Twin-width and polynomial kernels

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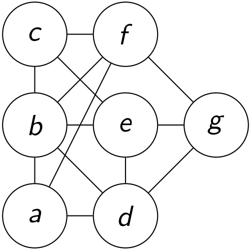
Contents

Twin-width and Contraction Sequences

Goal : contract near-twins, record neighbourhood errors.



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d-contraction sequence : maximal red degree d.

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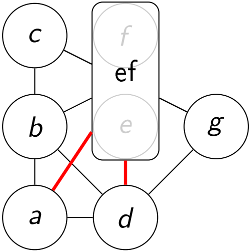

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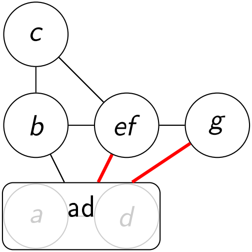

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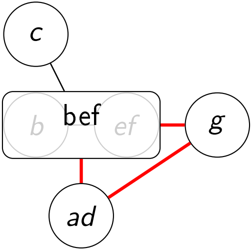

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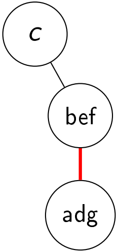

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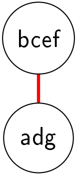

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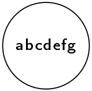

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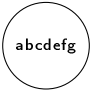

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> **Definition** *The twin-width of G is the least integer d such that G admits a d-contraction sequence.*

Bounded twin-width classes:
Bounded treewidth, rank-width, queue/stack number, Proper minor-closed, **Grids**, $\Omega(\log)$-subdivisions... Stable under FO-transductions.

Complexity definitions

Definition *Kernel for parameterized $\mathcal{Q}$: polynomial time reduction from $(I,k) \in \mathcal{Q}$ to an equivalent $(I',k') \in \mathcal{Q}$. A kernel for parameter k is polynomial if $|I'| = O(k^p)$.*

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Definition Vertex Cover: *instances (G,k), is there a set $S \subseteq V$ of size at most k covering E.*

- ***Connected V-C***: *require S to be connected in G.*
- ***Capacitated V-C***: *adds a capacity $c : V \to \mathbb{N}$ s.t. $E \to S$ without exceeding capacity of any $x \in S$.*

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Definition k-Dominating Set: *Instances (G,k), is there a set $S \subseteq V$ of size at most k dominating V.*

Twin-width : FPT Consequences

Theorem (Bonnet, Kim, Thomassé, Watrigant '20) *FO model checking is FPT with respects to formula size on classes of bounded twin-width given a contraction sequence.*

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- k-IS : $\exists x_1 ... \exists x_k \bigwedge_{1 \leqslant i \leqslant j \leqslant k} \neg(x_i = x_j \vee E(x_i, x_j))$,
- k-DS : $\exists x_1 ... \exists x_k \forall x \bigvee_{1 \leqslant i \leqslant k} (x = x_i) \vee \bigvee_{1 \leqslant i \leqslant k} E(x, x_i)$.

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k-DS admits $O(k^{(t+1)^2})$ kernels notably on sparse ($K_{t,t}$-free) classes (Philip, Raman, Sikdar '12), what about **bounded twin-width**?

Results overview : Kernelization

In this talk:

[1] even given a 4-sequence

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Unless coNP $\subseteq$ NP*/poly, k*-Dominating Set *on graphs of twin-width at most 4 does not admit a polynomial kernel* [1].

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Theorem (Bonnet, Kim, R., Thomassé, Watrigant '21)
Connected *k*-Vertex Cover *and* Capacitated *k*-Vertex Cover *admit a kernel with* $O(k^{\frac{3}{2}})$ *and* $O(k^2)$ *vertices respectively on classes of bounded twin-width, and even of VC-density 1.*

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Lemma (Bonnet, Kim, R., Thomassé, Watrigant '21)
There exists f such that for every G of tww(d) and $X \subseteq V(G)$, the number of distinct neighborhoods in X is at most $f(d)|X|$.

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Theorem (Bonnet, Kim, R., Thomassé, Watrigant '21)
One can decide in polynomial time if a graph has twin-width at most 1.

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Kernel pre-processing:

- Bounded tww G: take X 2-approx for VC,
- if $|X| \geqslant 2k+1$: no solution,
- $\rightarrow$ in X, number of distinct neighbourhoods $\leqslant f(d)k$

A (sub)quadratic kernel for C-VC on VC-density 1

Reduction rule: If there is $S \subseteq V(G) \setminus X$ with identical neighbourhood in X and $|S| > k$, delete a vertex of S.

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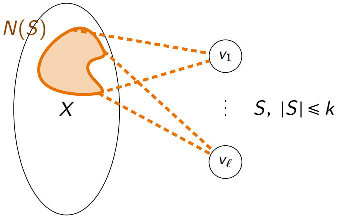

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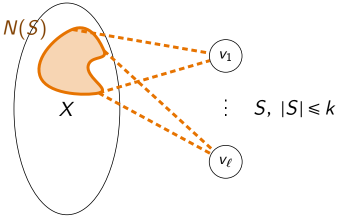

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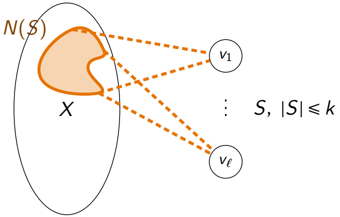

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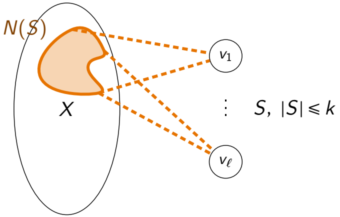

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If (G, k) has a solution, replace deletion with twin $\rightarrow$ reconnects T

If (G', k) has a solution T, T is also a solution for (G, k),

- Take deleted s, $S \backslash s \not\subseteq T$ as ind. set with $|S| \geqslant k$,
- Therefore $N(S) \subseteq T$, and s is covered by T in G.

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Theorem (Bonnet, Kim, R., Thomassé, Watrigant '21)
On classes of VC-density 1, Connected k-Vertex Cover *and* Capacitated k-Vertex Cover *admit kernels with $O(k^{1.5})$ and $O(k^2)$ vertices respectively. In particular, they also do in classes of bounded twin-width.*

Contents

Framework

OR-cross-composition, from $\mathscr{L}$ to parameterized $\mathscr{Q}$:
Polytime reduction the **OR** of t ("equivalent") instances of $\mathscr{L}$ to one instance of $\mathscr{Q}$.

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Polytime reduction the **OR** of t ("equivalent") instances of $\mathscr{L}$ to one instance of $\mathscr{Q}$.

> **Theorem (Bodlaender, Jansen, Kratsch '12)** *If an NP-hard language $\mathscr{L}$ admits an OR-cross-composition into a parameterized problem $\mathscr{Q}$, then $\mathscr{Q}$ does not admit a polynomial kernel unless* coNP $\subseteq$ NP*/poly.*

- $\mathscr{L}$: Tailored k-DS (next slide),
- $\mathscr{Q}$: k-DS with twin-width 4.

Planar 3SAT and Tailored k-DS

Theorem k-Dominating Set *remains* NP-*hard even when restricted to instances* $(G, k, \mathscr{B} = \{B_1, \ldots, B_k\})$ *such that:*

- $\mathscr{B}$ *partitions* $V(G)$*,* G *has 4-sequence* $\to G/\mathscr{B}$*,*
- $G/\mathscr{B}$ *is a spanning subgraph of a grid,*
- *every dominating set of* G *intersects each* B_i*, for* $i \in [k]$

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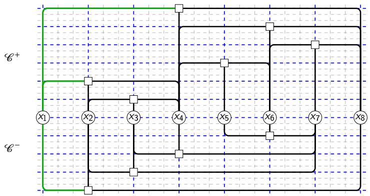

Figure: Reduction sketch

OR-composition sketch

Tailored k-**DS**, instances $(I_i)_{i \in [t]} \to k$-**DS**, $tww \leqslant 4$ instance (H, k)

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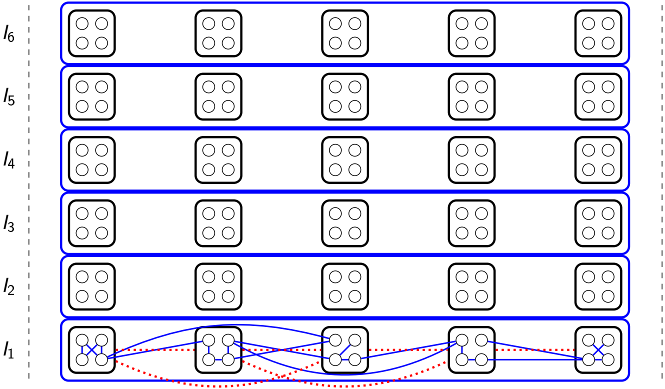

Figure: Instances ↔ layers, partition classes ↔ boxes.

OR-composition sketch

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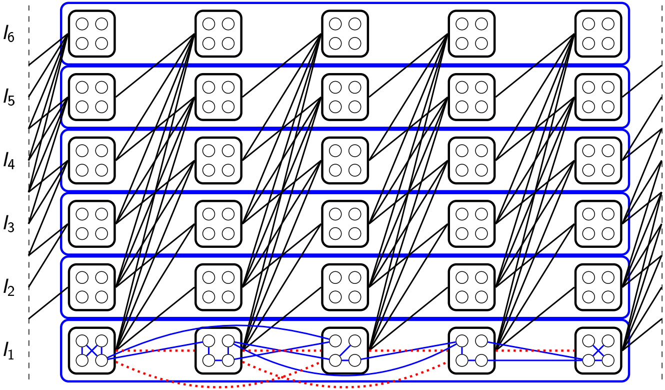
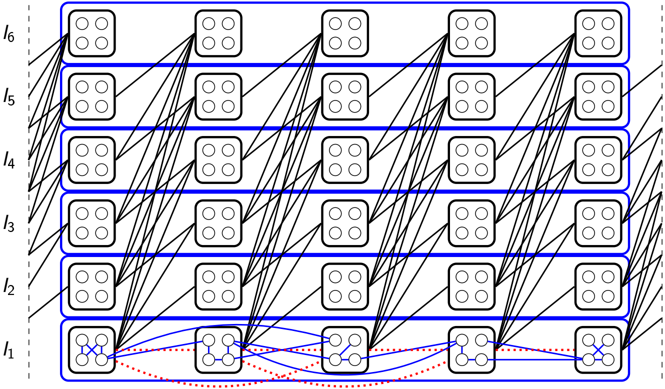


Figure: Instances $\leftrightarrow$ layers, partition classes $\leftrightarrow$ boxes.

In H**:** Top instance $\leftrightarrow$ dummy, half graphs cycle between classes

Positive Tailored instance ⇒ Positive composition

Assume one instance (I_4 here) is positive, pick its solution in H:



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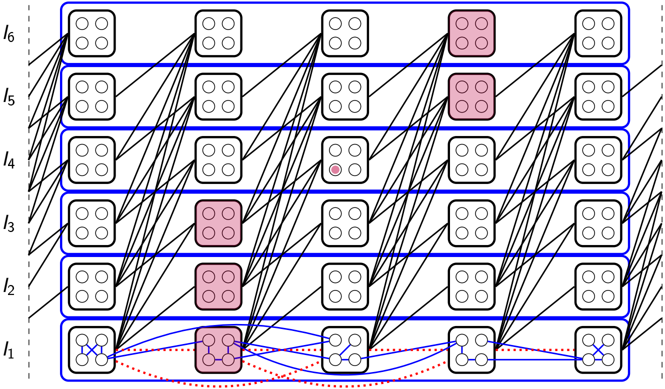

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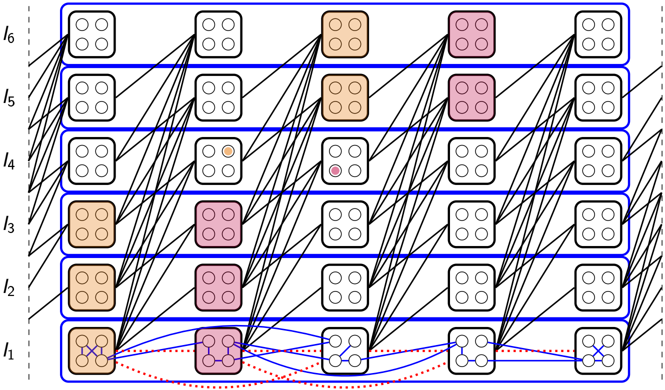

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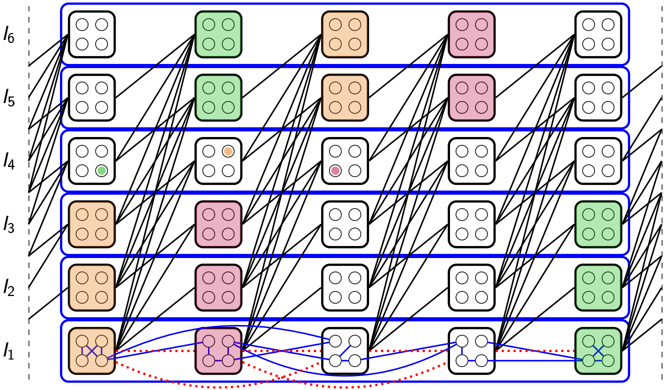

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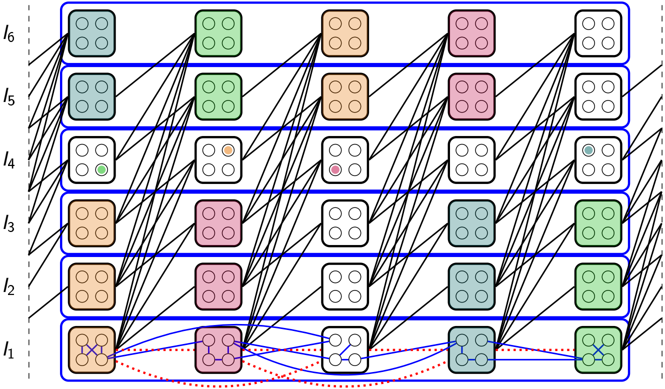

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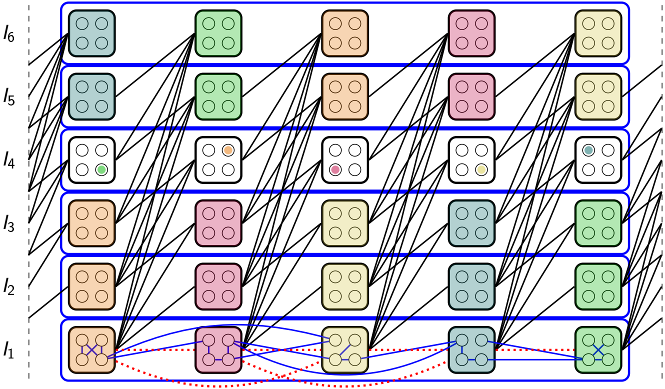

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Positive composition ⇒ Positive tailored instance

Assume (H, k) is positive:

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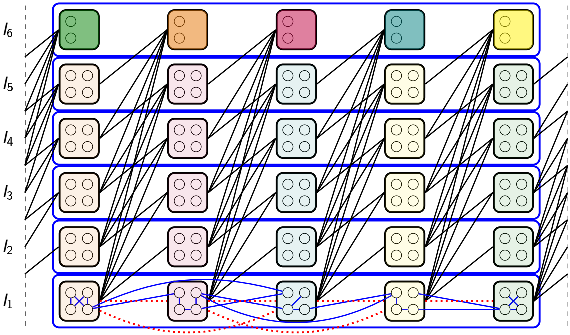

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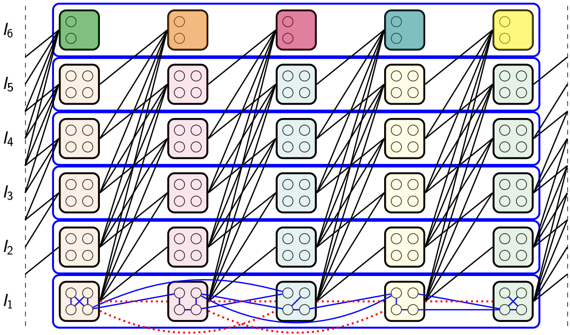

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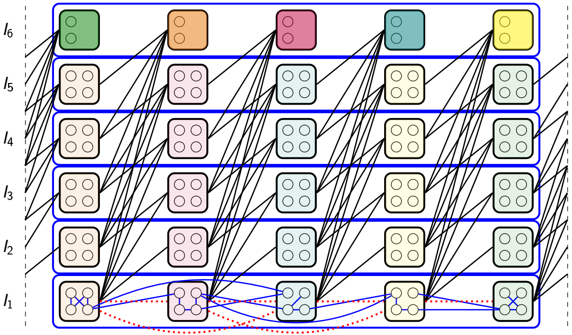
- Assume not...



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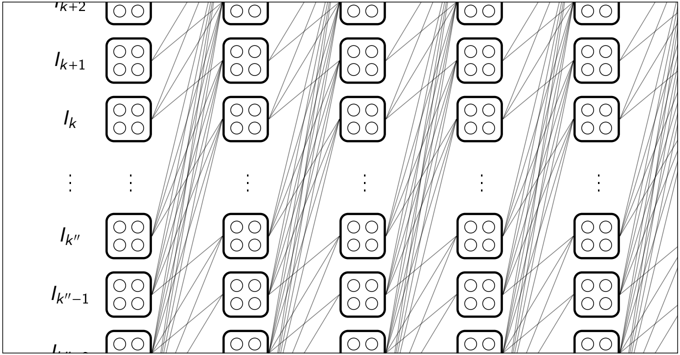
- Non-identical layer choices → domination gap,



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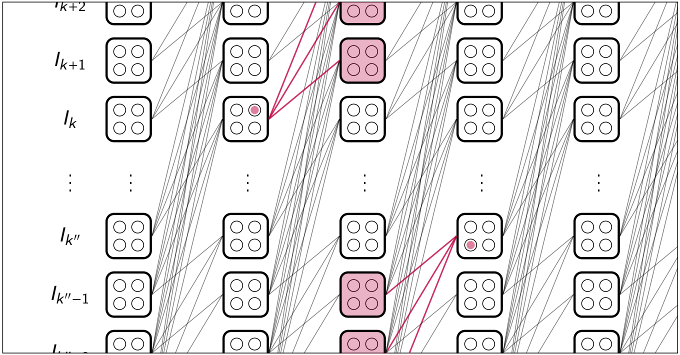
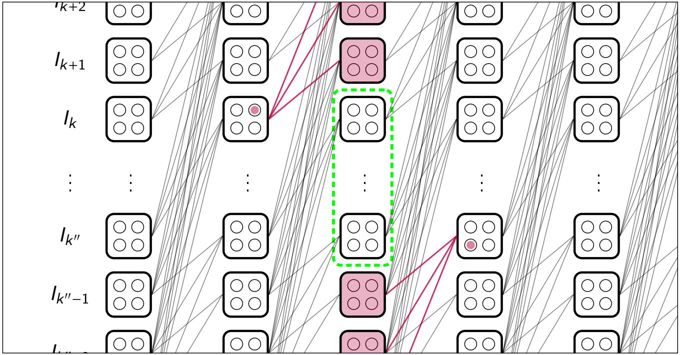


Positive composition ⇒ Positive tailored instance

- Non-identical layer choices → domination gap,
- At least two classes are not dominated by neighbouring columns,
- Only one choice left in this column → absurd.



Bounding the twin-width

- For any I_i, each $\mathscr{B}_{i,j}$ is a module in $H - I_i$ $\rightarrow$ contraction impacts only I_i,

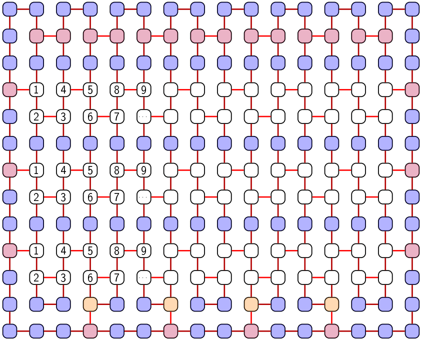


Figure: An instance after contracting classes.

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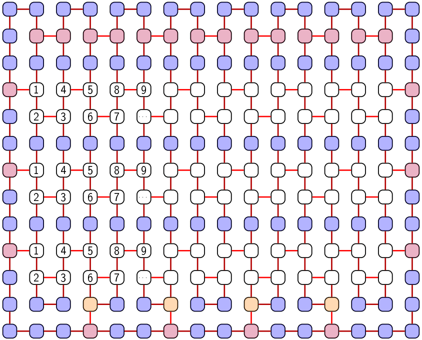
- For any I_i, each $\mathscr{B}_{i,j}$ is a module in $H - I_i$ $\rightarrow$ contraction impacts only I_i,
- Contract each partition class of each instance



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- For any I_i, each $\mathscr{B}_{i,j}$ is a module in $H - I_i$ $\rightarrow$ contraction impacts only I_i,
- Contract each partition class of each instance
- Result: red subgraph of grid (altered for *tww* 4).

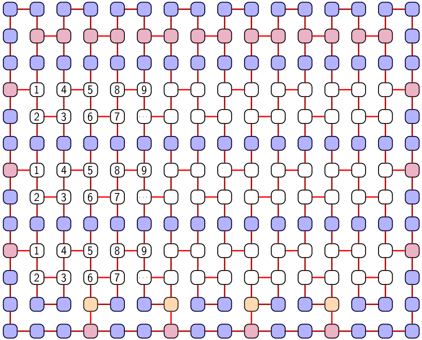
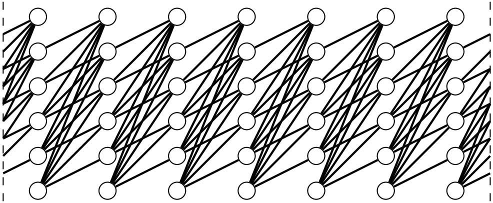


Figure: An instance after contracting classes.

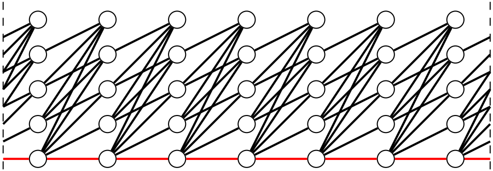
Bounding the twin-width

- Between instances : cycle of half graphs,



Bounding the twin-width

- Between instances : cycle of half graphs,
- Half graphs cycles have tww 3 : successively contract bottommost two layers



Bounding the twin-width

- In the Composition: half-graph cycle between "homogeneous" vertices, (at the same position in their respective instance).

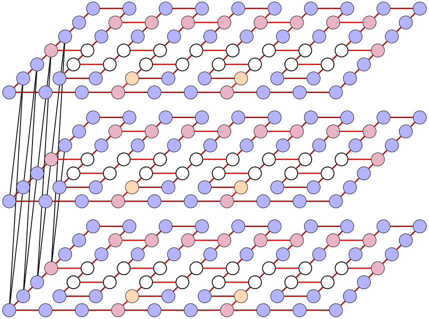


Figure: Instances and the half-graph cycle

Bounding the twin-width

- In the Composition: half-graph cycle between "homogeneous" vertices, (at the same position in their respective instance).
- Goal: Successively contract the bottommost two instances.

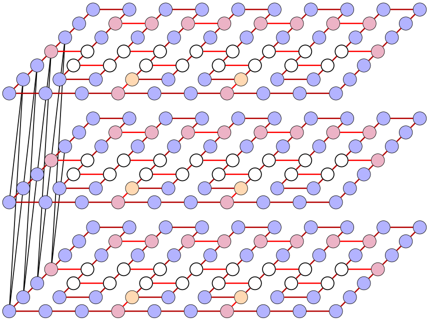


Figure: Instances and the half-graph cycle

Bounding the twin-width

- In the Composition: half-graph cycle between "homogeneous" vertices, (at the same position in their respective instance).
- Goal: Successively contract the bottommost two instances.
- Each contraction: creates red edges only locally → induction

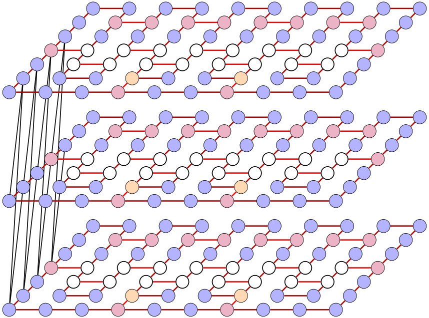


Figure: Instances and the half-graph cycle

Concluding

- NP-hard Tailored k-DS OR-cross-composes into k-DS,
- The composed instance has tww at most four.

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Theorem (Bonnet, Kim, R., Thomassé, Watrigant '21)
Unless coNP $\subseteq$ NP*/poly, k*-Dominating Set *on graphs of twin-width at most 4 does not admit a polynomial kernel, even if a 4-sequence of the graph is given.*

Contents

Open questions

- A *linear* kernel for Connected Vertex Cover on VC-density 1.
- Are there polynomial kernels for k-Dominating Set on twin-width 2 or 3?

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Thanks for your attention!