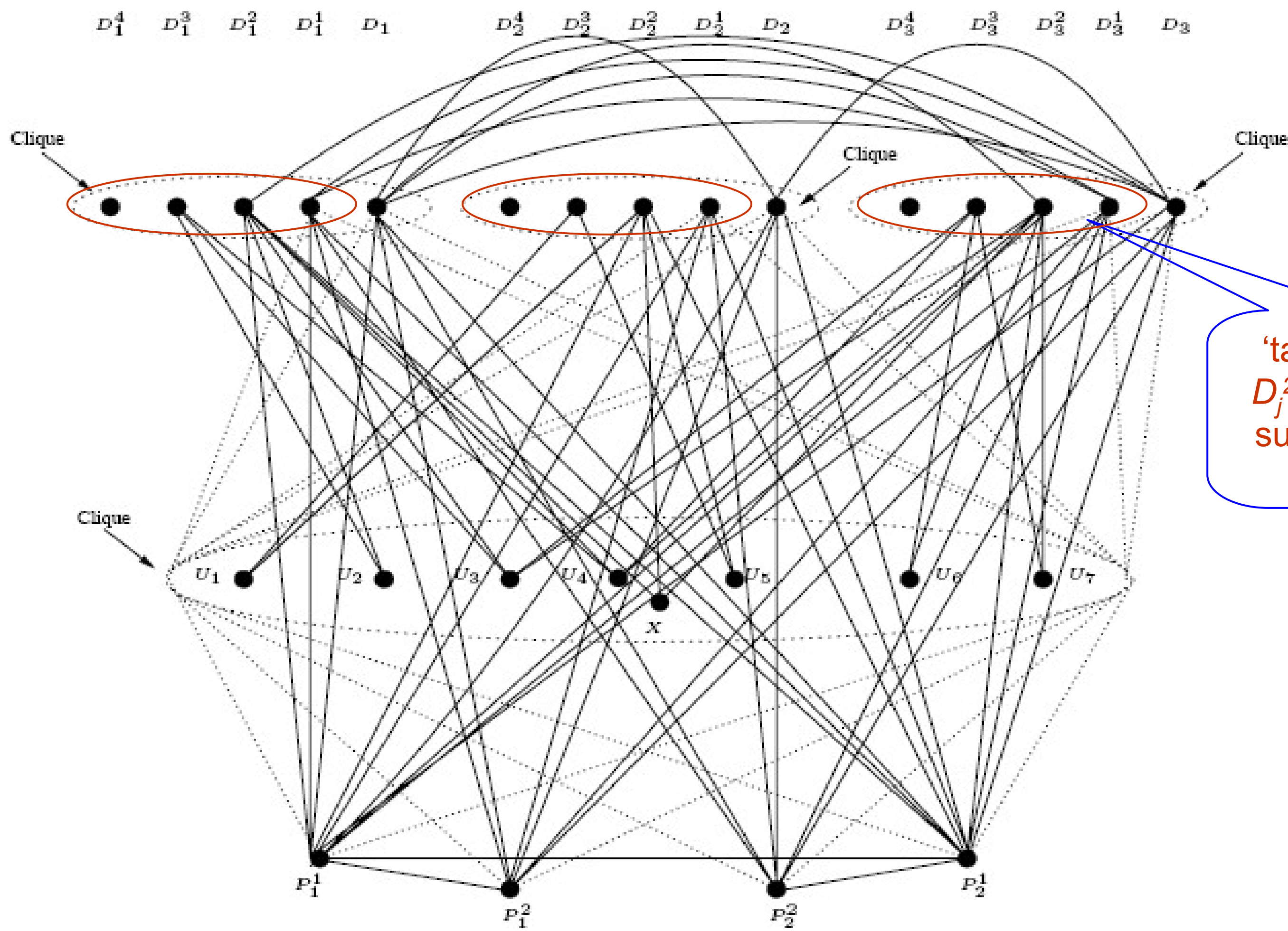


Each 'subset vertex' D_j corresponds to the subset d_j in D.

Given $S = \{u_1, \ldots, u_7\}$ and $D = \{d_1, d_2, d_3\}$ with

$d_1 = \{u_2, u_3, u_4\}$, $d_2 = \{u_1, u_5\}$, $d_3 = \{u_3, u_4, u_6, u_7\}$.

An instance $G = G(D,S)$ for 4-TH POWER OF BIPARTITE GRAPH



'tail vertices' D_j^1, D_j^2, D_j^3, D_j^4 of the subset vertex D_j .

Given $S = \{u_1, . . ., u_7\}$ and $D = \{d_1, d_2, d_3\}$ with
$d_1 = \{u_2, u_3, u_4\}$, $d_2 = \{u_1, u_5\}$, $d_3 = \{u_3, u_4, u_6, u_7\}$.

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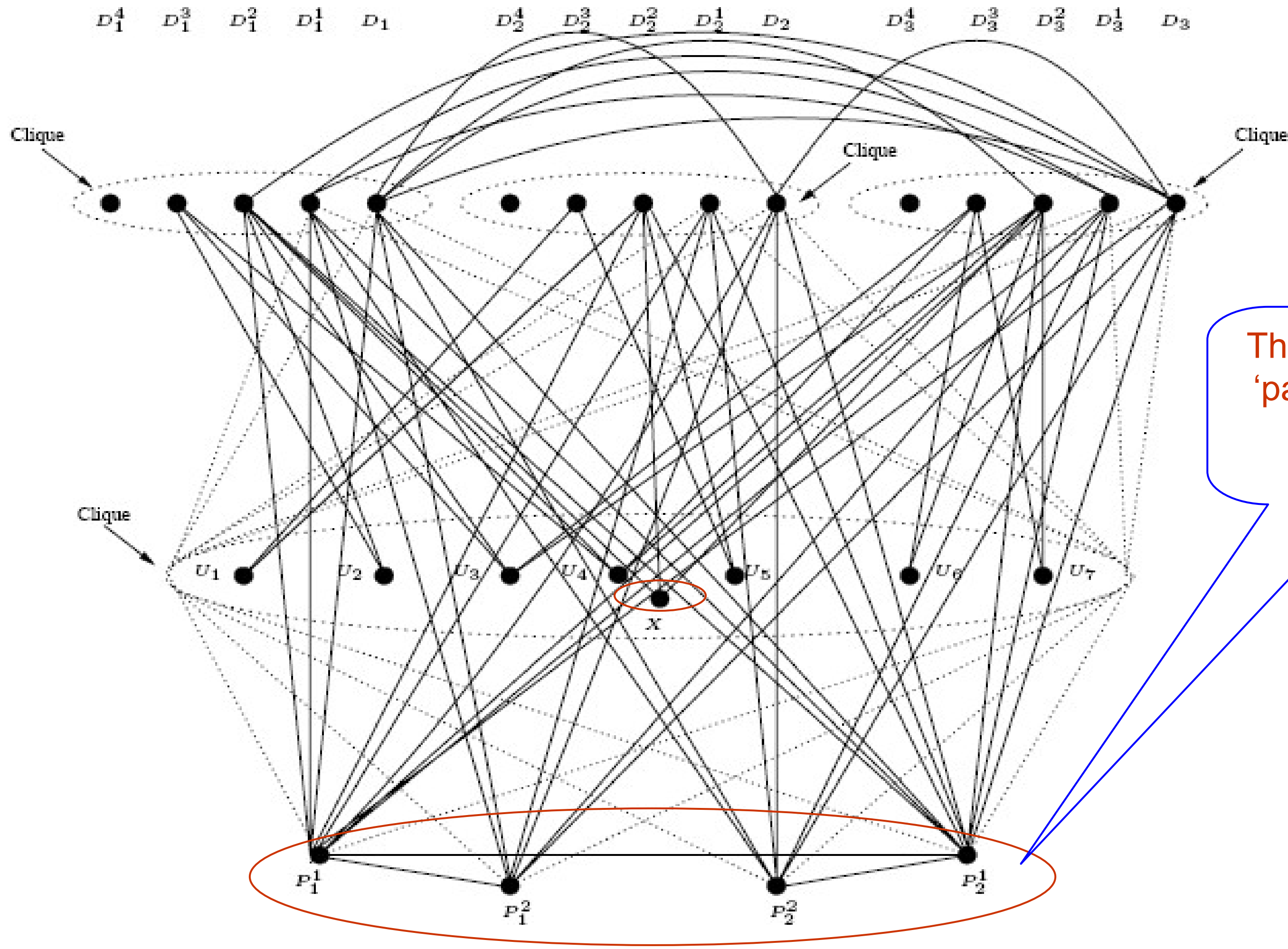

This is k-2 pairs of ‘partition vertices’

Given $S = \{u_1, . . ., u_7\}$ and $D = \{d_1, d_2, d_3\}$ with

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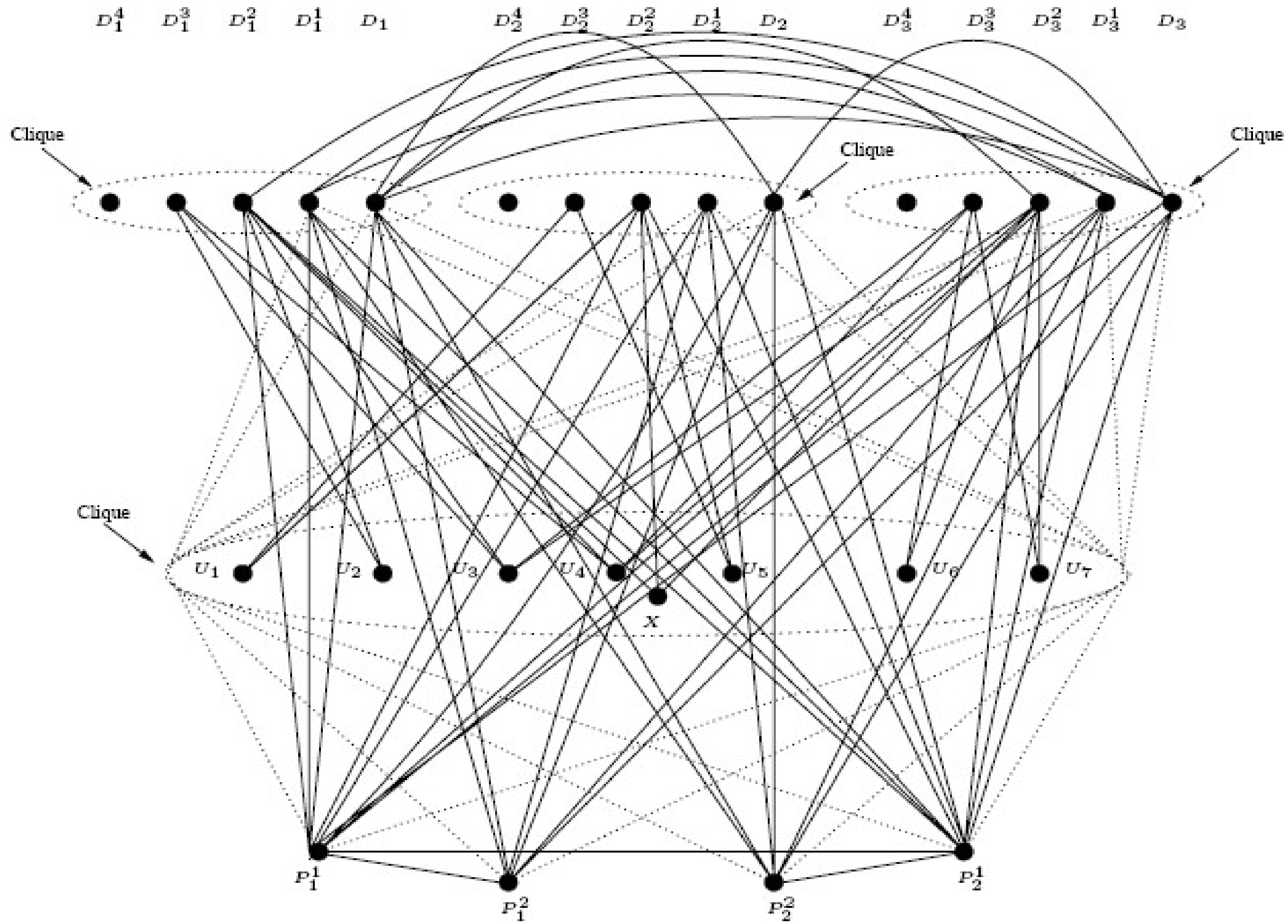

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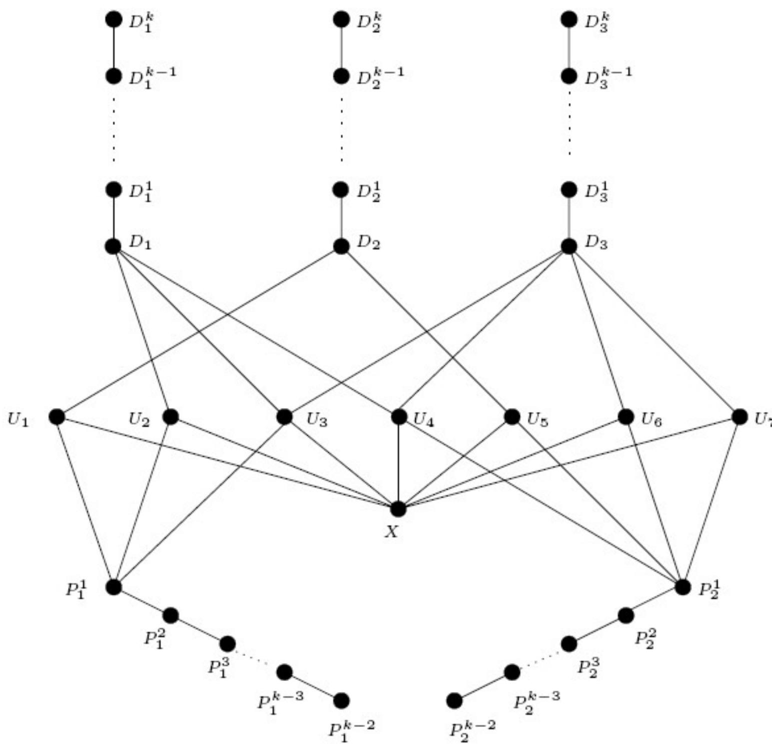
The bipartite k-th root H of G to the solution S_1, S_2

- Lemma 1: If there exists a partition of S into two disjoint subsets S_1 and S_2 such that each subset in D intersects both S_1 and S_2, then there exists a *bipartite graph* H such that $G=H^k$.



- $S_1=\{u_1, u_2, u_3\}$, $S_2=\{u_4, u_5, u_6, u_7\}$ is a solution of SET SPLITTING.

- Lemma 2: If H is a k-th root of G, then there exists a partition of S into two disjoint subsets S_1 and S_2 such that each subset in D intersects both S_1 and S_2 .

- Theorem: k-TH POWER OF BIPARTITE GRAPH is NP-C for fixed $k \geq 3$.

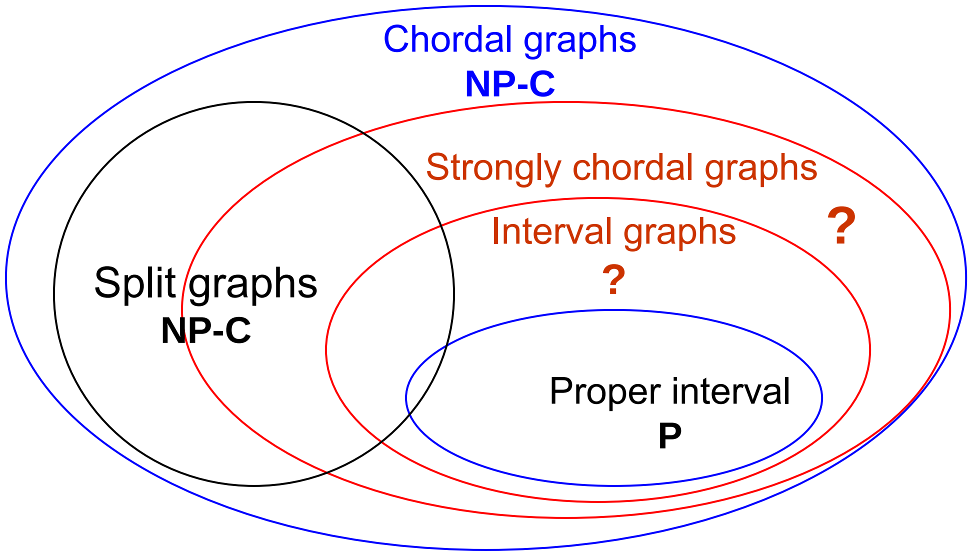
Hardness results

- Theorem 1: k-TH POWER OF BIPARTITE GRAPH is NP-complete for all fixed $k \geq 3$.
- Theorem 2: k-TH POWER OF GRAPH is NP-complete for all fixed $k \geq 2$.
- Theorem 3: k-TH POWER OF CHORDAL GRAPH is NP-complete for all fixed $k \geq 2$.

Squares of subclasses of chordal graphs

- What is the complexity of squares of
  - strongly chordal graphs ?
  - interval graphs ?
  - strongly chordal split graphs ?

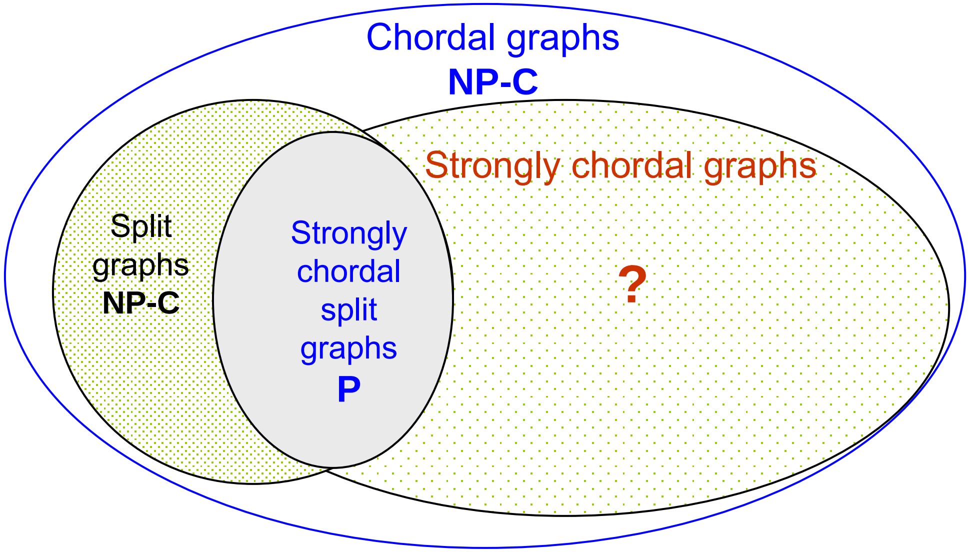
Recognizing squares of…



Squares of strongly chordal split graphs

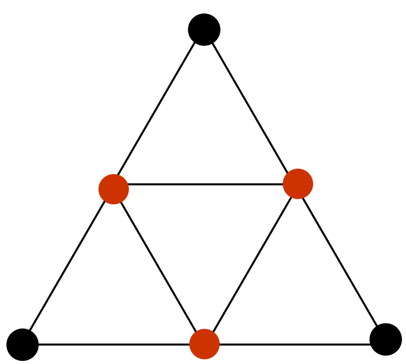
- There exists a good characterization for squares of strongly chordal split graphs that gives a recognition algorithm in time $O(\min\{n^2, m\log n\})$ for such squares.

Recognizing squares of…

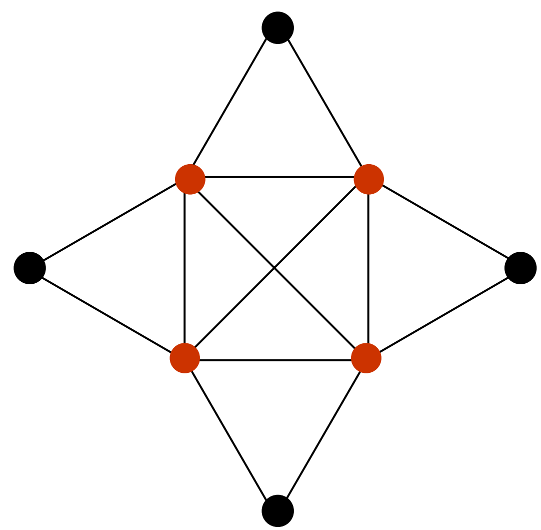


Squares of strongly chordal split graphs

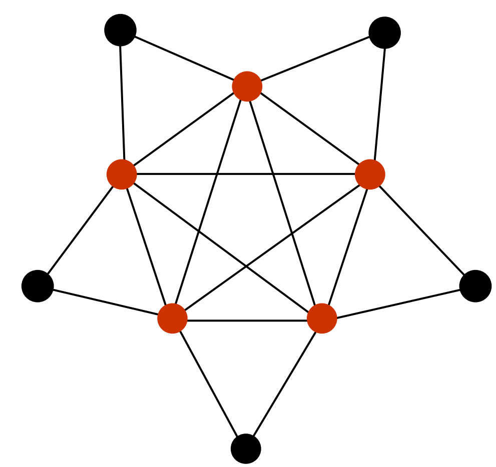
- Graph G is *chordal* if it is C_k-free for $k \geq 4$.
- An *ℓ-sun*, $\ell \geq 3$, consists of a stable set $\{u_1, \ldots, u_\ell\}$ and a clique $\{v_1, \ldots, v_\ell\}$ such that for $i \in \{1, \ldots, \ell\}$, u_i is adjacent to exactly v_i and v_{i+1} (index arithmetic modulo ℓ).



3-sun S_3



4-sun S_4



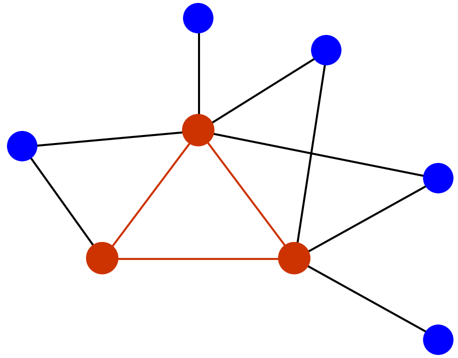
5-sun S_5

Squares of strongly chordal split graphs

- Graph G is *chordal* if it is C_k-free for $k \geq 4$.

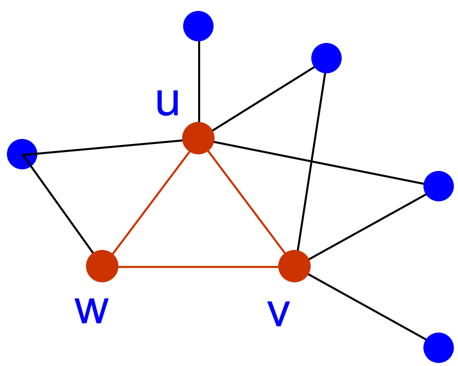
- A graph is *strongly chordal* if it is chordal and S_ℓ -free for all $\ell \geq 3$.
- Theorem [Lubiw 1982; Dahlhaus, Duchet 1987; Raychaudhuri 1992]
  For every $k \geq 2$, G is strongly chordal $\Rightarrow$ G^k is strongly chordal.

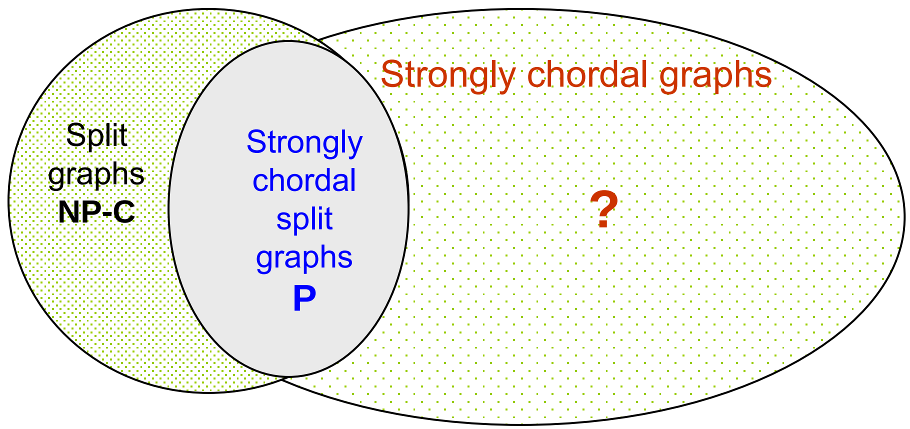
- A *split graph* is one whose vertex set can be partitioned into a clique and a stable set.



Squares of strongly chordal split graphs

- A graph G is *strongly chordal split graph* if it is strongly chordal and split graph .

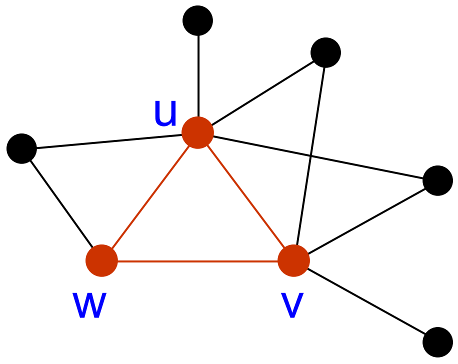


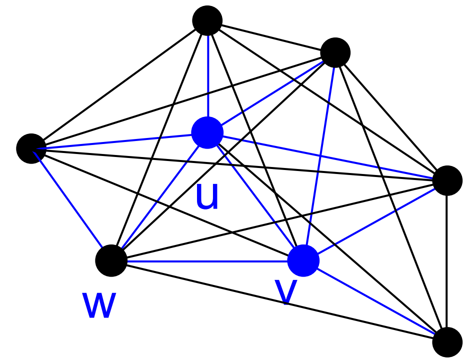


- In a graph, a vertex is *maximal* if its closed neighborhood is maximal.
- C(G) denotes the set of all maximal cliques of G.
- For split graphs $H = (V_H , E_H)$ we write $H = (C \cup S, E_H)$, meaning $V_H = C \cup S$ is a partition of the vertex set of H into a clique C and a stable set S.

Squares of strongly chordal split graphs

- Key Lemma: Let $H = (C \cup S, E_H)$ be a connected split graph without 3-sun. Then Q is a maximal clique in $G = H^2$ if and only if $Q = N_H[v]$ for some maximal vertex $v \in C$ of H.

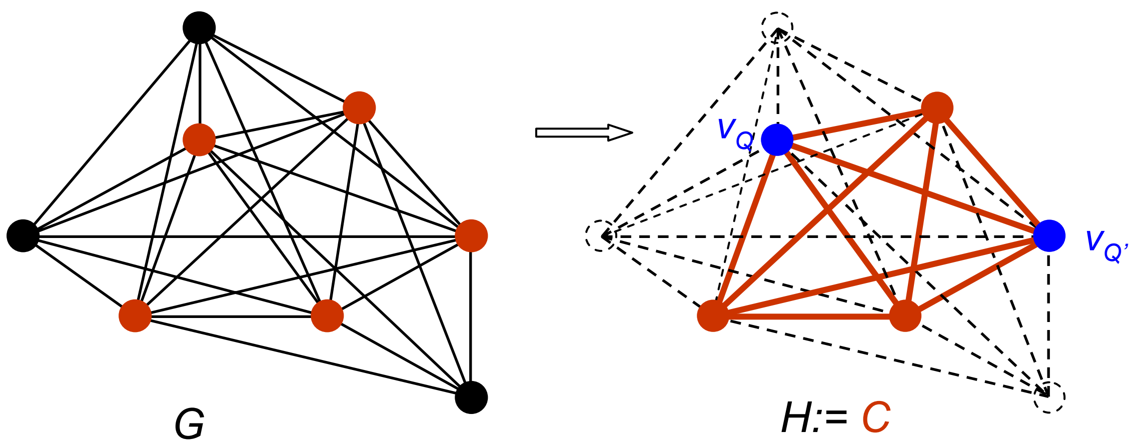




- Theorem 4: G is the square of a strongly chordal split graph if and only if G is strongly chordal and $|\bigcap_{Q \in C(G)} Q| \geq |C(G)|$.

Recognizing if *G* is the square of some s.c.s.g *H*

- Testing if *G is strongly chordal* can be done in time $O(\min\{n^2, m\log n\})$.
- Computing all maximal cliques of chordal *G* in linear time.
- Compute $C = \bigcap_{Q \in C(G)} Q$, for $Q \in C(G)$.
- If $|C| \geq |C(G)|$ then we construct H as follows:
- Put the clique *C* into *H*. For each Q, choose a unique vertex $v_Q \in C$ such that $v_Q \neq v_{Q'}$ if and only if $Q \neq Q'$.



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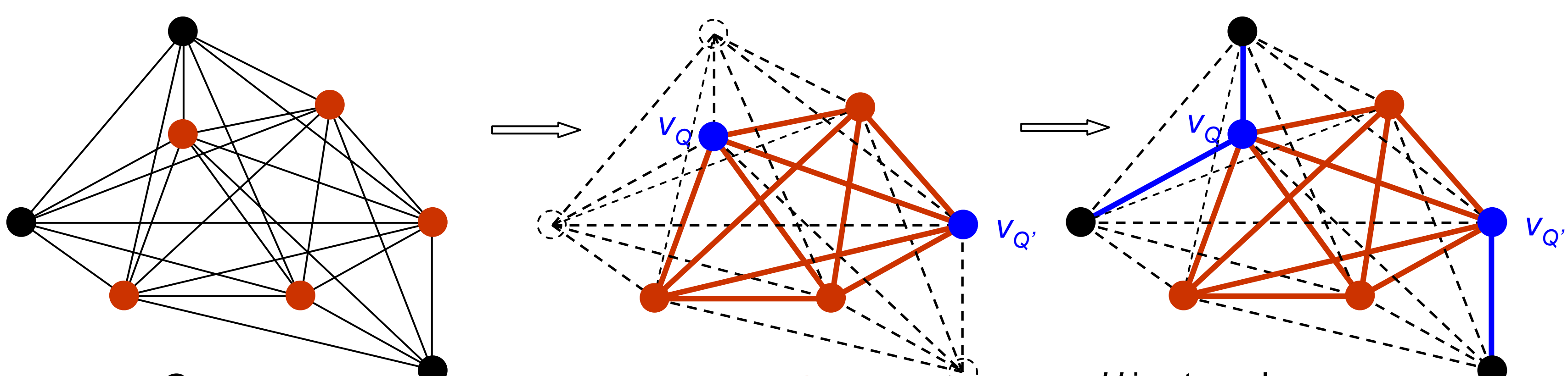
- Put the clique *C* into *H*. For each Q choose a unique vertex $v_Q \in C$ such that $v_Q \neq v_{Q'}$ if and only if $Q \neq Q'$.
- For each maximal clique *Q* of *G*, put the edges $v_Q v$, $v \in Q \setminus C$, into *H*.



G

H:= *C*

H is strongly chordal split square root of *G*

Recognizing if G is the square of some s.c.s.g H

Theorem 5: Given an n-vertex and m-edge graph $G=(V_G, E_G)$, recognizing if G is the square of some strongly chordal split graph can be done in time $O(\min\{n^2, m\log n\})$, and if any, such a square root H of G can be constructed in the same time.

Algorithm:

1. <u>**If**</u> G is strongly chordal <u>**then**</u>

2. compute all maximal cliques $Q_1, \ldots, Q_q$ of G

3. compute $C = \bigcap_{1 \le i \le q} Q_i$

4. <u>**If**</u> $|C| \ge q$ <u>**then**</u>

5. $V_H := V_G$; $E_H := \{ xy \mid x,y \in C \}$

6. <u>**for**</u> $i := 1$ to q <u>**do**</u>

7. choose a vertex $v_i \in C$ with $v_i \neq v_j$ for $i \neq j$

8. <u>**for**</u> $i := 1$ to q <u>**do**</u>

9. $E_H := E_H \cup \{ v_i v \mid v \in Q_i \setminus C \}$

10. **return** H

11. <u>**else**</u> **return** ' NO'

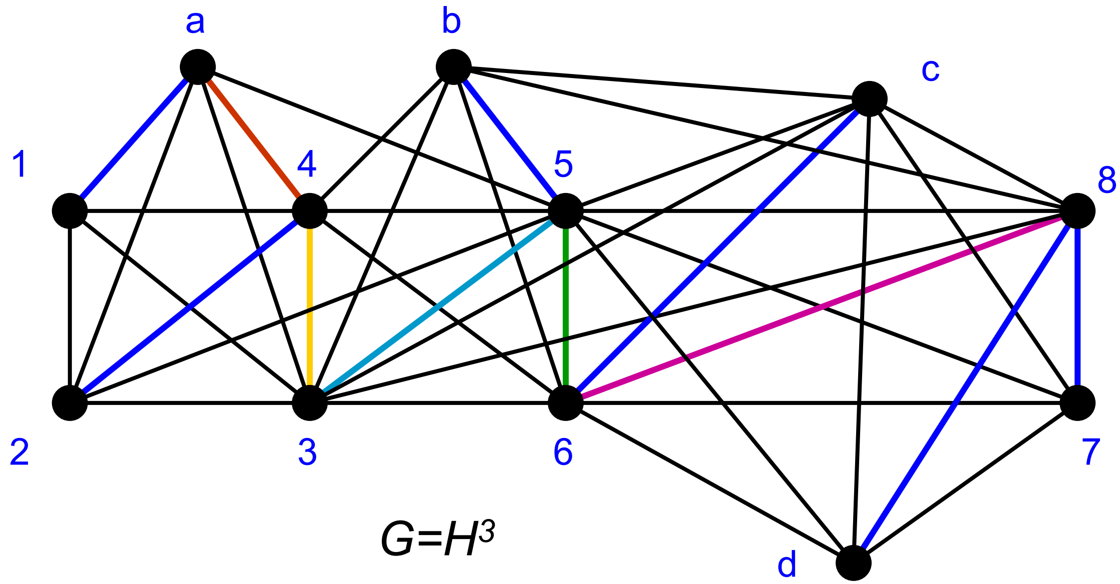
12. <u>**else**</u> **return** 'NO'

Cube of graphs with girth conditions

- Square roots with girth conditions have been considered by Farzad *et al.* [*Computing graph roots without short cycles,* STACS 2009].
- SQUARE OF GRAPH WITH GIRTH ≤ 4 is NP-complete, while SQUARE OF GRAPH WITH GIRTH ≥ 6 is polynomially solvable.
- Conjecture 3 [Farzad *et al*. STACS 2009]: k-TH POWER OF GRAPH WITH GIRTH $\geq 3k - 1$ is polynomially solvable.
- We show that CUBE OF GRAPH WITH GIRTH ≥ 10 is polynomially solvable.
  - Provide a good characterization of graphs that are cubes of a graph having girth at least 10.
  - Give a recognition algorithm in time $O(nm^2)$ for cubes of graphs with girth ≥ 10.

Maximal cliques in $G=H^3$

- **Key Lemma:** Let $G = (V, E_G)$ be a connected, non-complete graph such that $G=H^3$ for some graph $H= (V, E_H)$ with girth at least 10. Then $Q \subseteq V$ is a maximal clique in G iff $Q = N_H[u,v]$ for some edge $uv \in E_H$ with $\deg_H(u) \geq 2$ and $\deg_H(v) \geq 2$.



$Q_1=1234a = N_H[a,4]$

$Q_2=2345a = N_H[3,4]$

$Q_3=3456b = N_H[3,5]$

$Q_4=3568bc = N_H[5,6]$

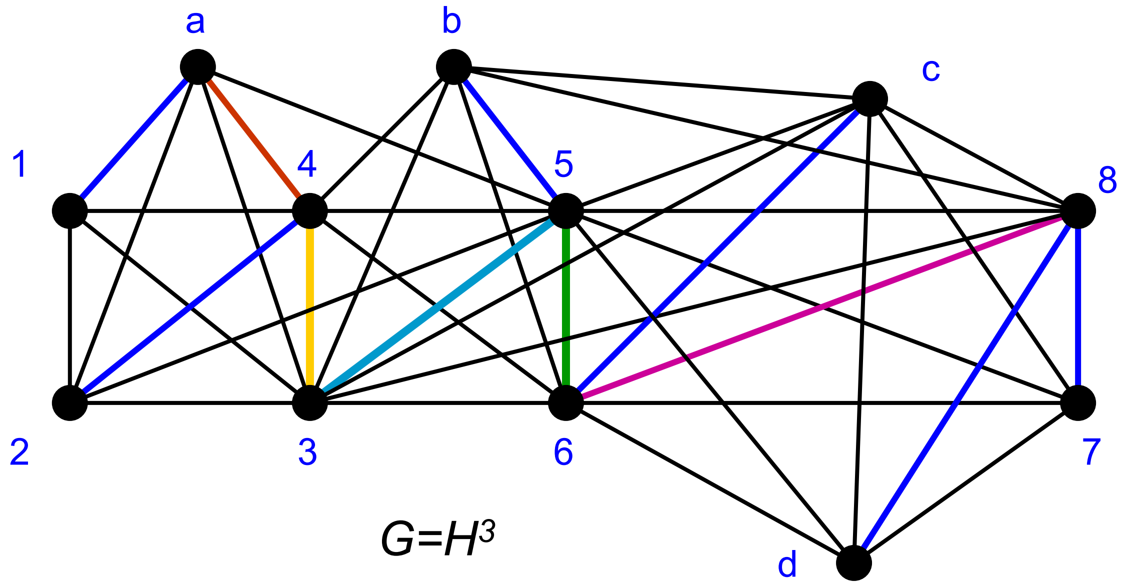
$Q_5=5678cd = N_H[6,8]$

- **Corollary:** If $G= (V_G, E_G)$ is the cube of some graph with girth at least 10, then G has at most $|E_G|$ maximal cliques.

Forced edges in G

- **Definition:** Let G be an arbitrary graph. An edge e of G is called *forced* if e is the intersection of two distinct maximal cliques in G.

- **Observation:** Let $G=H^3$ for some graph H with girth at least 10. Then, an edge of G is forced iff it is the mid-edge of a P_6 in H.



$Q_1=1234a$, $Q_2=2345a$

$Q_3=3456b$, $Q_4=3568bc$

$Q_5=5678cd$

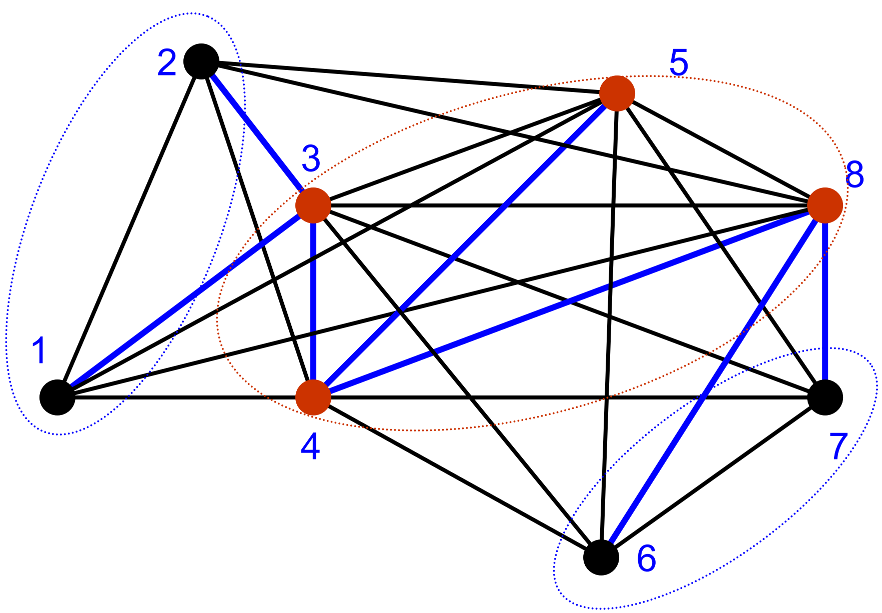
$34 = Q_1 \cap Q_3$

$35 = Q_2 \cap Q_4$

$56 = Q_3 \cap Q_5$

Trivial graphs

- Definition: A connected graph G is *trivial* if it contains a non-empty clique C such that $G \setminus C$ is the disjoint union of at most $|C|$-1 cliques and every vertex in C is adjacent to every vertex in $G \setminus C$.
- Observation:
  - A graph is trivial iff it is the cube of some tree of diameter at most 4.
  - Trivial graphs can be recognized in linear time.



- C_e consists of all maximal cliques containing e

Characterization of cubes of…

Theorem 6: Let G be a connected non-trivial graph. Let F be the subgraph of G consisting of all forced edges in G. Then, G is the cube of a graph with girth at least 10 iff the following conditions hold.

(i) For each $e \in F$, there exists a unique maximal clique $Q_e \in C_e$ such that

- (a) For every two distinct non-disjoint forced edges e and e', $e \cup e' \subseteq Q_e \cap Q_{e'}$
- (b) For every $Q \in C(G) \setminus \{Q_e \mid e \in F\}$ and for all forced edges e_1, e_2 in Q, $Q_{e1} \cap Q = Q_{e2} \cap Q$

(ii) For each $e \in F$, $C_e \setminus \{Q_e\}$ can be partitioned into non-empty disjoint sets $\mathcal{A}_e$ and $\mathcal{B}_e$ with

- (a) $Q \cap Q' = e$ iff $Q \in \mathcal{A}_e$ and $Q' \in \mathcal{B}_e$ or vice versa
- (b) setting $A_e = \cap_{Q \in \mathcal{A}e} Q$, $B_e = \cap_{Q \in \mathcal{B}e} Q$, all pairs of maximal cliques in $\mathcal{A}_e$ have the same intersection A_e , all pairs of maximal cliques in $\mathcal{B}_e$ have the same intersection B_e
- (c) $Q_e = A_e \cup B_e$ and $|A_e| \geq |\mathcal{A}_e| + 2$, $|B_e| \geq |\mathcal{B}_e| + 2$
- (d) $F[A_e \cap V_F]$ and $F[B_e \cap V_F]$ are stars with distinct universal vertices in e

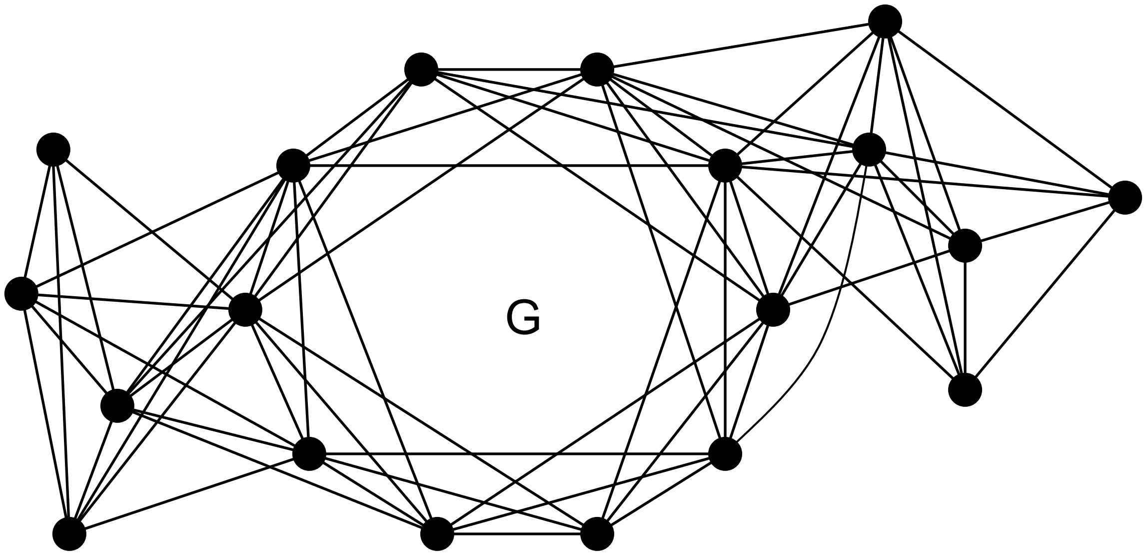
(iii) $C(G) = \cup_{e \in F} C_e$

(iv) $V_G \setminus \cup_{e \in F} Q_e$ consists of exactly the simplicial vertices of G

(v) F is connected and have girth at least 10.

Recognizing if *G* is the cube of some *H* with girth...

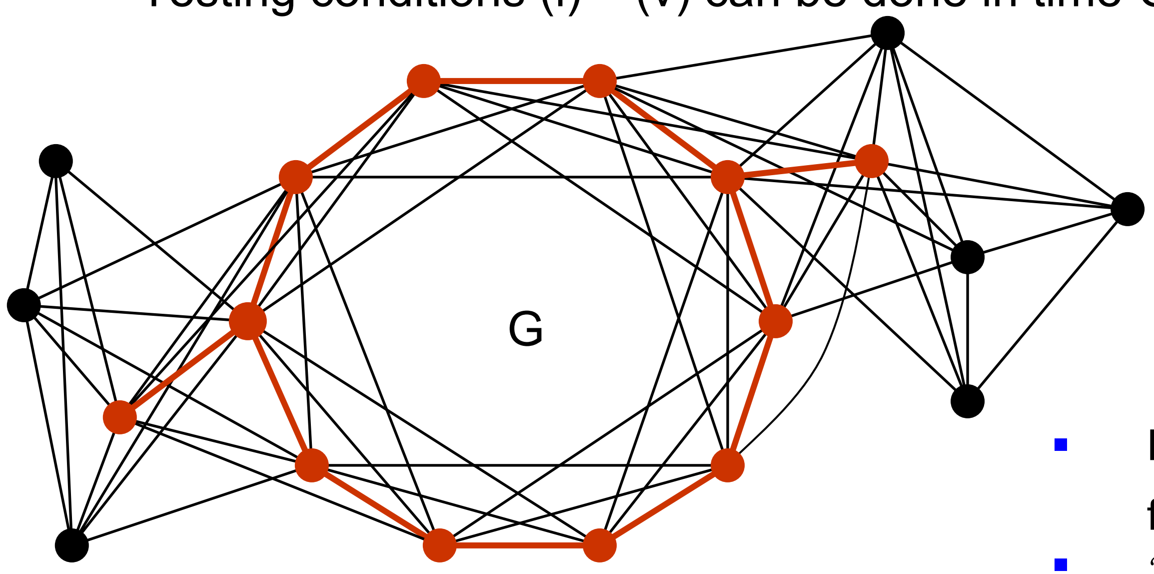
- List all maximal cliques of *G* in time $O(nm^2)$



• C_e consists of all maximal cliques containing e

Recognizing if G is the cube of some H with girth...

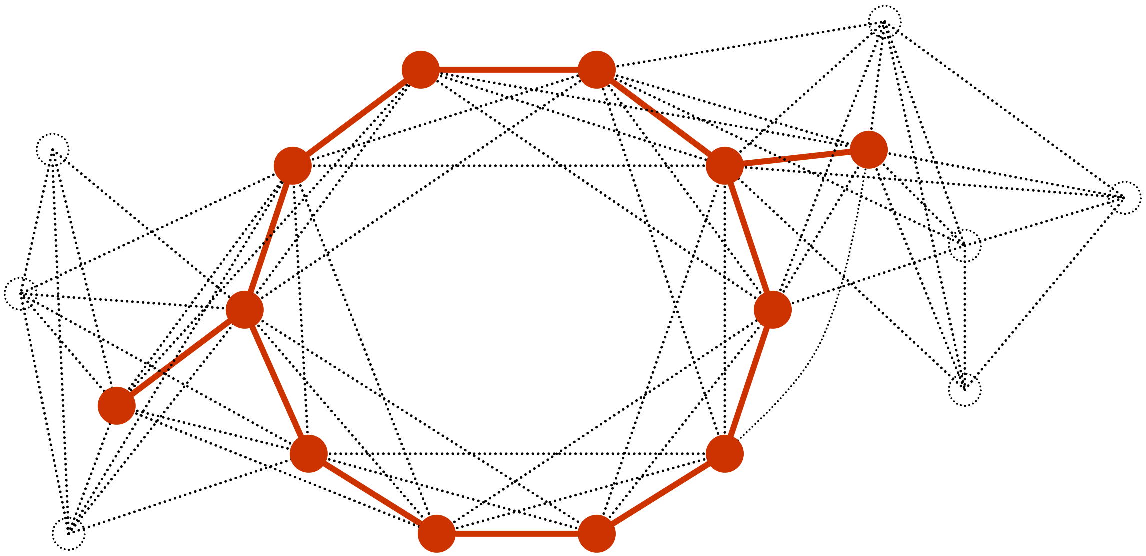
- List all maximal cliques of G in time $O(nm^2)$
- Compute the forced edges of G to form the subgraph F of G in time $O(m^2)$
- The lists C_e for each $e \in F$ can be computed in time $O(m^2)$
- Testing conditions (i) – (v) can be done in time $O(nm^2)$



- Partition $C_e = \mathcal{A}_e \cup \{Q_e\} \cup \mathcal{B}_e$ for each $e \in F$
- $\mathcal{K} := \{Q_e \mid e \in F\}$
- $A_e = \cap_{Q \in \mathcal{A}e} Q$, $B_e = \cap_{Q \in \mathcal{B}e} Q$

Recognizing if *G* is the cube of some *H* with girth...

- Testing conditions (i) – (v) can be done in time $O(nm^2)$
- If conditions (i) – (v) are satisfied then construct *H* is constructed as follows:
  - Put *F* into *H*



Recognizing if *G* is the cube of some *H* with girth...

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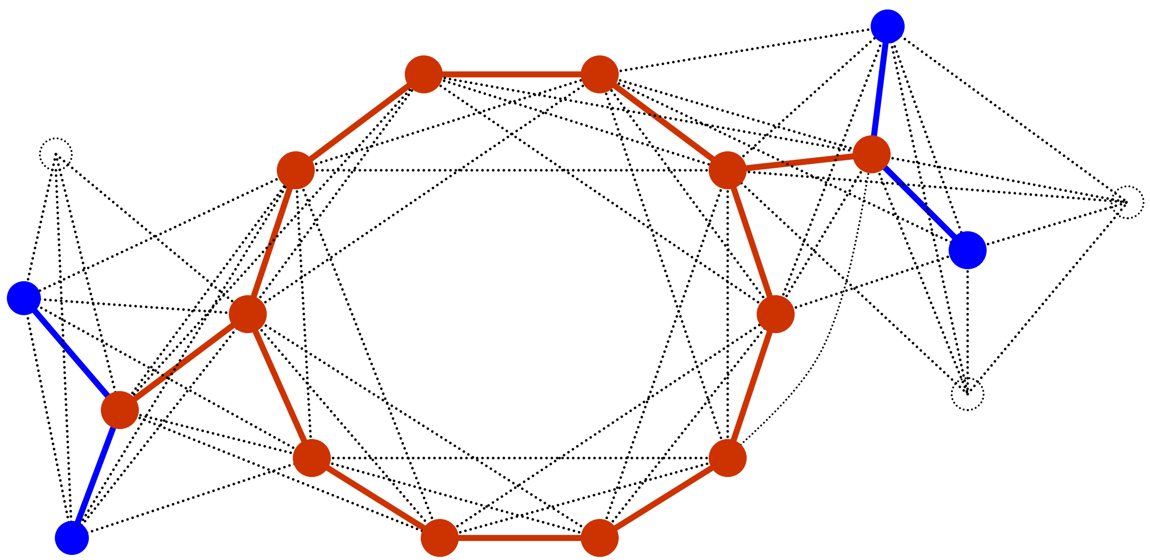
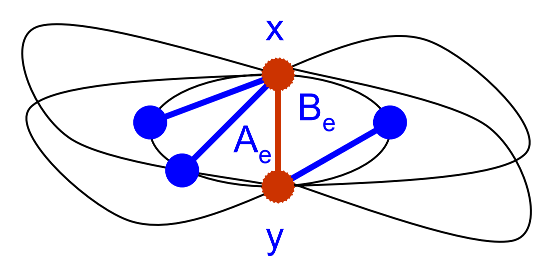
- $A_e = \cap_{Q \in \mathcal{A}e} Q$, $B_e = \cap_{Q \in \mathcal{B}e} Q$

- Put *F* into *H*
- For each $Q_e \in \mathcal{K}$ with $e = xy$, put all edges xu, $u \in A_e \setminus V_F$ and yv, $v \in B_e \setminus V_F$, into *H*





Recognizing if G is the cube of some H with girth...

- Partition $C_e = \mathcal{A}_e \cup \{Q_e\} \cup \mathcal{B}_e$ for each $e \in F$.
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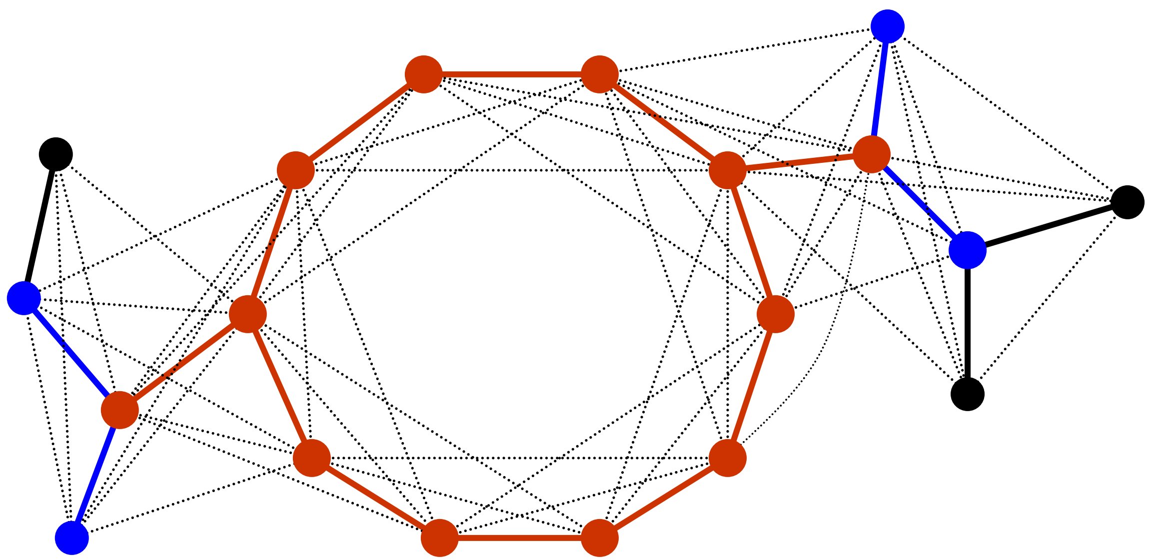
- Put F into H
- For each $Q_e \in \mathcal{K}$ with $e=xy$, put all edges xu, $u \in A_e \setminus V_F$ and yv, $v \in B_e \setminus V_F$, into H



- For each $Q \notin \mathcal{K}$ containing forced edge e
  - Choose a vertex $c_Q \in (Q \cap Q_e) \setminus V_F$
  - Put all edges $c_Q v$, $v \in Q \setminus V_H$, into H.

Corollaries

In the characterization for cubes of graphs with girth at least 10, if we replace the condition (v) by " F is a (C_4,C_6,C_8)-free bipartite" or " F is a tree " then:

- There is a good characterization and an $O(nm^2)$-time recognition for cubes of (C_4,C_6,C_8)-free bipartite, while CUBE OF BIPARITE GRAPH is NP-C.
- There is a good characterization and an $O(nm^2)$-time recognition for cubes of trees.

Conclusion

Hardness results:

- *k*-TH POWER OF BIPARTITE GRAPH is NP-complete for all fixed $k \geq 3$.
- *k*-TH POWER OF GRAPH is NP-complete for all fixed $k \geq 2$.
- *k*-TH POWER OF CHORDAL GRAPH is NP-complete for all fixed $k \geq 2$.

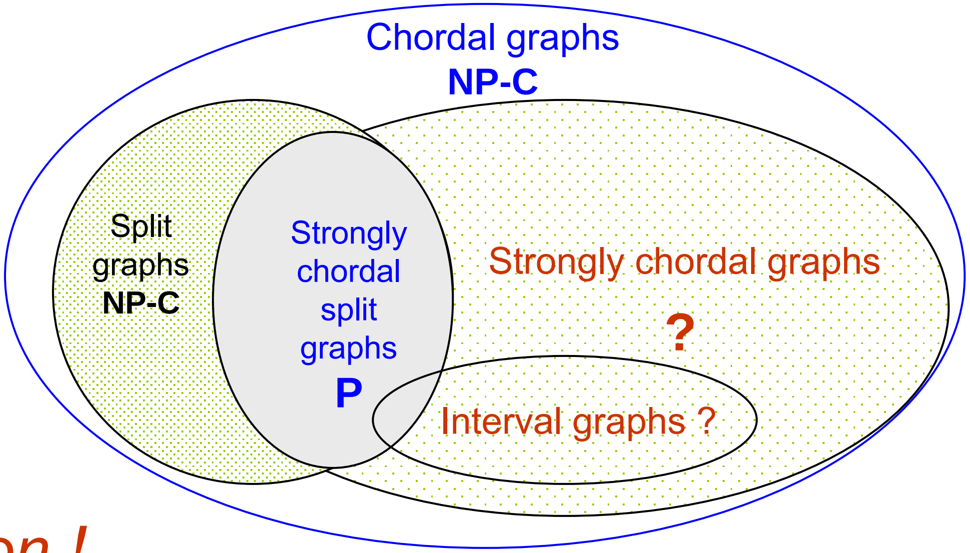
Efficient algorithms:

- Provide a good characterization of squares of strongly chordal split graphs that gives a recognition algorithm in time $O(\min\{n^2, m\log n\})$ for such squares.
- Give a good characterization of cubes of a graph with girth at least ten that leads to a recognition algorithm in time $O(nm^2)$ for such cubes.

Open Problems

- What is the complexity of recognizing powers of
  - strongly chordal graphs?
  - interval graphs?
  - chordal bipartite graphs (bipartite graphs without cycles of length at least six)?
  - Graphs with large girth ?

Recognizing squares of



Thank you for your attention !