# Twin-width: Forbidden subdivisions & Polynomial kernels

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ENS de Lyon

*joint work with*
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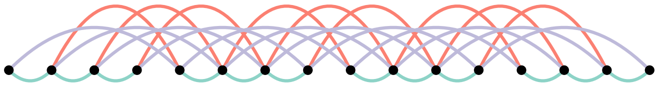

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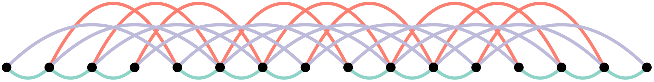

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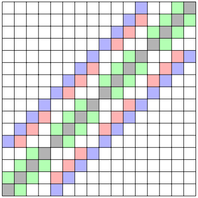
**Matricial:** Adjacency matrix coloured by $k$ increasing zones



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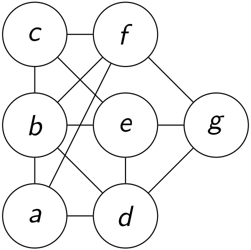
Structural def: contract near-twins, record neighbourhood errors.

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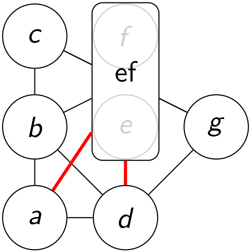


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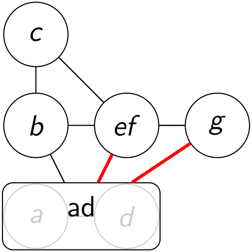


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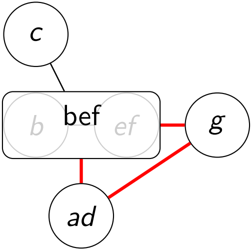


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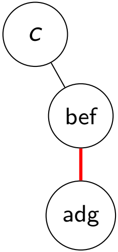


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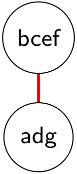


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**Definition** *The twin-width of $G$ is the least integer $d$ such that $G$ admits a $d$-contraction sequence.*

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$\phi$ can capture: Dominating Set, Vertex Cover, Independent Set...

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2. Better complexity: existence of polynomial kernels?

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**Theorem (Bonnet, Kim, R., Thomassé, Watrigant '21)**
*On bounded tww, Connected k-VC admits a $O(k^{\frac{3}{2}})$ kernel.*

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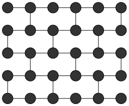

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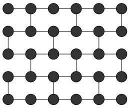

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**Theorem (Twin-width: B,K,T,W '20)**
*Proper minor-closed classes have bounded twin-width.*

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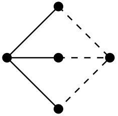

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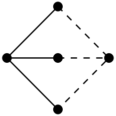
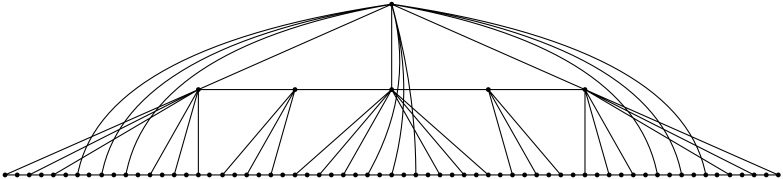


**Theorem (Sintiari, Trotignon '19)**
*There exist graphs forbidding induced thetas, having arbitrarily large girth and treewidth.*



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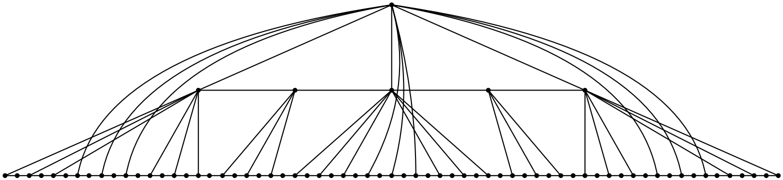

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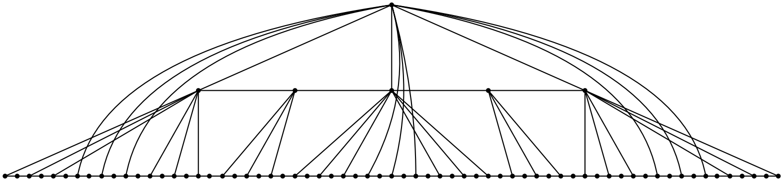

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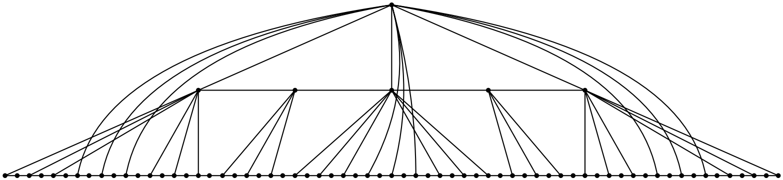

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Do such graphs have bounded twin-width?

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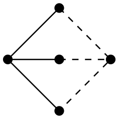
Our class: Theta-free, girth 5.



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Little information on **connected** subgraphs, hard finding thetas.

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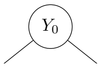

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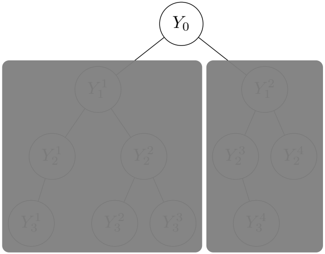

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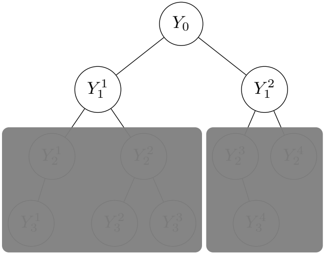

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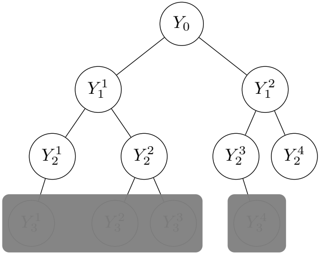

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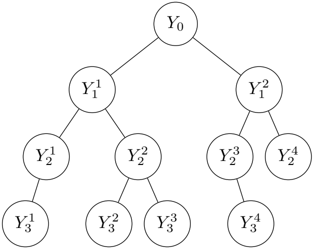

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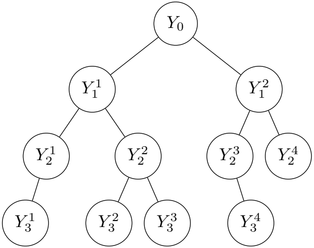

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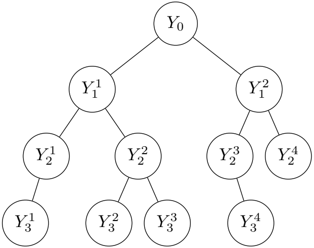


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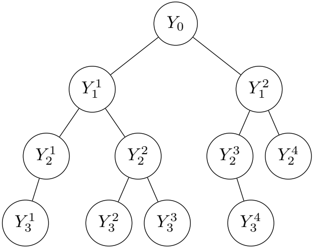


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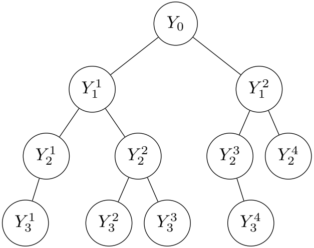


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Global structure: **tree** relating components $Y_i^j$.

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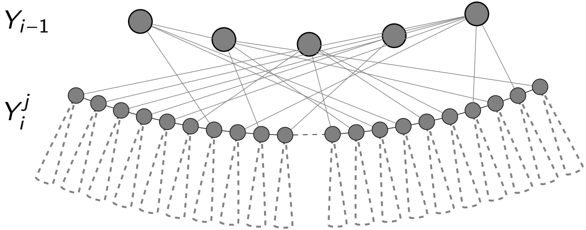
# Component-Antecedent Structure

How do successors of **different** vertices of $Y_{i-1}$ relate in $Y_i$?



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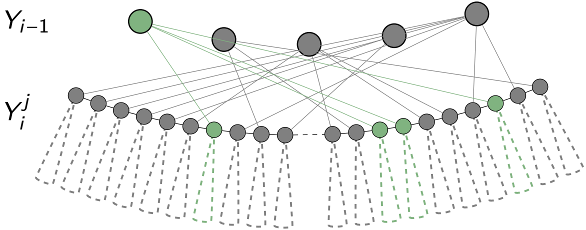

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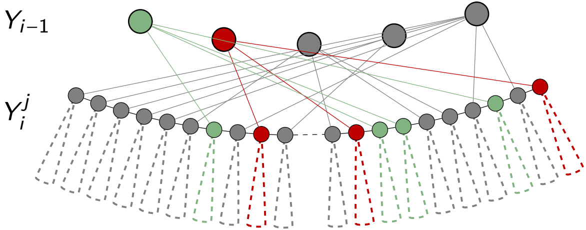

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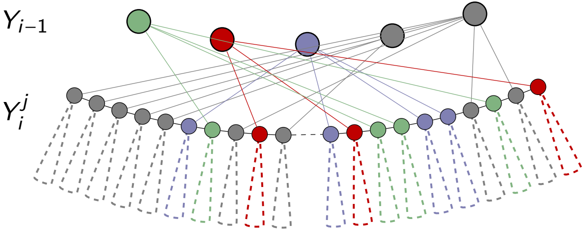

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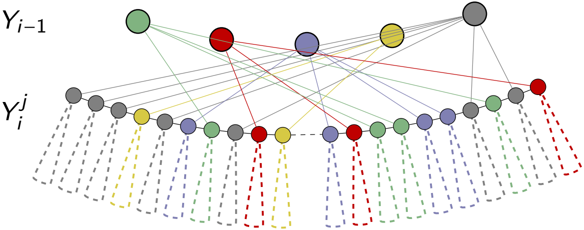

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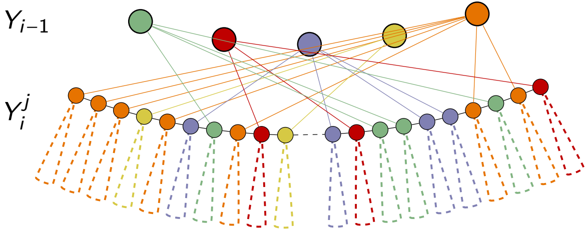

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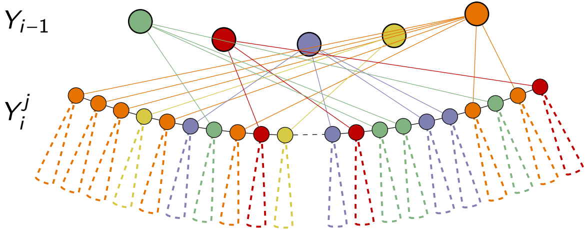


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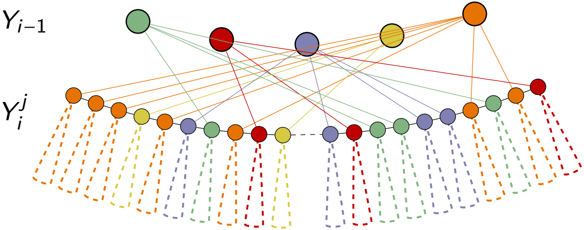


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Natural order: DFS exhausting private branches following $P_i^j$.

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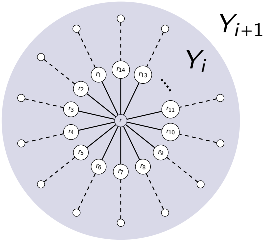

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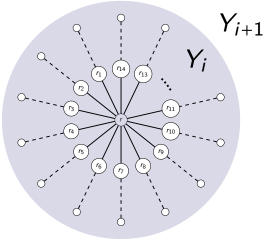

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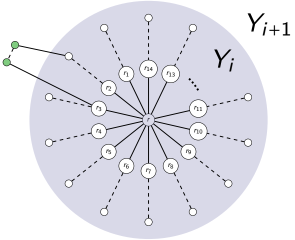

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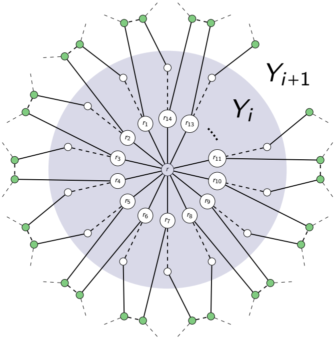

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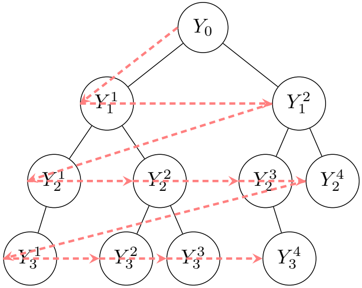


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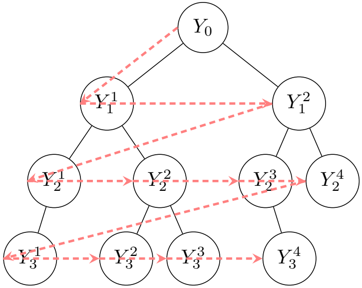


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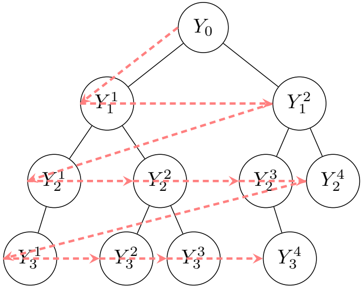


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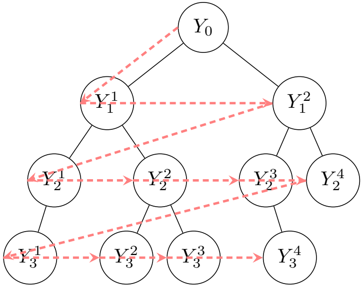


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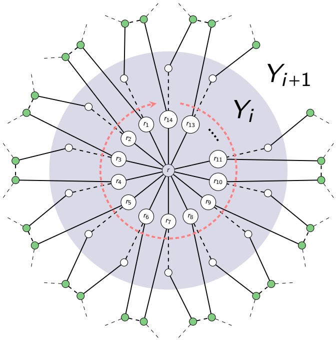


Fig. Intra-component DFS

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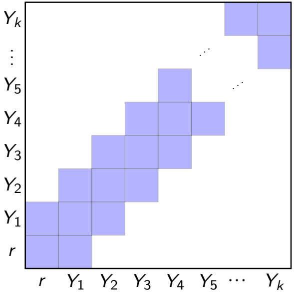

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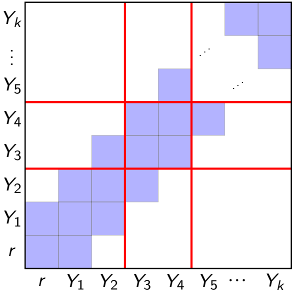

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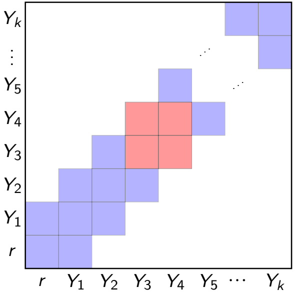

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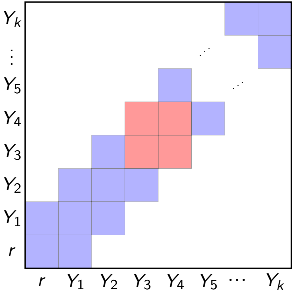


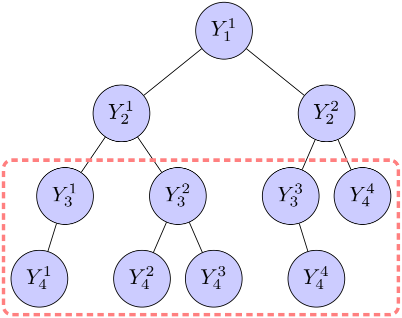

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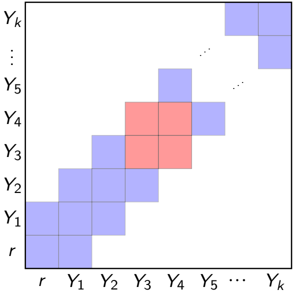


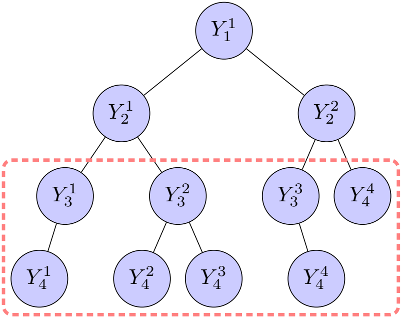


**Lemma** *If matrices $M_<$ have arbitrarily large grid minors, some submatrices indexed by $(Y_i \cup Y_{i+1})^2$ also do.*

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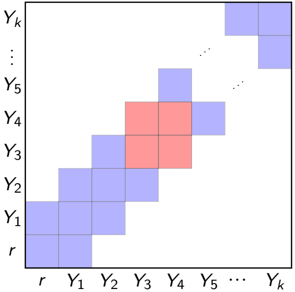

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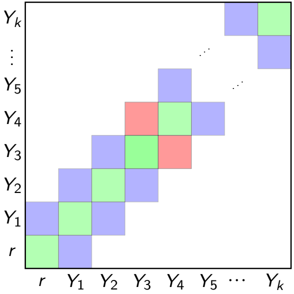

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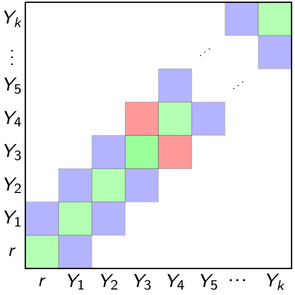


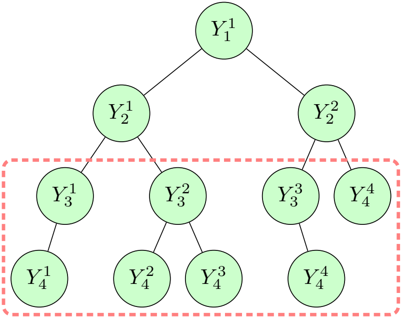

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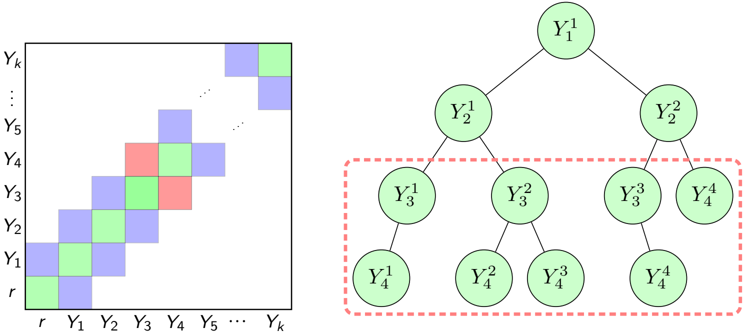


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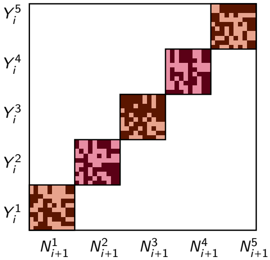

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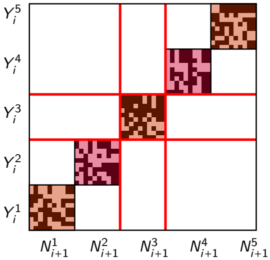

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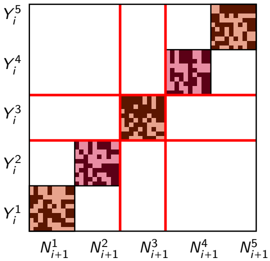


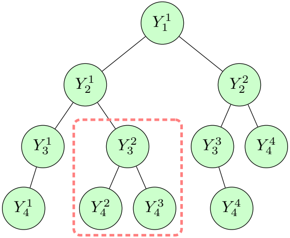


**Lemma** *If $M_<$ admit arbitrarily large grid minors, some submatrices indexed by $Y_i^j$ and its successors do too.*

# Excluding Grid Minors

Assume large grid minor on $Y_i^j$ and successors:

| | | | | | | |
|---|---|---|---|---|---|---|
| $R_t$ | 1 | 1 | 1 | 1 | 1 | 1 |
| $\vdots$ | | | | $\vdots$ | | 1 |
| $R_3$ | 1 | 1 | 1 | 1 | $\cdots$ | 1 |
| $R_2$ | 1 | 1 | 1 | 1 | | 1 |
| $R_1$ | 1 | 1 | 1 | 1 | | 1 |
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| ⋮ | | | | ⋮ | | 1 |
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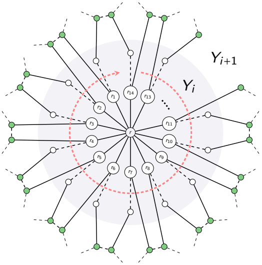

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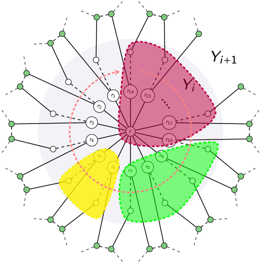

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| | | | | | | |
|---|---|---|---|---|---|---|
| $R_t$ | 1 | 1 | 1 | 1 | 1 | 1 |
| $\vdots$ | | | | $\vdots$ | | 1 |
| $R_3$ | 1 | 1 | 1 | 1 | $\cdots$ | 1 |
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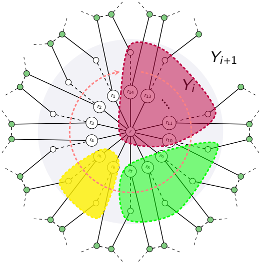
| | $C_1$ | $C_1$ | $C_2$ | $C_3$ | $\cdots$ | $C_t$ |



# Excluding Grid Minors

Assume large grid minor on $Y_i^j$ and successors:

- Row sets ↔ subforests of $Y_i^j$,
- Column sets ↔ subpaths of $Y_{i+1}$,
- Non-zero entries → **consecutivities**.

Proof scheme:

- Concentrate rows around $r \in Y_i^j$,
- Intertwining around $r$ **violates** our order

**Theorem (B,K,R,T,W '21)**
*Theta-free graphs of girth at least 5 have bounded twin-width.*

| | | | | | | |
|---|---|---|---|---|---|---|
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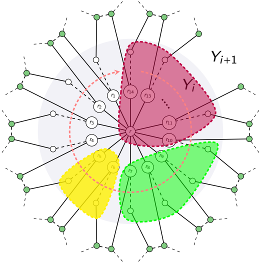

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Questions ?

$\hookrightarrow$ $k$-DS admits no polynomial kernels

# Complexity definitions

**Definition** *Kernel for parameterized problem $\mathcal{Q}$: polytime reduction from $(I, k) \in \mathcal{Q}$ to equivalent $(I', k') \in \mathcal{Q}$.*
*A kernel for parameter $k$ is polynomial if $|I'| = O(k^p)$.*

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# Twin-width : FPT Consequences

**Theorem (Bonnet, Kim, Thomassé, Watrigant '20)**
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- $k$-IS : $\exists x_1 \ldots \exists x_k \bigwedge_{1 \leqslant i \leqslant j \leqslant k} \neg(x_i = x_j \vee E(x_i, x_j))$,
- $k$-DS : $\exists x_1 \ldots \exists x_k \forall x \bigvee_{1 \leqslant i \leqslant k} (x = x_i) \vee \bigvee_{1 \leqslant i \leqslant k} E(x, x_i)$.

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$k$-DS admits $O(k^{(t+1)^2})$ kernels notably on sparse ($K_{t,t}$-free) classes (Philip, Raman, Sikdar '12), what about **bounded twin-width**?

# Lower bounds framework

**OR-cross-composition**, from $\mathscr{L}$ to parameterized $\mathscr{Q}$:
Polytime reduction the **OR** of $t$ ("equivalent") instances of $\mathscr{L}$ to one instance of $\mathscr{Q}$.

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> **Theorem (Bodlaender, Jansen, Kratsch '12)**
> *If an NP-hard language $\mathscr{L}$ admits an OR-cross-composition into a parameterized problem $\mathscr{Q}$, then $\mathscr{Q}$ does not admit a polynomial kernel unless* coNP $\subseteq$ NP*/poly.*

- $\mathscr{L}$ : Tailored $k$-DS (next slide),
- $\mathscr{Q}$ : $k$-DS with twin-width 4.

# Tailored k-DS on grid-like graphs

**Theorem**

$k$-Dominating Set *remains* NP*-hard even when restricted to instances* $(G, k, \mathscr{B} = \{B_1, \ldots, B_k\})$ *such that:*

- $\mathscr{B}$ *partitions* $V(G)$*,* $G$ *has 4-sequence* $\rightarrow G/\mathscr{B}$*,*
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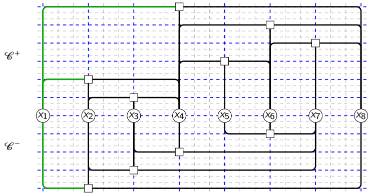


Fig. Reduction sketch

# OR-composition sketch

**Tailored** $k$-**DS**, instances $(I_i)_{i \in [t]} \rightarrow k$-**DS**, $tww \leqslant 4$ instance $(H, k)$

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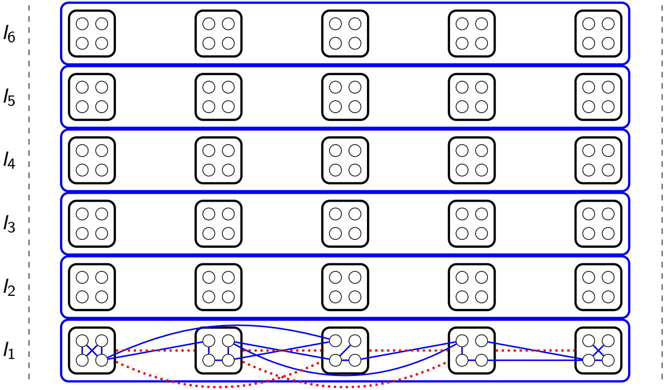


Fig. Instances ↔ layers, partition classes ↔ boxes.

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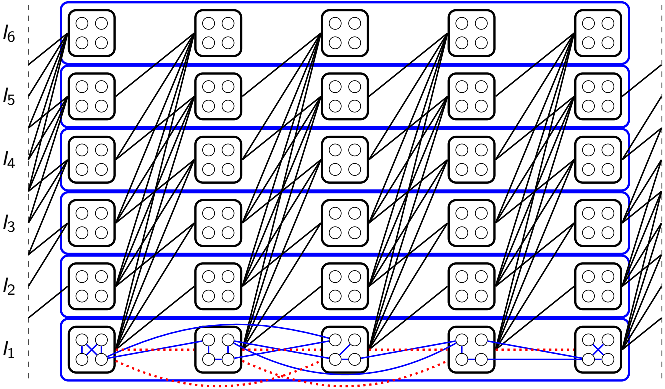
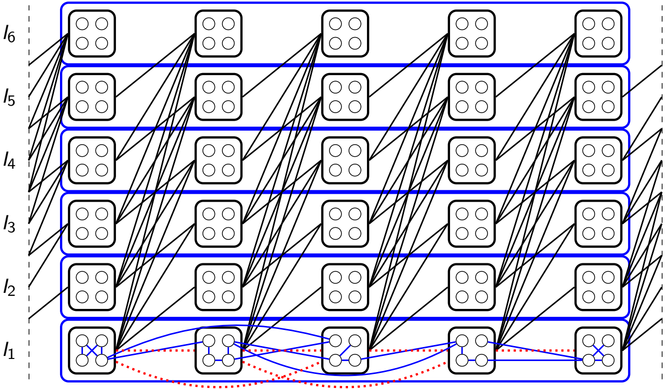



Fig. Instances ↔ layers, partition classes ↔ boxes.
**In** $H$**:** Top instance ↔ dummy, half graphs cycle between classes

# Positive Tailored instance ⇒ Positive composition

Assume one instance ($I_4$ here) is positive, pick its solution in $H$:

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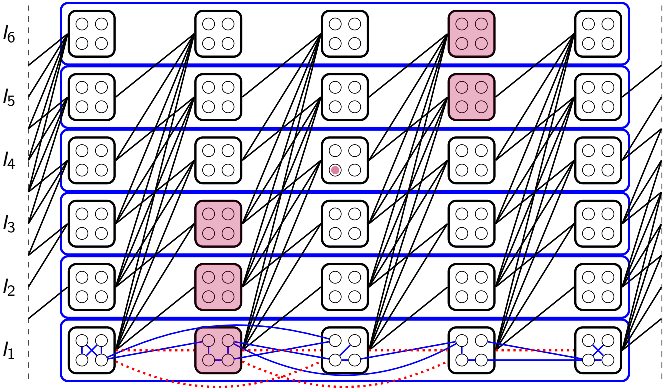


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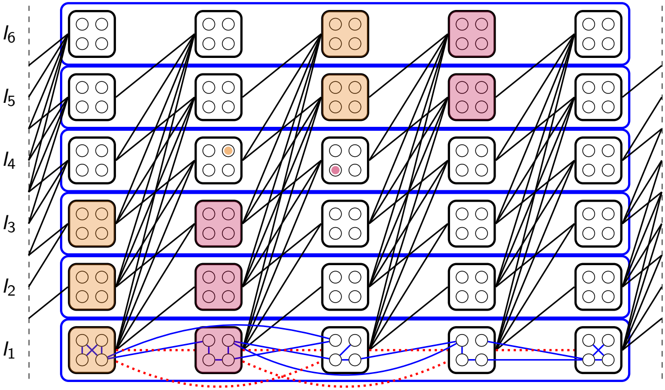


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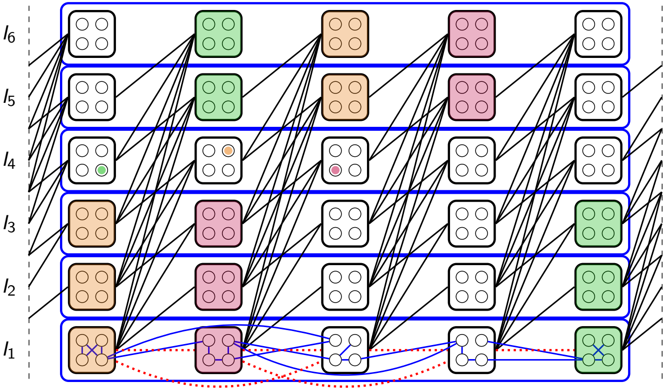


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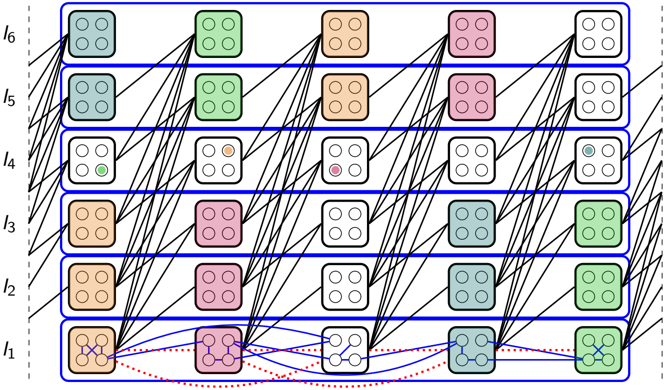


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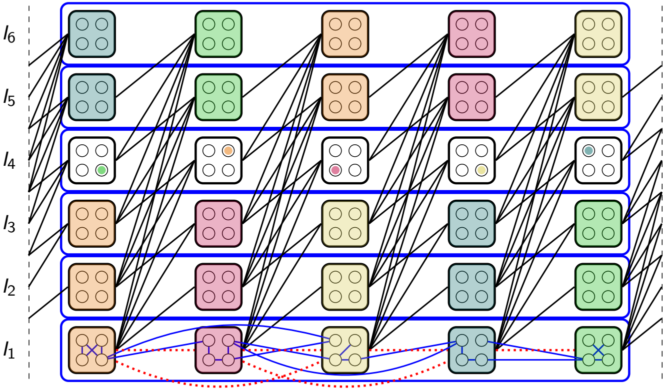


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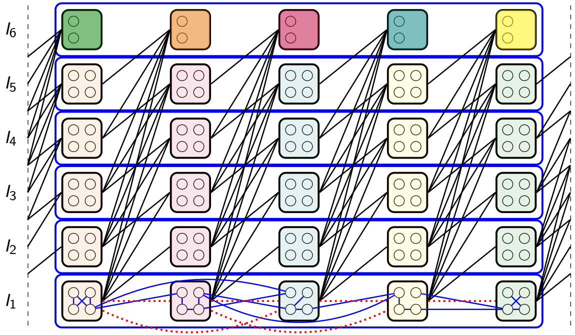
Assume $(H, k)$ is positive:

- Dummy instance $I_6$ forces one vertex per column (figure),



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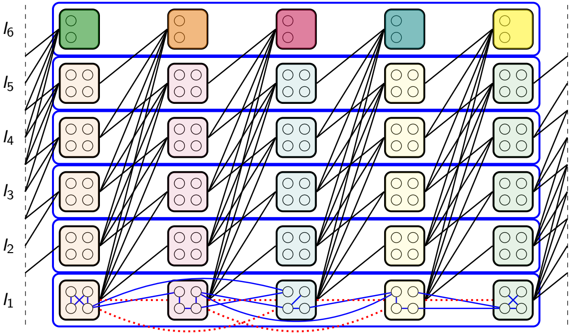

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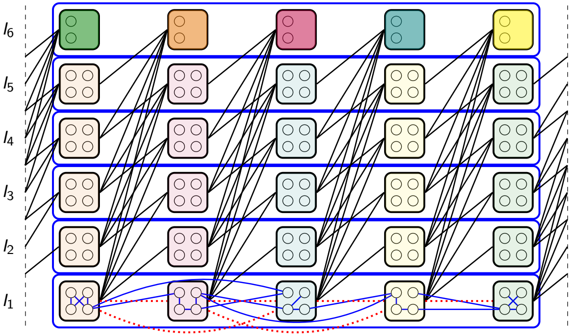

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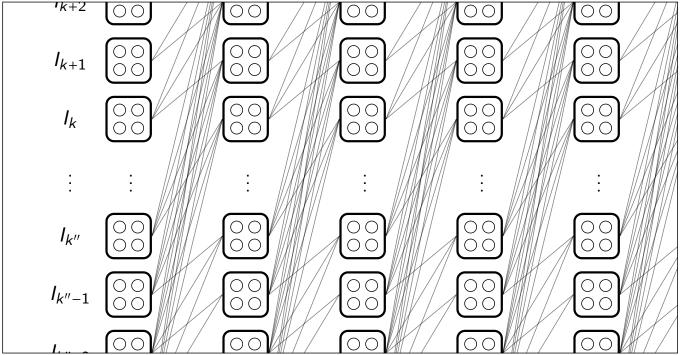

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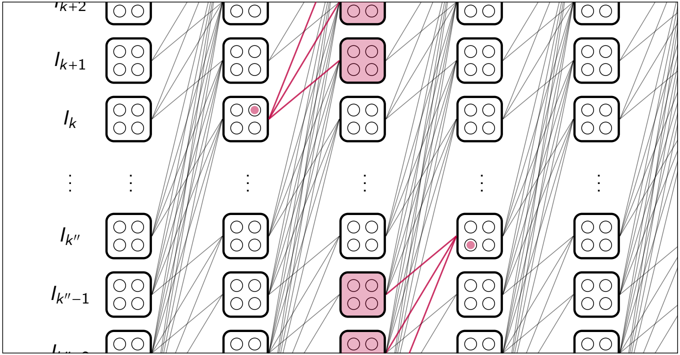
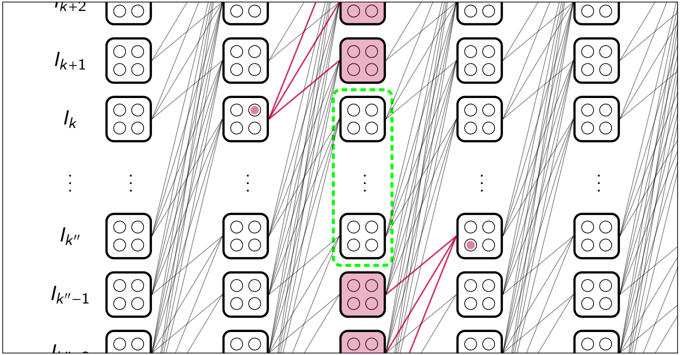


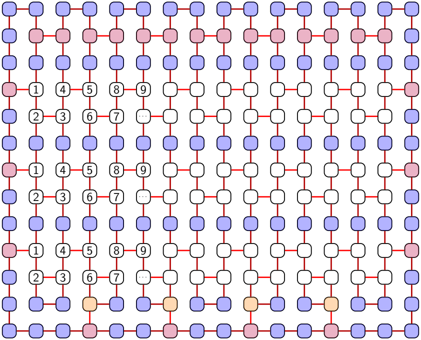
# Positive composition ⇒ Positive tailored instance

- Non-identical layer choices → domination gap,
- At least two classes are not dominated by neighbouring columns,
- Only one choice left in this column → absurd.



# Bounding the twin-width

- For any $I_i$, each $\mathscr{B}_{i,j}$ is a module in $H - I_i$ → contraction impacts only $I_i$,

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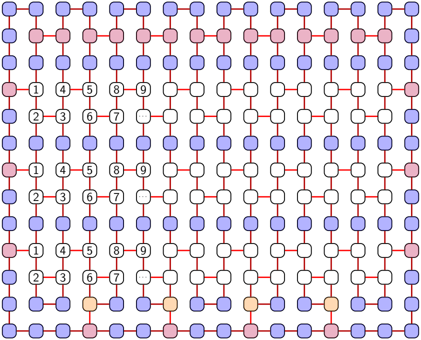


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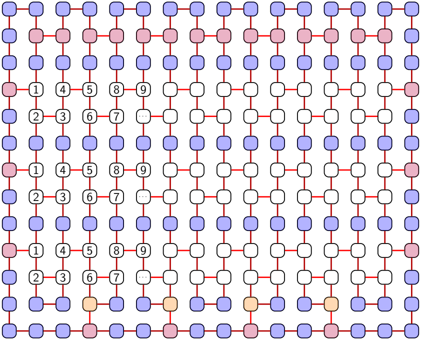
- Result: red subgraph of grid (altered for *tww* 4).

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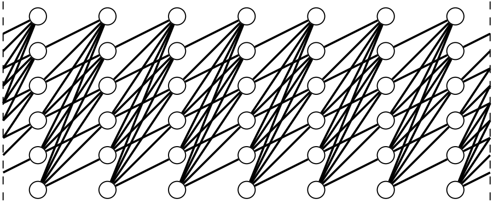
- Between instances : cycle of half graphs,



# Bounding the twin-width

- Between instances : cycle of half graphs,
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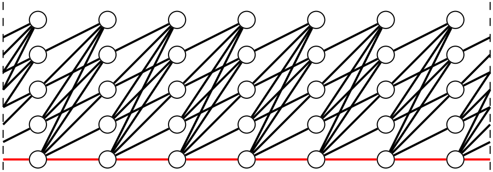

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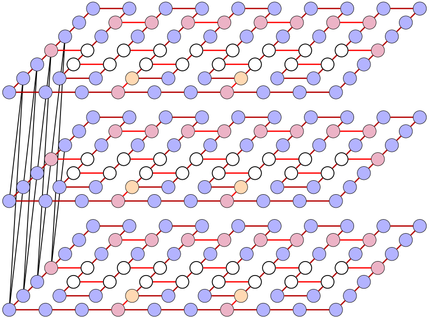

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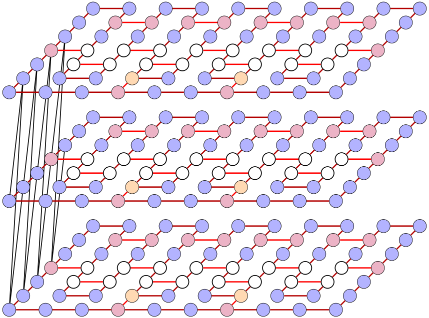

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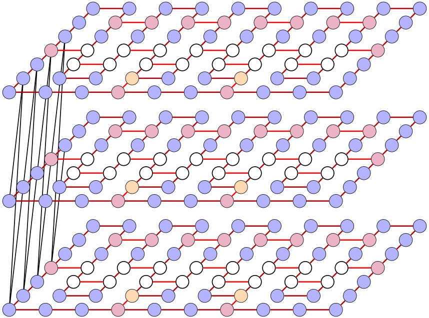
- Each contraction: creates red edges only locally → induction



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**Theorem (Bonnet, Kim, R., Thomassé, Watrigant '21)**
*Unless* coNP $\subseteq$ NP/*poly, $k$-*DOMINATING SET *on graphs of twin-width at most 4 does not admit a polynomial kernel, even if a 4-sequence of the graph is given.*

# A (sub)quadratic kernel for C-VC on VC-density 1

**Lemma (Bonnet, Kim, R., Thomassé, Watrigant '21)**
*There exists $f$ s.t. for any $G$ of tww( $d$) and $X \subseteq V(G)$: number of distinct neighborhoods in $X$ is at most $f(d)|X|$.*

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- $\rightarrow$ in $X$, number of distinct neighbourhoods $\leqslant f(d)k$

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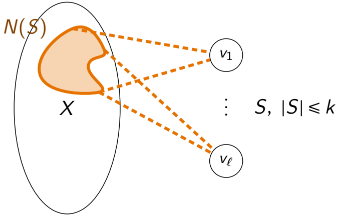


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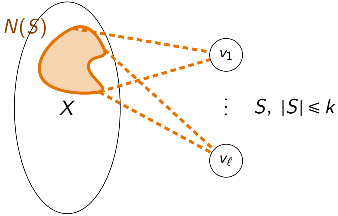


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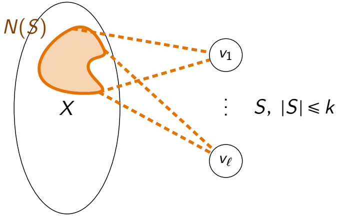


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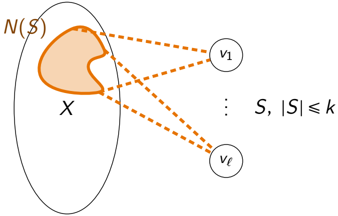


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If $(G',k)$ has a solution $T$, $T$ is also a solution for $(G,k)$,

- Take deleted $s$, $S \backslash s \not\subseteq T$ as ind. set with $|S| \geqslant k$,
- Therefore $N(S) \subseteq T$, and $s$ is covered by $T$ in $G$.

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> **Theorem (Bonnet, Kim, R., Thomassé, Watrigant '21)**
> *On classes of VC-density 1,* Connected *$k$*-Vertex Cover *admits kernels with $O(k^2)$ vertices (and even $O(k^{1.5})$).*

# Further Directions

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Thank you!

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