Twin-width and polynomial kernels

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joint work with

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Stéphan Thomassé, Rémi Watrigant

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[1] some slides have been recklessly stolen from Édouard

Overview

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Question: What problems admit polynomial kernels for classes of small twin-width ?

Our results

Theorem (Bonnet, Kim, R., Thomassé, Watrigant '21)
On classes of bounded twin-width, CONNECTED k-VERTEX COVER *admits kernels with* $O(k^{1.5})$ *vertices.*

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In this talk:

Theorem (Bonnet, Kim, R., Thomassé, Watrigant '21)
Unless coNP $\subseteq$ NP/*poly,* k-Dominating Set *does not admit a polynomial kernel on graphs of twin-width at most 4.*

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- **Solution size**,
- Measures of "structural complexity" for graphs: diameter, genus, treewidth...

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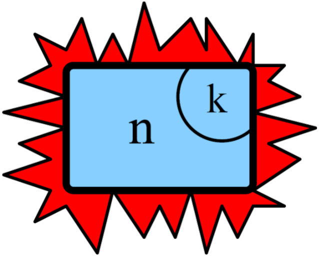

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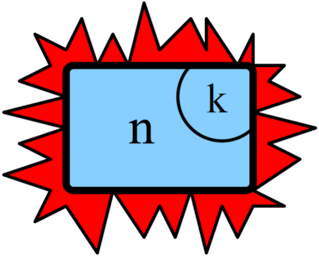

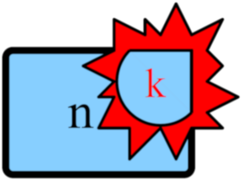

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- Hamiltonian Cycle parameterized by treewidth tw:
  $\hookrightarrow$ instance (G, tw) solvable in time $O(tw^{tw}|G|)$

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Definition ***Kernel*** *for parameterized problem* $\mathcal{Q}$:

- *Polytime reduction from instance* (I, k) *for* $\mathcal{Q}$ *to equivalent instance* (I', k') *for* $\mathcal{Q}$.
- *output instance has size* $|I'| = h(k)$, *and* k' *is bounded by a function of* k.

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- k-Vertex-Cover, k-Dominating-Set on sparse graphs ($K_{t,t}$-free).

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↪ can we prove that a problem has no polykernel?

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- G' is (very) small: $|G'| \leqslant k^c \leqslant n^c < t$: **less than one bit per initial instance**,
- kernelization must have dismissed / "solved" one G_i entirely... in polynomial time → absurd.

Ruling out polynomial kernels

Formalization: given problem $\mathscr{L}$, and parameterized problem $\mathscr{Q}$.

Definition 2.1 (OR-composition) Polytime reduction:

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Definition 2.3 (OR-composition) Polytime reduction:

- Input: Instances $I_1, \ldots, I_t$ for $\mathscr{L}$,
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Definition 2.4 (OR-composition) Polytime reduction:

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Definition 2.5 (OR-composition) Polytime reduction:

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Theorem (Bodlaender, Jansen, Kratsch '12)
If an NP-hard problem $\mathscr{L}$ admits an OR-composition into a parameterized problem $\mathscr{Q}$, then $\mathscr{Q}$ does not admit a polynomial kernel unless coNP $\subseteq$ NP/*poly.*

Graph invariants and FPT

"Meta-theorems" for problems expressible by logic formulas:

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> **Theorem 1 (Courcelle '90)** *Given a **MSO formula** ϕ on a graph G, deciding whether ϕ is satisfied by G is **FPT** in $|\phi|$ on graphs of **bounded tree-width**[2].*

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- Efficiency depends on how "**near**" vertices are to being twins → twin-**width**.

Graphs and trigraphs

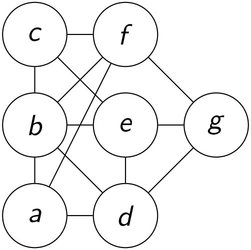

Fig. Graph: edges, non-edges

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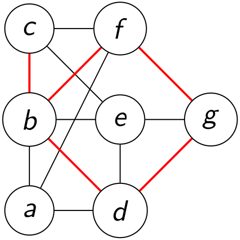
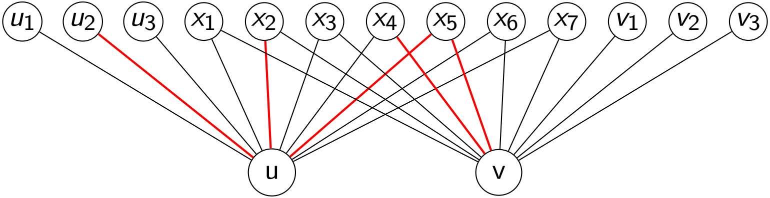


Fig. Trigraph: edges, non-edges, or **red** edges

Contractions in trigraphs

Intuitive goal: group near-twins together.



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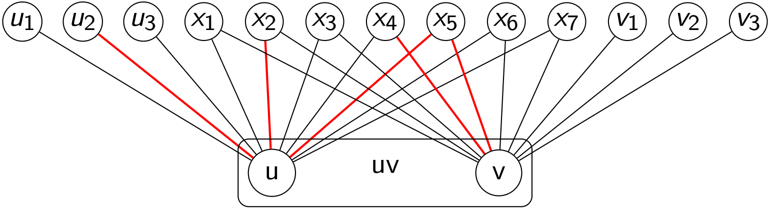

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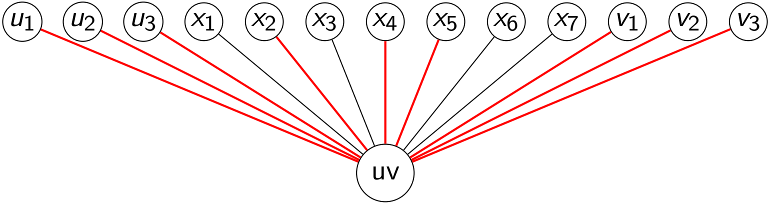
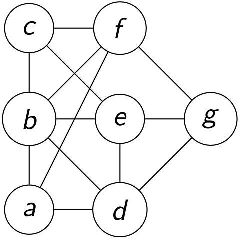


edges to $N(u) \triangle N(v)$ turn red, for $N(u) \cap N(v)$ red is absorbing

Twin-width

Definition (Twin-width) *tww(G): minimal d such that G admits a* ***contraction sequence*** *where all trigraphs have* ***maximum red degree*** *at most d.*



Maximum red degree $= 0$

overall maximum red degree = 0

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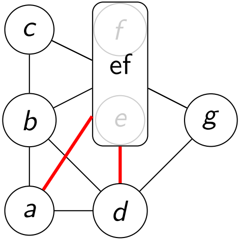

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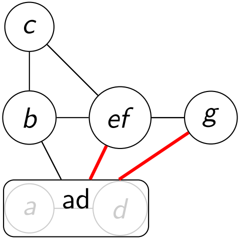

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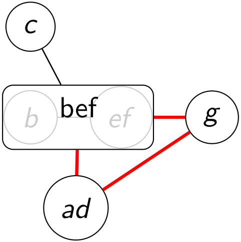

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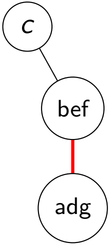

Maximum red degree = 1

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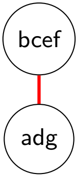

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Maximum red degree $= 0$
overall maximum red degree = 2 $\rightarrow tww(G) \leqslant 2$.

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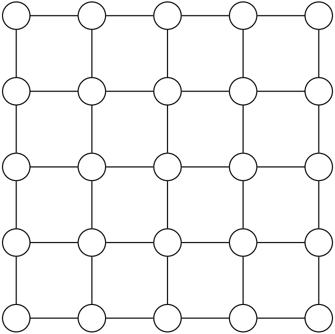
- Contraction sequences → dynamic programming
- bounded red degree → few "complicated" updates.

Example: twin-width of grids

Grids have unbounded tree-width... what about twin-width?



Maximum red degree = 0

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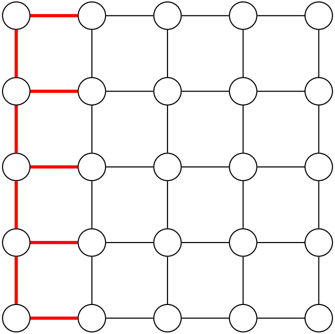

Maximum red degree = 3

overall maximum red degree = 3

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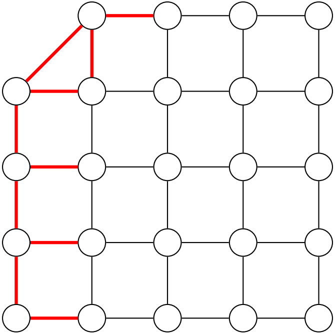

Maximum red degree = 4

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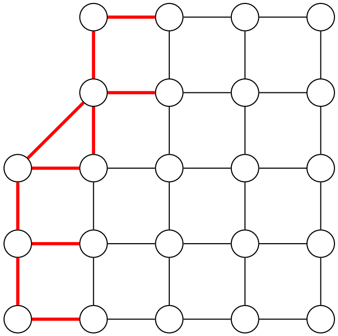

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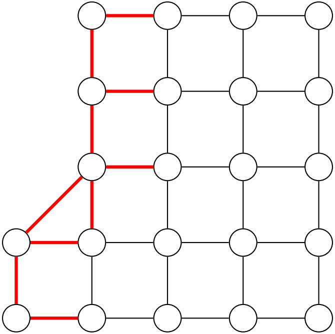

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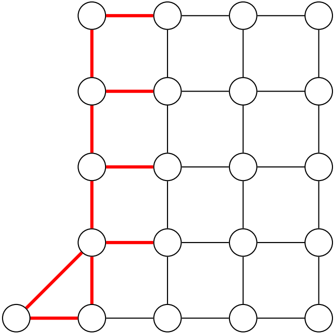

Maximum red degree = 4

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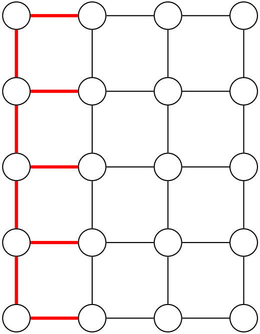

Maximum red degree = 3
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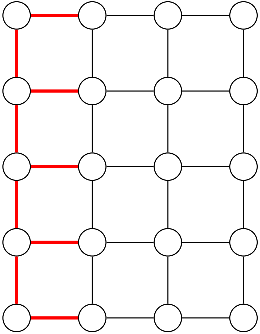
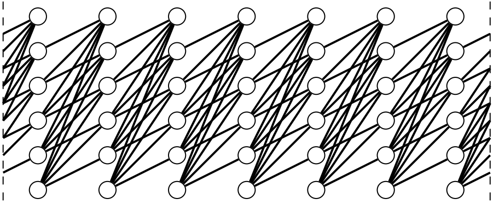


Maximum red degree = 3

overall maximum red degree = 4 ⇒ twin-width $\leqslant 4$

Example: Half-Graph Cycles

A dense graph with bounded twin-width:



Maximum red degree = 0

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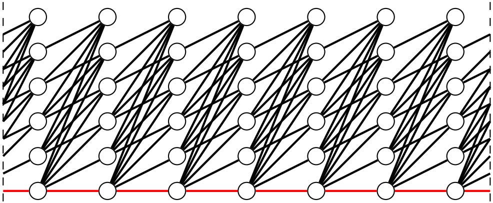

Maximum red degree = 2

overall maximum red degree = 2

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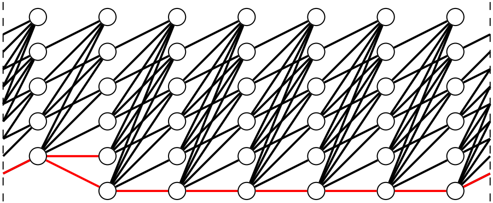

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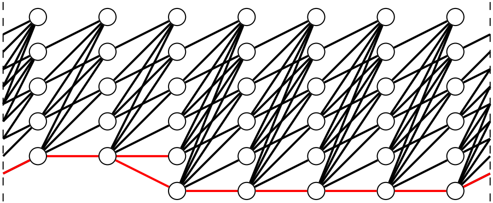

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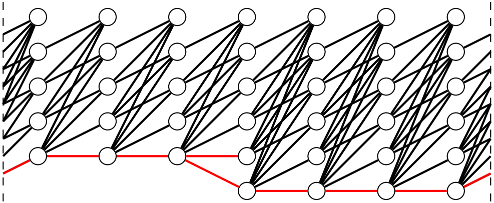

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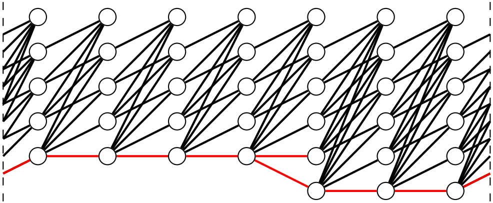

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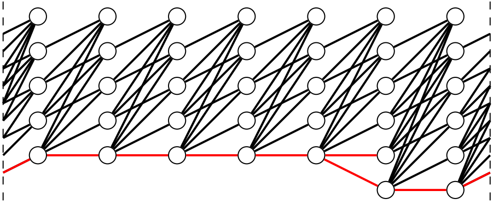

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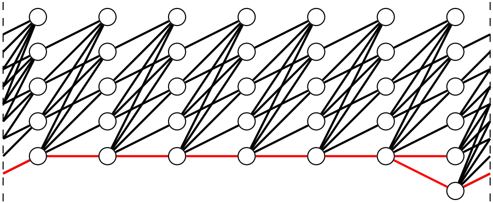

Maximum red degree $= 3$

overall maximum red degree = 3

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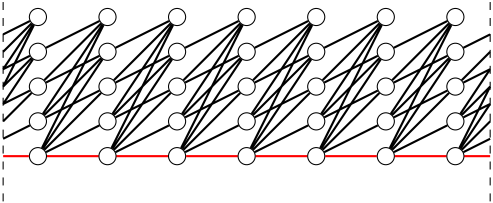

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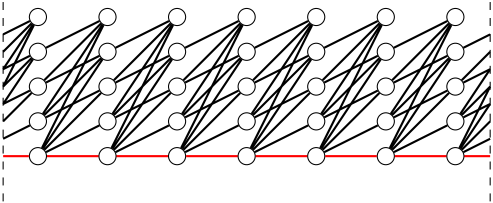

Maximum red degree $= 2$

overall maximum red degree = 3 $\Rightarrow$ **twin-width** $\leqslant 3$

Twin-width : FPT first order problems

Recall Courcelle for MSO and tree-width:

[3] given a tree decomposition
[4] given a contraction sequence

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Theorem (Courcelle '90)
*Given a **MSO formula** ϕ on a graph G, deciding ϕ on G is **FPT** in $|\phi|$ on graphs of **bounded tree-width**[3].*

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Analogue: FO and twin-width:

Theorem (Bonnet, Kim, Thomassé, Watrigant '20)
*Given a **FO formula** ϕ on a graph G, deciding ϕ on G is **FPT** with respects to $|\phi|$ on classes of **bounded twin-width**[4].*

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Consequences

Some **problems** expressible in FO:

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↪ very broad !

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Admits polykernels on sparse graphs... **but not on bounded twin-width.**

OR-composition for k-DS

Recall the tool to prove no polykernels:

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Theorem (Bodlaender, Jansen, Kratsch '12)
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- $\mathscr{Q}$: k-DS on a class of bounded twin-width ($\leqslant 4$).
- $\mathscr{L}$: NP-hard problem tailored to be "easily" composable into $\mathscr{Q}$. A version of k-DS on graphs of bounded twin-width.

Tailored k-DS on grid-like graphs

NP-hard $\mathscr{L}$: k-DS on instances $(G, k, \mathscr{B} = \{B_1, \dots, B_k\})$ s.t:

- $\mathscr{B}$ partitions $V(G)$, G has 4-sequence $\rightarrow G/\mathscr{B}$,
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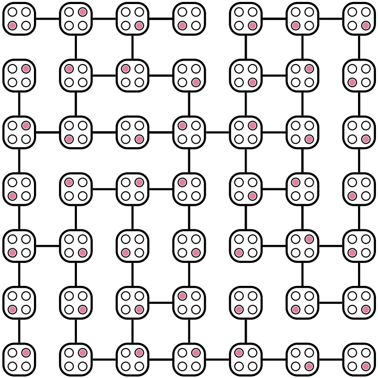

OR-composition

Tailored k-**DS**, instances $(I_i)_{i \in [t]} \rightarrow k$-**DS**, $tww \leqslant 4$ instance (H, k)

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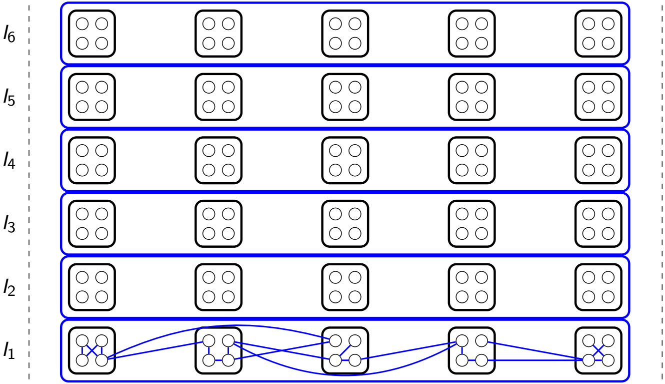

Fig. Instances ↔ layers, partition classes ↔ boxes.

OR-composition

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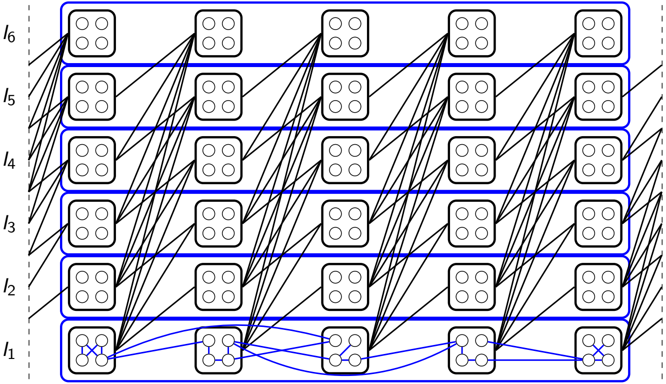
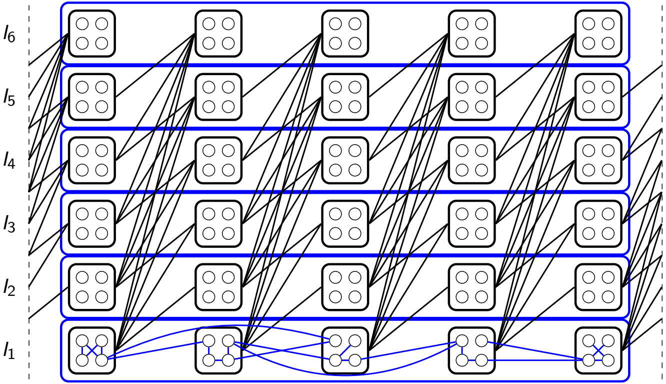


Fig. Instances ↔ layers, partition classes ↔ boxes.

In H**:** Top instance ↔ dummy, half graphs cycle between classes

Positive Tailored instance ⇒ Positive composition

Assume one instance (I_4 here) is positive, pick its solution in H:



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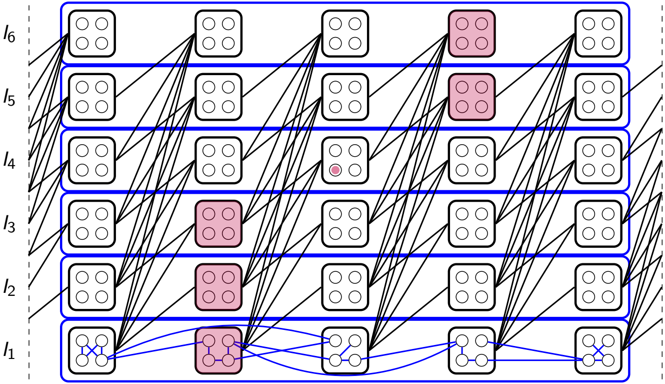

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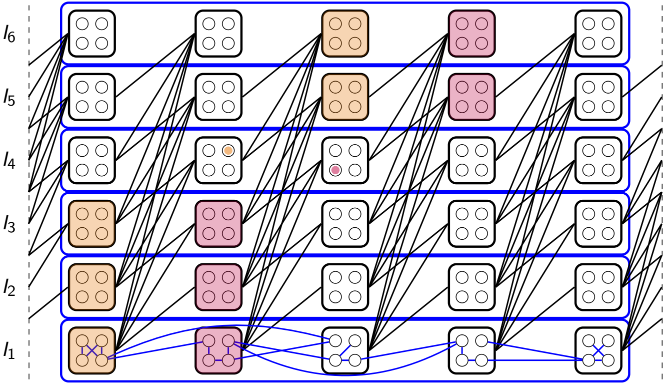

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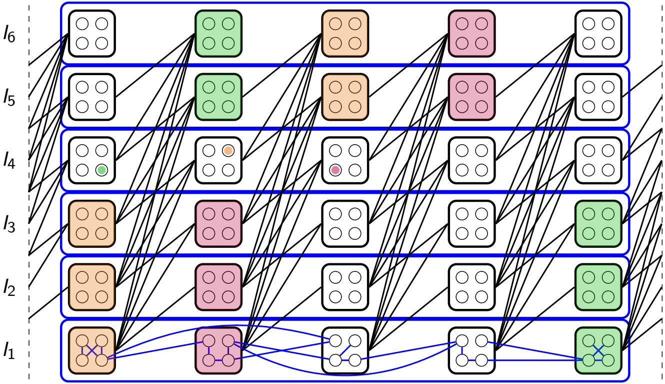

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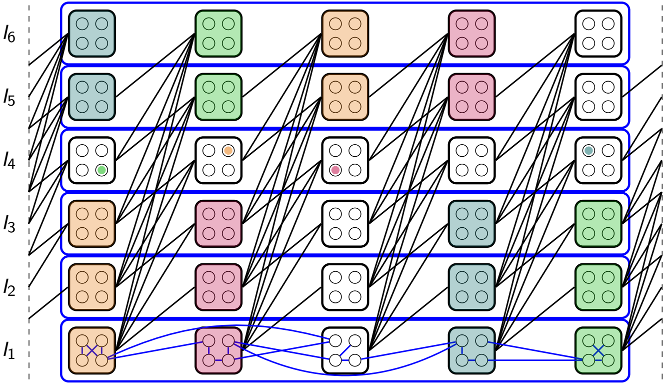

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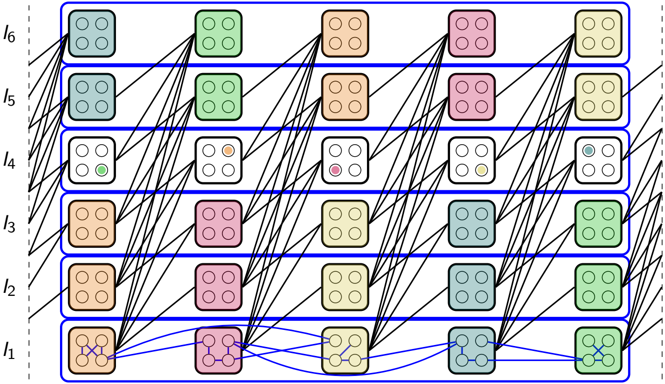

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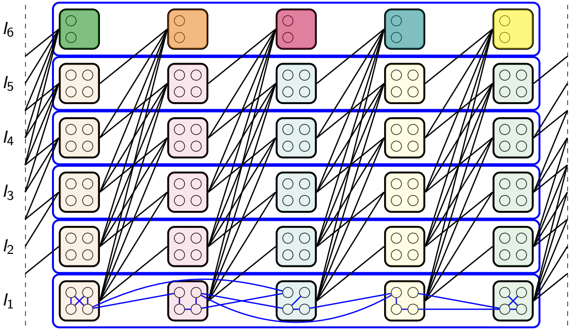

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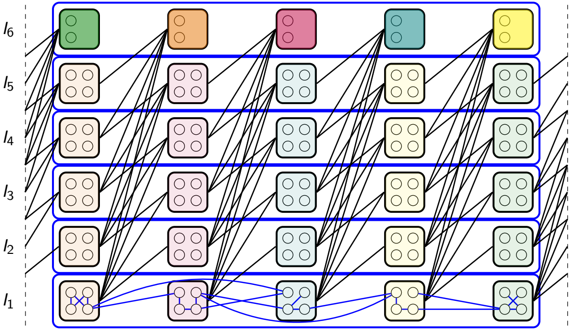

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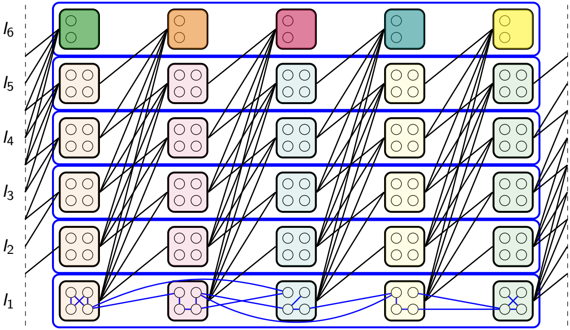
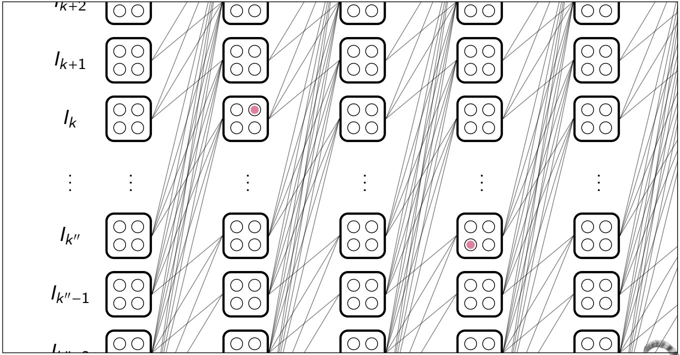
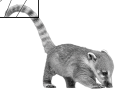


Positive composition ⇒ Positive tailored instance

- Non-identical layer choices → domination "gap" on C_j,





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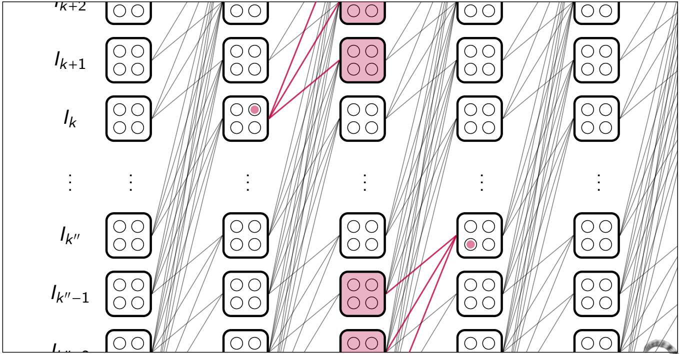

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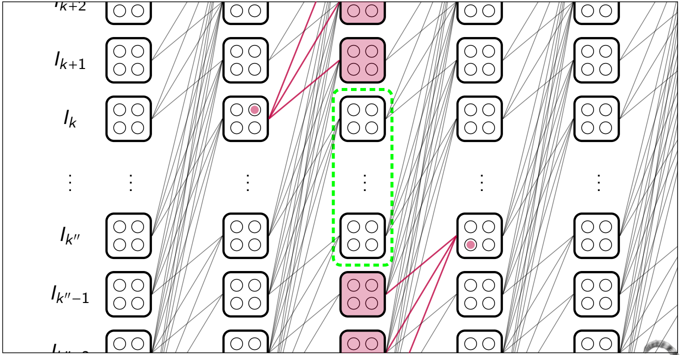

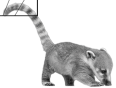

Positive composition ⇒ Positive tailored instance

- Non-identical layer choices → domination "gap" on C_j,
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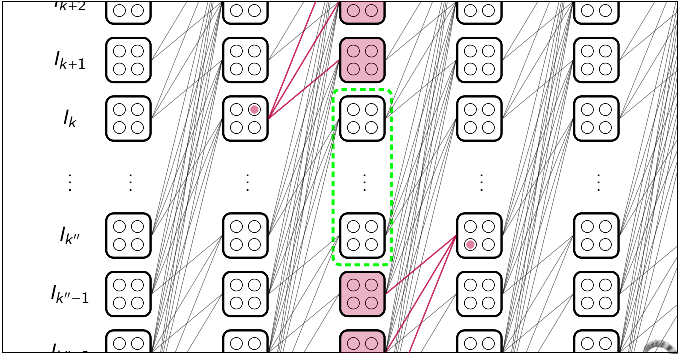
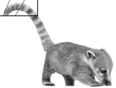
- Only one choice left in C_j → absurd.





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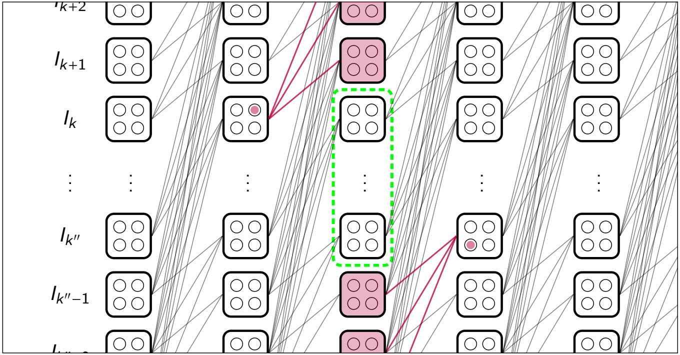

Solution lies on a single layer → corresponding instance is dominated.

Bounding tww: contracting partition classes

- For any I_i: each partition class has same neighbours in $H - I_i$,

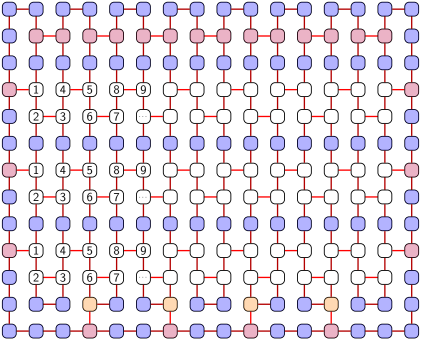

Fig. An instance after contracting classes.

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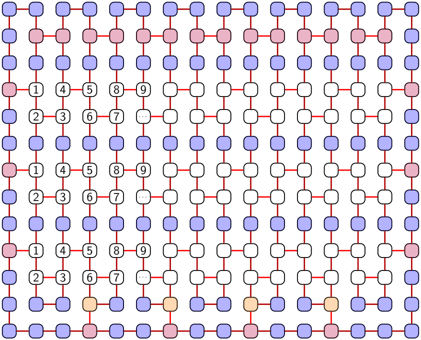
- For any I_i: each partition class has same neighbours in $H - I_i$,
- Contract each class → red edges created only in I_i



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- Result: red subgraph of grid (altered for *tww* 4).

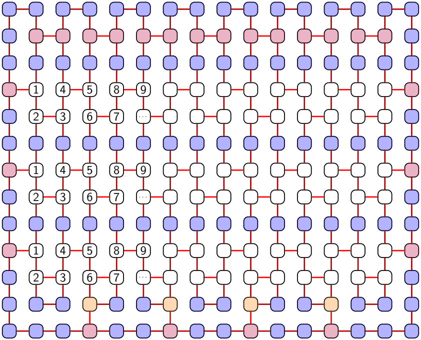


Fig. An instance after contracting classes.

Bounding tww: contracting instances together

- Between instances: half-graph cycle along vertices at the same position.

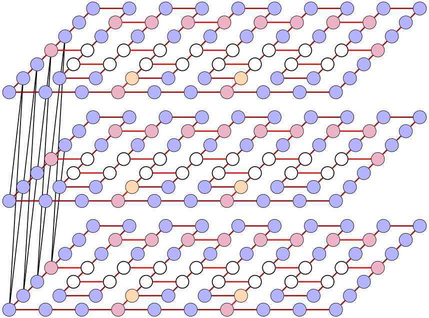


Fig. Instances and the half-graph cycle

Bounding tww: contracting instances together

- Between instances: half-graph cycle along vertices at the same position.
- Goal: Successively contract the bottommost two instances.

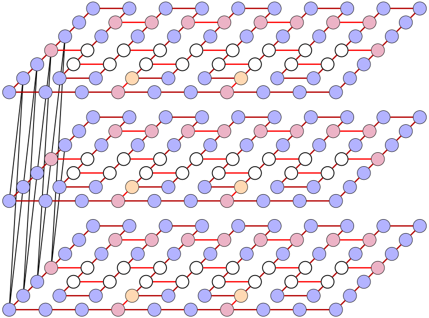
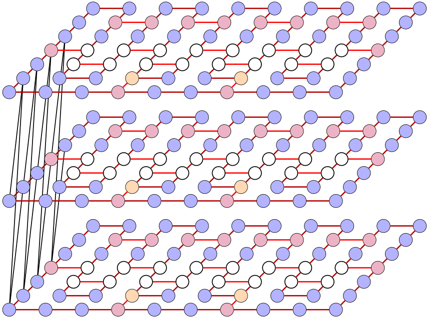


Fig. Instances and the half-graph cycle

Bounding tww: contracting instances together

- Between instances: half-graph cycle along vertices at the same position.
- Goal: Successively contract the bottommost two instances.
- Contracting two vertices at the **same position** creates red edges only **locally** → induction



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Theorem (Bonnet, Kim, R., Thomassé, Watrigant '21)
Unless coNP $\subseteq$ NP/*poly, k*-DOMINATING SET *on graphs of twin-width at most 4 does not admit a polynomial kernel*[5]*.*

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Further Directions

- Does k-DS on tww $\leqslant 3$ admit polynomial kernels?

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Thank you!

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A (sub)quadratic kernel for C-VC on VC-density 1

Lemma (Bonnet, Kim, R., Thomassé, Watrigant '21)
There exists f s.t. for any G of tww(d) and $X \subseteq V(G)$: number of distinct neighborhoods in X is at most $f(d)|X|$.

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- $\rightarrow$ in X, number of distinct neighbourhoods $\leqslant f(d)k$

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Reduction rule: If there is $S \subseteq V(G) \setminus X$ with identical neighbourhood in X and $|S| > k$, delete a vertex of S.

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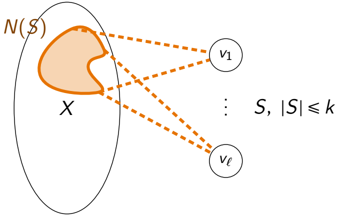

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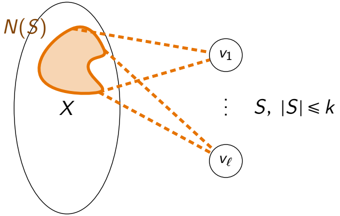

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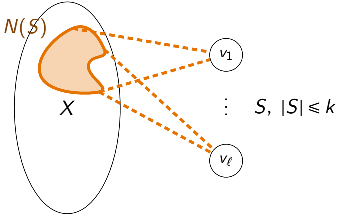

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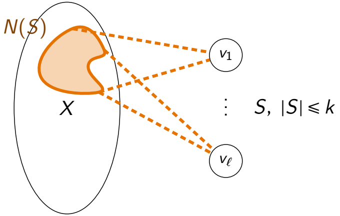

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If (G, k) has a solution, replace deletion with twin → reconnects T

If (G', k) has a solution T, T is also a solution for (G, k),

- Take deleted s, $S \setminus s \not\subseteq T$ as ind. set with $|S| \geqslant k$,
- Therefore $N(S) \subseteq T$, and s is covered by T in G.

Concluding

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Theorem (Bonnet, Kim, R., Thomassé, Watrigant '21)
On classes of VC-density 1, CONNECTED k-VERTEX COVER *admits kernels with $O(k^2)$ vertices (and even $O(k^{1.5})$).*